

**ON FINITE GROUPS WITH EXACTLY ONE
NONCOMMUTATOR**S.V. SKRESANOV *Communicated by P.P. PETROV*

Abstract: An element x of a group G is a *commutator* if it can be expressed in the form $x = a^{-1}b^{-1}ab$ for some $a, b \in G$. In 2010 MacHale posed the following problem in the Kourovka notebook: does there exist a finite group G , with $|G| > 2$, such that there is exactly one element of G which is not a commutator? We answer this question in the affirmative and provide an infinite series of such groups, the smallest group in our construction having size 16609443840.

Keywords: finite group, perfect group, commutator.

1 Introduction

The study of commutators in finite groups has a long history, see, for instance, a survey [6]. It is well known that the derived subgroup of a finite group need not consist solely of commutators; the smallest such example having size 96, see [2]. One can construct a perfect group with a non-commutator, see [4] and an improvement in [3]. In contrast, by the positive solution to the Ore conjecture [7], every element of a nonabelian finite simple group is a commutator.

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In the 20th edition of the Kourovka notebook [9] MacHale asked the following question.

Question 17.76. *Does there exist a finite group G , with $|G| > 2$, such that there is exactly one element in G which is not a commutator?*

Let G be a finite group with exactly one noncommutator. Since every element outside the derived subgroup $[G, G]$ is not a commutator, we must have $|G \setminus [G, G]| \leq 1$. Hence either G is a cyclic group of order 2, which is excluded from MacHale's question, or G is perfect. If t is the noncommutator element of G , then t^{-1} and $g^{-1}tg$, $g \in G$, are also noncommutators. It follows that $t^2 = 1$ and t lies in the center $Z(G)$ of G .

Using GAP [1], Staroletov [11] showed that there are no groups with one noncommutator among perfect groups of order at most 10^6 , with the possible exception of some group orders which were not present in the GAP perfect groups library at the time. He also showed that there are no such examples among finite quasisimple groups, i.e. among groups G such that $G/Z(G)$ is a nonabelian simple group (see also [8, Table 1]).

We show that there is, in fact, a perfect group with exactly one noncommutator, so MacHale's question can be answered in the affirmative. Our construction is inspired by a theorem of Isaacs [4]. He showed that if A is an abelian group and M is a nonabelian group, then the derived subgroup of the wreath product $A \wr M$, where M is assumed to act regularly, contains a noncommutator under some conditions (which were substantially relaxed by Guralnick in [3]). In our construction, let M be the Mathieu group M_{11} acting transitively on 22 points; the point stabilizer will be isomorphic to the alternating group of degree 6. We consider the wreath product $H = C_2 \wr M$ of the cyclic group of order 2 and M . Note that M no longer acts regularly, so the results [3, 4] do not apply, and $|H| = 2^{22} \cdot |M| = 2^{22} \cdot 7920$. Finally, we set $G_{MacHale} = [H, H]$.

Theorem. *The group $G_{MacHale}$ has order 16609443840 and contains exactly one noncommutator.*

The proof of this theorem relies on a computation in GAP, but because of the size of the group a direct "bruteforce" computation is not feasible. In the next section we describe how one can check that $G_{MacHale}$ has one noncommutator in reasonable time.

Note also that if G is a group with one noncommutator and N is a nontrivial normal subgroup of G , then G/N consists entirely of commutators. In particular, one cannot reduce the size of our example by considering proper quotients of $G_{MacHale}$.

The main result implies that there is an infinite series of groups with one noncommutator.

Corollary. *There are infinitely many finite groups with exactly one noncommutator.*

Proof. Let G be a finite nonabelian group with exactly one noncommutator t . Let T be a normal subgroup of $G \times G$ generated by (t, t) , and set $K = (G \times G)/T$. Notice that $|K| > |G|$. We will prove that K also has exactly one noncommutator.

Suppose that $(t, 1)T \in K$ is a commutator. Then there exist elements $a, b \in G \times G$ such that $(t, 1) \in [a, b]T$. If $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then $(t, 1) = ([a_1, b_1], [a_2, b_2])T$. Hence either $t = [a_1, b_1]$, which is impossible since t is not a commutator in G , or $1 = [a_2, b_2]t$. But that implies $t = [a_2, b_2]$ which is again a contradiction. Therefore $(t, 1)$ is not a commutator in K .

Let $(x, y)T \in K$ be an element distinct from $(t, 1)T$. We may assume that $x, y \neq t$. Indeed, if $x = t$, then $y \neq 1$, since $(x, y) \notin (t, 1)T$. So we may replace the representative (x, y) by $(xt, yt) = (1, yt)$, where $1, yt \neq t$. Similar reasoning applies in the case when $y = t$. Now, if $x, y \neq t$, there exist elements $a_1, a_2, b_1, b_2 \in G$ such that $x = [a_1, b_1]$, $y = [a_2, b_2]$, and hence $(x, y)T = [a, b]T$ is a commutator in K , where $a = (a_1, a_2)$, $b = (b_1, b_2)$.

Given a nonabelian group G with one noncommutator we constructed a larger group K with one noncommutator. By applying this procedure repeatedly to the original group $G_{MacHale}$, we obtain an infinite series of groups with one noncommutator. \square

It follows immediately from the Corollary and the compactness principle that there is an infinite group of bounded exponent with exactly one noncommutator.

2 How to check properties of $G_{MacHale}$

We gave an explicit construction of $G = G_{MacHale}$ in the previous section, but for GAP computations we can represent it as a permutation group of degree 44 generated by 4 elements:

$$\begin{aligned} G := \text{Group}([& (1, 2)(43, 44), \\ & (1, 2)(21, 22), \\ & (1, 39, 13, 43, 25)(2, 40, 14, 44, 26)(3, 37, 15, 41, 27)(4, 38, 16, 42, 28) \\ & (5, 35, 23, 17, 31)(6, 36, 24, 18, 32)(7, 33, 21, 19, 29)(8, 34, 22, 20, 30), \\ & (1, 23, 27, 11, 41)(2, 24, 28, 12, 42)(3, 22, 26, 10, 44, 4, 21, 25, 9, 43) \\ & (5, 16, 20, 40, 32, 6, 15, 19, 39, 31)(7, 13, 17, 37, 29)(8, 14, 18, 38, 30)]); \end{aligned}$$

To aid the reader with required computations, we provide an ancillary GAP file, see [10]. The file contains the definition of the group G alongside with other data and code required.

It can be easily checked in GAP that the group G has order 16609443840 and its center has order 2. Let t be the nonidentity element of the center.

We need to verify that all elements of G besides t are commutators. Our group has 1280 conjugacy classes and it suffices to check whether a representative of each conjugacy class (besides t) is a commutator. In the ancillary file we provide GAP code which defines a list `commPairs` containing

1279 pairs of permutations. Each pair a, b defines a commutator $[a, b]$ and it can be checked that these commutators give representatives of 1279 conjugacy classes of G .

The supplementary GAP file also contains a function `CheckCommutators` which can be used to check this property; it takes the author's computer about 15 minutes to perform this check. The list `commPairs` was found by generating random commutators until all conjugacy classes (besides t) were covered.

Finally, in order to check that t is not a commutator, note that t acts on the set of conjugacy classes of G by left multiplication. Indeed, if g^G is a conjugacy class, then $t \cdot g^G = (tg)^G$ is also a conjugacy class. Moreover, t is a commutator if and only if it stabilizes some conjugacy class: if $t = g^{-1}x^{-1}gx$ for some $x, g \in G$, then $t \cdot g^G = g^G$ and vice versa. This property can be easily tested in GAP by considering all conjugacy classes of G ; in the supplementary file we provide a function `CheckCentral` which performs this check.

We note that the character table of G determines its set of commutators, see, for instance, [5, Problem 3.10]. It is possible to compute the full character table of G in GAP and verify the required properties, but this computation takes much longer than the method proposed above.

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