Multi-agent Logics with Frozen States, Admissibility via Projectivity

Vladimir V. Rybakov
Institute of Mathematics and Computer Science,
Siberian Federal University, Krasnoyarsk;
and Institute of Informatics Systems Ac. Sci., Novosibirsk
Vladimir_Rybakov@mail.ru

Subject Classification: 510.64,510.65,510.66

Abstract. [1]

We study multi-agent temporal logics with models which allow frozen states as information nodes (information states). This models similar to standard ones for non-classical multi-modal temporal logics, but the current time clusters may posses states with broken connections of information channels (or chains of agents' connections with restricted possibilities). Such models imitates real situations which may happen with networks of agents' information. In order to solve open problems of decidability in such logics and problems of recognizing admissible rules we use elements of unification and projective formulas techniques. Based at such tools we find positive solutions of these open problems.

Key words: *temporal logics, multi-agent logics, information, knowledge, deciding algorithms*

1 Introduction

Multi-agent logics has become an active research area in AI and social sciences. E.g. the book [1] suggests approach which uses multi-dimensional Kripke models $M = \langle W, R_1, \ldots, R_n, V \rangle$, where W represents some amount of states and relations R_j models agents' accessibility relations between states. In this approach modal-like formula K_iA may be interpreted as – the agent i knows the information (or the knowledge) described by formula A. That technique may be extended by using instead the formula K_iA a formula which says that the agent is simply informed (a formula $A_i(x)$ illustrates that *the agent i knows (may know) information x*). That ideas were developed in Rybakov [2], in particular,

[1]This work was supported by the Russian Science Foundation and Krasnoyarsk Regional Fund of Science (Project No 25-21-20011, https://rscf.ru/en/project/25-21-20011/).

there we find algorithm for decidability such logics. To date, many works on application of non-standard logic to AI and computation logical information appeared (cf. for instance [3, 4, 5, 6, 7, 8]). In this our paper we develop some new logical techniques in order to solve open problems for the case of multi-agent logics whose models allow frozen states as information nodes (information states).

The models look as standard multi-modal temporal logics, but the current time clusters may possess states with broken connections of information channels. All accessibility relations of agents' - relations R_j – in each time cluster $C(i)$ defined separately and independently. This approach studies possible conflicts with networks of agents' information. In order to solve open problems of decidability in such logics and problems of recognizing admissible rules we use some elements of unification and projective formulas. These tools allow to find efficient and short proofs and to solve open problems.

2 Preliminary Information, Notation

We start by short recall known technique which we will use in this paper. Basic knowledge in non-classical logic is assumed. We need to report basics of unification. For a set of propositional letters P a substitution is a mapping ε of P into set of all formulas For; any such substitution may be extended to set of all formulas by $\varepsilon(\varphi(x_1,\ldots,x_n)) := \varphi(\varepsilon(x_1),\ldots,\varepsilon(x_n))$. A formula is said to be unifiable in a logic L if there is a substitution ε (which is said to be an unifier for φ) such that $\varepsilon(\varphi) \in L$.

Definition 2.1. *A unifier ε for a formula φ in a logic L is said to be more general then a unifier ε_1 if there is a substitution δ such that for any letter x, $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$.*

If a logic L is decidable, to verify unifiability of a formula is an easy task (in theoretical means, since complexity of computation may be very high) : actually it is sufficient to verify only substitution of formulas from the set $\{\bot, \top\}$ instead propositional letters - variables. At the same time, the question of finding all unifiers is not simple.

Definition 2.2. *A set of unifiers CU for a formula φ in a logic L is complete set of unifiers if the following holds. For any unifier σ of φ in L, there is a unifier σ_1 from CU, such that σ_1 is more general then σ.*

For modal logics, for example extending modal logic $S4$, the definition of projective formulas is as follows:

Definition 2.3. *A formula φ is projective in a logic L if the following holds. There exists a substitution σ (which is called projective substitution for formula φ) such that σ is a unifier for φ in L and $\Box\varphi \rightarrow [x_i \equiv \sigma(x_i)] \in L$ for any propositional letter x_i from φ.*

Lemma 2.4. *If a substitution σ_p is projective for φ in L, then $\{\sigma_p\}$ is a complete set of unifiers for φ (that is σ_p is more general unifier).*

Proof. Assume that σ is some unifier for φ in L. Since σ_p is projective for φ in L, we get $\Box\varphi \rightarrow [x_i \equiv \sigma_p(x_i)] \in L$ for any letter x_i from φ. Applying σ to formulas above we get $\sigma(\Box\varphi) \rightarrow [\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in L$, that is $\sigma(x_i) \equiv \sigma(\sigma_p(x_i)) \in L$. Q.E.D.

Notice that, if for a decidable modal logic L, we can verify if any given formula φ is projective and can compute its projective unifier, then **the admissibility problem in logic L is decidable**. Indeed, recall *the definition.* An inference rule $\varphi_1, \dots \varphi_n/\psi$ is admissible in L iff for any unifier ϵ for $\varphi_1 \wedge \dots \wedge \varphi_n$ this ϵ is also a unifier for ψ.

To show decidability of the admissibility problem then, admit that an inference rule $\varphi_1, \dots \varphi_n/\psi$ is admissible in L, Admit that there is a unifier for $\varphi_1, \dots \varphi_n$. Then by our assumption above we may compute its projective unifier and it is enough just to verify if this projective unifier (which is then most general unifier) is a unifier for ψ, and the problem is solved. This fruitful idea was found by Silvio Ghilardi [9,10]. In this our paper we will apply elements of this technique below. We assume that the reader has some initial knowledge about work of rational models in non-classical logic, though we also recall some necessary definitions below.

3 Decidability by Projectivity

As we noted above, applications of projective technique was found by Silvio Ghilardi [9,10] and later on it was applied to the linear modal logics in works of Wojciech Dzik and Piotr Wojtylak for modal logics extending S.4.3 cf. [11]. Recently we used similar technique for multi-modal logics studying transfer information and its' reliability (cf. [12]). We start now by definition of semantics for our logic - Kripke-like frames of our logic. By N we denote the set of all natural numbers. J is some finite set contained the notations (names) of agents.

Definition 3.1. *Frames $F := \langle \bigcup_{i\in N} C(i), \leq, Prev, R_j, j \in J\rangle$ are relational structures where sets $C(i)$ for $i \in N$ form infinite sequence of not-intersecting non-empty sets $C(i)$ of states (which may be thought as clusters of time).*

For $j \in J$, each R_j is a binary reflexive and transitive (agents') accessibility relation on each $C(i)$. Relation $\leq$ is the the binary linear–time like relation (imitating time order): for all $a, b \in \bigcup_{i\in N} C(i)$, $a \leq b$ iff $a \in C(i)$ for some $C(i)$ and $b \in C(i_1)$, where $i \leq i_1$.

$Prev$ is the binary relation (introduced to distinguish the previous time cluster), $\forall a, b \in \bigcup_{i\in N} C(i)$, $a\ Prev\ b \Leftrightarrow b \in C(i+1), a \in C(i)$ or $a = b$.

Here, each R_j imitates an agents binary reflexive and transitive accessibility relation on each set of states $C(i)$. The binary relation $\leq$ imitates the run of time. $Prev$ is a binary relation introduced to distinguish the previous time cluster (we admit also that $a\ Prev\ a$, because it is important to our research technique would work properly).

A model M for our logic is a frame F with a valuation V of a set of propositional letters in states of the frame. A valuation V of a set of propositional letters (variables) P in F is a mapping V of P in the set of all subsets of $\bigcup_{i\in N} C(i)$, so for every $p \in P$, $V(p) \subseteq \bigcup_{i\in N} C(i)$. We write for $a \in F$ that $a \Vdash_V p$ iff $a \in V(P)$, and we say then that p is true at a w.r.t. V.

The language of our temporal multi-agent logic consists of propositional letters, usual Boolean logical operations, temporal-modal operations $\Box$ and $\Diamond$, and agents' operations $\Box_i, \Diamond_j, j \in J$ and the previous-time unary operation $\Diamond_{Prev}$. Rules for constructions of formulas by these operations are standard.

Below we consider the states a from the base set $\bigcup_{i\in N} C(i)$ of our model, that is we assume $a \in \bigcup_{i\in N} C(i)$. For any Kripke model M, the truth values can be extended from propositions letters P to arbitrary formulas constructed from these propositions as follows.

$\forall p \in Prop\ (M,a) \Vdash_V p \Leftrightarrow a \in \bigcup_{i\in N} C(i) \wedge\ a \in V(p);$

$(M,a) \Vdash_V (\varphi \wedge \psi) \Leftrightarrow (M,a) \Vdash_V \varphi\ \wedge\ (M,a) \Vdash_V \psi;$

$(M,a) \Vdash_V (\varphi \vee \psi) \Leftrightarrow (M,a) \Vdash_V \varphi\ \vee\ (M,a) \Vdash_V \psi;$

$(M,a) \Vdash_V \neg\varphi \Leftrightarrow ([(M,a) \Vdash_V \varphi]\ \textit{is not true});$

$(M,a) \Vdash_V \Diamond\varphi \Leftrightarrow \exists b[(a \le b)\&(M,b) \Vdash_V \varphi];$

$(M,a) \Vdash_V \Box\varphi \Leftrightarrow \forall b[(a \le b) \Rightarrow (M,b) \Vdash_V \varphi];$

$(M,a) \Vdash_V \Diamond_j\varphi \Leftrightarrow \exists b[(aR_jb)\&(M,b) \Vdash_V \varphi];$

$(M,a) \Vdash_V \Box_j\varphi \Leftrightarrow \forall b[(aR_jb) \Rightarrow (M,b) \Vdash_V \varphi];$

$(M,a) \Vdash_V \Diamond_{Prev}\varphi \Leftrightarrow \exists b[(bPreva)\&(M,b) \Vdash_V \varphi];$

For a Kripke model $M := \langle F, V\rangle$ and a formula φ with letters from the domain of V, φ is true in M (denotation – $M \Vdash \varphi$) if, for any state b from F, the formula φ is true at b (that is: $(M,b) \Vdash_V \varphi$). For a frame $F := \langle W, R\rangle$, we say that a formula φ is true at F, (and we will write $F \Vdash \varphi$) if φ is true at any model based at F.

We now impose some additional requirement on the structure of all models M of our logic in order to express possible situations for agents accessibility relations.

Definition 3.2. *Let $P_n := \{p_1, \ldots, p_n\}$. Let for all $Y \subseteq P_n$,*

$$\psi(Y) := \bigwedge_{p_i\in Y} p_i \wedge \bigwedge_{p_i\notin Y} \neg p_i).$$

We say that a model M with a valuation V for P_n has frozen states w.r.t. P_n if for any $C(i)$ there is some non-empty set of subsets $Y_1, \ldots, Y_m$ of P_n (some Y_j may be empty) with the following properties.

(1) for any Y_j there is $a \in C(i)$ such that $b \Vdash_V \psi(Y_j)$; for any $a \in C(i)$ there is Y_j such that $b \Vdash_V \psi(Y_j)$.

(2) For any $a \in C(i)$ where $a \Vdash_V \psi(Y_j)$ and any $b \in (C(i)$ and any agents' accessibility relation R_h $(h \in J)$, if aR_hb then $b \Vdash_V \psi(Y_j)$.

So, states $a \in C(i)$ have frozen agents' accessibility relations R_h, so to say broken information channels. All accessibility relations of agents' - relations R_h – in each time cluster $C(i)$ are defined separately and independently. This approach corresponds close to real world situation with networks of agents' information.

Our logic L is *the set of all formulas $\varphi(p_1, \dots, p_n)$ which are true at any state of any model M with frozen sets w.r.t. $P_n := \{p_1, \dots, p_n\}$ with any given valuation.*

Lemma 3.3. *L is closed w.r.t. substitutions; so indeed is a logic. That is if $\varphi(p_1, \dots, p_m) \in L$ and $\alpha_1, \dots, \alpha_m$ are some formulas then $\varphi(\alpha_1, \dots, \alpha_m) \in L$.*

Proof. Let $\varphi(p_1, \dots, p_m) \in L$ but for some model with frozen states for letters from $\varphi(\alpha_1, \dots, \alpha_m)$ we have $(M, a) \Vdash_V \neg\varphi(\alpha_1, \dots, \alpha_m)$. Let $a \in C(i)$ and let the sets of formulas $Y_1, \dots, Y_m$ are taken from the definition of frozen model M for P_k where P_k are all letters of formulas $\alpha_1, \dots, \alpha_m$.

Let for any Y_j, $[Y_j]$ be the set of all states from $C(i)$ of M where the formula $\psi(Y_j)$ (recall $\psi(Y_j) := \bigwedge_{p_i \in Y_j} p_i \wedge \bigwedge_{p_i \notin Y_j} \neg p_i$) is true w.r.t. V. Then for any formula ψ constructed from letters of the set P_k by Boolean operations and all possible agents' modalities $\Diamond_h (h \in J)$ the truth value of ψ is the same for all states from $[Y_j]$.

This immediately follows from definitions and standard induction on the length of formulas. The same holds for all formulas ψ constructed from letters of the set P_k using all possible $\Diamond_h (h \in J)$ and in addition the time operation $\Diamond$; the inductive step of proof is evident also for operation $\Diamond$ because for all $c, d \in C(i)$ we have $c \leq d$.

And now we define the truth values of letters $p_1, \dots, p_m$ in our model M by a new valuation V_1 as follows. At all states from the set $[Y_j]$ for any $[Y_j]$ (recall that by (1) in the definition of models with frozen states, the union of all $[Y_j]$ gives the all set $C(i)$) we set that the truth value w.r.t. V_1 of any letter from $p_1, \dots, p_m$ is the same as the truth value of the corresponding formula $\alpha_1, \dots, \alpha_m$ in M w.r.t. the original valuation V. So then the model with V_1 is a model with frozen states, and $\varphi(p_1, \dots, p_m)$ is not true in this model. Lemma is proved. Q.E.D.

Recall that a formula $\varphi(p_1, \dots, p_n)$ is said to be unifiable in a logic L if there is a substitution $\varepsilon(x_i) := \beta_i$ arbitrary formulas β_i instead of propositional letters x_i such that $\varphi(\beta_1, \dots, \beta_n) \in L$.

Possible case for choice of our models for the logic L is the model M_0 when for all i $C(i) = \{a_{0,i}\}$ and when the valuation is $V_0(p_k) = \{a_{0,i} \mid 0, i\}$ for all p_k. That is for all $a_{0,i}$, $a_{0,i} \Vdash_{V_0} p_k$ for all k.

Lemma 3.4. *There is an algorithm, which for any formula $\varphi(p_1, \dots, p_n)$, constructed out of letters $p_1, \dots, p_n$, verifies if this formula is unifiable in L. If the formula is unifiable, then we may construct a substitution for letters $p_1, \dots, p_n$ which replaces p_j on y_j which is a unifier for $\varphi(p_1, \dots p_n)$ in L and any y_j is either $\top$ or $\bot$. We will denote in the sequel $\epsilon(p_j) := y_j$.*

Proof. If a formula $\varphi(p_1, \dots, p_n)$ is unifiable in L then there is a substitution replacing letters p_j by some formulas ψ_j, such that the formula $\varphi(\psi_1, \dots, \psi_n)$ is true by any valuation V at any state of any model for L with frozen sets of letters from $\varphi(\psi_1, \dots, \psi_n)$.

In particular this holds at the model M_o with frozen set P_n of all propositional letters from $\varphi(\psi_1, \dots, \psi_n)$, for all $a_{0,i}$, $(M_o a_{0,i}) \Vdash \varphi(\psi_1, \dots, \psi_n)$.

Then the formula $\varphi(\epsilon(p_1), \dots, \epsilon(p_n))$ is true in the one element model with evident accessibility relations w.r.t. the valuation V_1, where the truth value of any letter p_j is $\epsilon(p_j)$ where $\epsilon(p_j) = \top$ or $\epsilon(p_j) = \bot$ w.r.t. how ψ_j was true in M_o at any state.

For some substitutions $\epsilon_1(p_j) \in \{\top, \bot\}$, if the formula $\varphi(\epsilon_1(p_1), \dots, \epsilon_1(p_n))$ is true in the one element model it is easy to calculate. If we found a such substitution, then the formula $\varphi(\epsilon_1(p_1), \dots, \epsilon_1(p_n)$ is true in any model with frozen sets, so φ is unifiable. Q.E.D.

Theorem 3.5. *Any unifiable formula φ is projective in L. There is an algorithm verifying unifiability in L and constructing a projective unifier for φ if φ is unifiable (so, then the admissibility of inference rules in L is decidable).*

Proof. Given a formula $\varphi(p_1, \dots, p_n)$ constructed out of propositional letters p_i. By **Lemma 3.4.** we verify if this formula is unifiable in L and if yes we construct a shown in this Lemma unifier $\epsilon(p_j) := y_j$. We now will try to prove that this formula is projective.

Let $Var(\Box\varphi)$ be the set $\{p_1, \dots, p_n\}$ of all propositional letters from $\Box\varphi$ (so in other notation, $Var(\Box\varphi) = P_n$). Let, as earlier, for all $Y \subseteq P_n$, $\psi(Y) := (\bigwedge_{p_i \in Y} p_i \wedge \bigwedge_{p_i \notin Y} \neg p_i)$.

Assume that we have a model M with frozen states w.r.t. the set P_n. That means that for any $C(i)$ of our model there is an non-empty set of subsets SS — $Y_1, \dots, Y_m$ of P_n (that is any $Y_j \subseteq P_n$; some Y_j may be empty) with the following properties.

(a) for any Y_j there is $a \in C(i)$ such that $a \Vdash_V \psi(Y_j)$; for any $a \in C(i)$ there is Y_j such that $b \Vdash_V \psi(Y_j)$.

(b) For any $a \in C(i)$ where $a \Vdash_V \psi(Y_j)$ and any $b \in (C(i)$ and any agents' relation R_h, if aR_hb then $a \Vdash_V \psi(Y_j)$.

Let we fix strictly an order of all sets from SS as $Y_1, \dots, Y_m$. The real essence of the order is not important, but we need a fixed precise ranging formulas to choose the necessary single one for definition of the substitution below. At the beginning for any Y_j and letter p_i from $\{p_1, \dots, p_n\}$, we set $tv(p_i, Y_j) := \top$ if

p_i occurs in Y_j positively or $tv(p_i, Y_j) := \bot$ in opposite. We introduce now our substitution. For any letter p_i from P_n, let

$$\sigma(p_i) := [(\Box\varphi) \wedge p_i] \vee [\neg\Diamond\Box\varphi \wedge \epsilon(p_i)] \vee$$

$$[\neg\Box \wedge \Diamond[\Box\varphi \wedge \Diamond_{Prev}\neg\Box\varphi \wedge$$

$$[\bigvee_{k\leq m}[\bigwedge_{i\leq k}[\neg\psi(Y_i) \wedge \psi(Y_k) \wedge tv(p_i, Y_k)]]]]].$$

Now we will prove that this substitution is a projective unifier for φ in logic L. Choose any model $M = \langle F, V\rangle$ of our logic with a frame F and given valuation of the set of all letters P_n from φ, which is frozen w.r.t. P_n. Take any state $a \in F$.

Consider first the case when there is a such that $(F, a) \Vdash_V \neg\Diamond\Box\varphi$. By our definition of the unifier $\sigma(p_i)$ and Lemma 3.3. we have $\sigma(p_i)$ is equivalent to $\epsilon(p_i)$, where $\epsilon(p_i) = y_i$, and then σ it is a unifier for formula φ in our logic. So,

$$\forall b \in F,\ (F, b) \Vdash_V \sigma(\varphi)).$$

Let now $(F, a) \Vdash_V \Box\varphi$ for all a. Then the substitution σ does not change truth values of letters p_i in a and all b, where $a \leq b$, comparing with their original truth values w.r.t. V. Therefore

$$(F, a) \Vdash_V p_i \Leftrightarrow (F, a) \Vdash_V \sigma(p_i),\ \forall b \geq a\ (F, b) \Vdash_V p_i \Leftrightarrow (F, b) \Vdash_V \sigma(p_i).$$

This implies

$$(F, a) \Vdash_V \sigma(\Box\varphi).$$

It remains only to consider the case when there is a such that

$$(F, a) \Vdash_V \neg\Box\varphi \wedge \Diamond\Box\varphi.$$

Then let we look close to our model. The model M is based at F is a model with frozen states w.r.t. P_n, therefore, as we noted above in the beginning the proof of our theorem, for any $C(i)$ there are formulas $Y_1, \ldots, Y_m$ with requited properties.

Let $C(i)$ be the cluster where the formula $\Box\varphi$ is true w.r.t. the original valuation V but for all $C(j1)$ with $j_1 < i$ the formula $\Box\varphi$ is false at some its state. As we noted earlier in the beginning the prof of our theorem, there is a sequence $Y_1, \ldots, Y_m$ of subsets of P_n satisfying (a) and (b) for $C(i)$.

Therefore there is $b \in C(i)$ such that $b \Vdash_V \psi(Y_k)$ where Y_k is from definition of the substitution $\sigma(p_i)$ for letters p_i.

Then the truth value of any letter p_i w.r.t. σ at any $c \in C(j1)$ and any $d \in C(j2)$ for $j2 \leq j1$ is the same as the truth value of p_i in the state b from

$C(i)$ w.r.t. the original valuation. Then by (a) and (b) the truth value of any formula α constructed from letters $p_1, \ldots, p_n$, Boolean logical operations and agents' operations $\Box_j$ is the same at b and at any $c \in C(j1)$ and any $d \in C(j2)$ for $j2 \leq j1$. This follows by standard trivial induction on the length of formula. Now it only remains to note that the same holds if we will also use time operation $\Box$ (again simply to use induction of length the formula).

Thus, we obtain that the formula $\varphi(\sigma(p_1), \ldots, \sigma(p_n))$ is true at any state of our model M w.r.t. the original valuation and we proved that σ is a unifier for $\varphi(p_1, \ldots, p_n)$. The theorem is proved.

So, σ is a unifier for φ and it is evident that σ is projective for φ (that is $\Box\varphi \to [x_i \equiv \sigma(x_i)] \in L$). Theorem is proved.

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