# STABILITY RESULTS FOR NONLINEAR PARABOLIC EQUATIONS BACKWARD IN TIME

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**Abstract:** Let $T$ be a given positive constant. Denote $D := \{(x,t) : 0 < x < 1, 0 < t < T\}$. We establish stability estimates of Hölder type for general nonlinear parabolic equations backward in time

$$u_t = (a(x,t)\Phi(u_x))_x + \gamma u u_x + f(t,u), \quad (x,t) \in D,$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T,$$
$$\|u(\cdot,T) - \chi\| \leq \varepsilon,$$

under the condition

$$\max_{(x,t)\in\overline{D}} \{|u_x|, |u_{xt}|\} \leqslant E$$

with $E$ being some given positive number. Here $\gamma \geq 0$ is a constant, $a(x,t)$ is a smooth function satisfying the conditions $0 < a_0 \leq a(x,t)$, $|a_t(x,t)| \leq M$, $(x,t) \in \overline{D}$, the function $f$ satisfies the Lipschitz condition $\|f(t,\omega_1) - f(t,\omega_2)\| \leq k\|\omega_1 - \omega_2\|$ and $\chi$ is a given function.

**Keywords:** Nonlinear parabolic equations backward in time, stability estimates, log-convexity method.

## 1 Introduction

Let $T$ be a given positive constant. Denote $D := \{(x,t) : \ 0 < x < 1, \ 0 < t < T\}$. Furthermore, let $\gamma \geq 0$ be a given constant, $a(x,t)$ a smooth function satisfying the conditions

$$0 < a_0 \leq a(x,t), \quad |a_t(x,t)| \leq M, \ (x,t) \in \overline{D},$$

the function $f$ satisfying the Lipschitz condition

$$\|f(t,\omega_1) - f(t,\omega_2)\| \leq k\|\omega_1 - \omega_2\|, \tag{1}$$

and $\chi$ a given function.

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Consider nonlinear parabolic equations backward in time

$$u_t = (a(x,t)\Phi(u_x))_x + \gamma u u_x + f(t,u), \quad (x,t) \in D, \tag{2}$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{3}$$
$$\|u(\cdot,T) - \chi\| \leq \varepsilon. \tag{4}$$

The backward problem (2)–(4) has many applications in practice, for example in data assimilation [22, 27], in heat transfer processes [1, 3, 11], or in image processing [2, 5]. However, it is ill-posed as a small perturbation in $u(T)$ may cause arbitrarily large errors in the solution. One of the important questions in inverse and ill-posed problems is establishing stability estimates [19, 24]. Some results for nonlinear parabolic equations backward in time can be found in [4, 5, 7, 8, 10, 14, 15, 16, 18, 20, 21, 23, 25, 29, 30, 31, 32]. This estimates help us to know the degree of ill-posedness of the problem and based on them to develop efficient stable numerical methods. Despite many results on stability estimates for linear parabolic equations backward in time, those for fully nonlinear cases are very few. To our knowledge, there has been no result on stability estimates for the fully nonlinear problem (2)-(4) even for the case $\gamma = 0$ and $f \equiv 0$.

In this paper, by using the log-convexity method we obtain stability estimates for the problem (2)–(4) with $\Phi$ being a nonlinear function such as polynomials of higher degree, fractional functions, roots, etc ... We note that when $\Phi(s) = s$, our results imply stability estimates for backward Burgers' equations, backward semi-linear parabolic equations with time-dependent coefficients and backward linear parabolic equations with time-dependent coefficients. Furthermore, for backward Burgers' equations (see, e.g. [4, 9, 10, 14, 16, 25], we establish stability estimates of Hölder type for more general equations and under weaker conditions than those by Carasso [4] and Ponomarev [25]. Even if for the general nonlinear problem (2)–(4), our conditions are also weaker than those required by Carasso [4] and Ponomarev [25] for backward Burgers' equations.

It is interesting to remark that although there are several regularization methods for the semi-linear parabolic equation backward in time with time-independent coefficients ([13], [20], [21], [23], [31, 32]), the results for fully nonlinear cases seem to be lacking. Furthermore, to stabilize the solutions of semi-linear parabolic equations backward in time, researchers usually impose very strong conditions on the solutions of these problems. For examples, in [31, 32], D. D. Trong and N. H. Tuan used the method of integral equations to regularize the one-dimensional semi-linear heat equation

$$\begin{cases} u_t - u_{xx} = f(x,t,u(x,t)), \ (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0, \ t \in (0,T). \end{cases} \tag{5}$$

To get a stability estimate of Hölder type, they required that

$$\sum_{n=1}^{\infty} \lambda_n^4 e^{2T\lambda_n^2} |(u(t), \phi_n)|^2 < \infty, \forall t \in [0, T], \tag{6}$$

where $\phi_n = sin(nx)$ and $\lambda_n = n$.

In [23], P. T. Nam considered the ill-posed semi-linear parabolic equation backward in time with time-independent coefficients in a Hilbert space $H$

$$\begin{cases} u_t + Au = f(t, u(t)), & 0 < t < T, \\ u(T) = g \end{cases} \tag{7}$$

where the datum $g \in H$ is approximately given and $A$ is a positive self-adjoint unbounded linear operator which admits an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ in $H$, associated with the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots, \quad \text{and} \quad \lim_{i\to+\infty} \lambda_i = +\infty \tag{8}$$

and $f$ satisfies the Lipschitz condition

$$\|f(t, w_1) - f(t, w_2)\| \leqslant k\|w_1 - w_2\|$$

for some constant $k$ independent of $t, w_1, w_2$. In order to get a stability estimate of Hölder type, P. T. Nam ([23]) had to impose one of the following conditions:

$$\sum_{n=1}^{\infty} e^{2\lambda_n \min(t,\beta)} |(u(t), \phi_n)|^2 \leqslant E_0^2, \tag{9}$$

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta'} e^{2\lambda_n \min(t,\beta)} |(u(t), \phi_n)|^2 \leqslant E_1^2, \tag{10}$$

$$\sum_{n=1}^{\infty} e^{2\lambda_n} |(u(t), \phi_n)|^2 \leqslant E_2^2 \tag{11}$$

for all $t \in [0, T]$, where $\beta$, $\beta'$ stand for positive constants.

In [26] Quan *et. al.* considered the semi-linear parabolic equation backward in time with time-dependent coefficients

$$\begin{aligned} &u_t = (a(x, t)u_x)_x + f(x, t, u, u_x, u_{xx}), \quad (x, t) \in \mathbb{R} \times [0, T), \\ &u(x, T) = g(x), \quad x \in \mathbb{R}, \end{aligned}$$

where $f(x, t, u, u_x, u_x x)$ and $a(x, t)$ are given such that there exist $p, q, L > 0$ satisfying

$$0 < p \leq a(x, t) \leq q$$

and

$$|f(x, t, u_1, v_1, w_1) - f(x, t, u_2, v_2, w_2)| \leq L(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|).$$

Quan *et. al.* required the following condition:

$$\sup\left\{\int_{\mathbb{R}} e^{2m'|\xi|^{4+\alpha}}|\mathcal{F}(u)(\xi,t)|^2 d\xi : t\in[0,T]\right\}<\infty \tag{12}$$

where $m' = \eta(T) + m + \dfrac{3}{2}$ with $m$ is a positive number and

$$\eta(T) = \int_0^T k(s)ds,$$
$$k(s) = \lim_{x\to\infty} a(x,s),$$

$\mathcal{F} : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R})$ is the Fourier transform that is defined by

$$\mathcal{F}(h)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h(x)e^{-ix\xi}dx.$$

The conditions given in the next section are much more reasonable than those above.

## 2 Stability estimates

Suppose that the function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$\Phi(s_1) - \Phi(s_2) = (s_1 - s_2)\varphi(s_1, s_2),$$

where $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ fulfills
i) $\varphi(s_1, s_2) \geq C_0 > 0$, for all $s_1, s_2 \in \mathbb{R}$,
ii) there exist constants $C_i, i = \overline{1,4}, p, q$ such that

$$|\varphi_t(\alpha,\beta)| \leq C_1\big(C_2(|\alpha| + |\beta|)^p + C_3(|\alpha_t| + |\beta_t|)^q + C_4\big),$$

with $\alpha(t), \beta(t)$ being smooth functions.

Suppose that $E$ is a given positive number. Set

$$a_1(t) = \max_{x\in[0,1]}\left(\frac{a_t(x,t)}{a(x,t)} + \frac{E^2}{C_0a_0} + \frac{C_5}{C_0}\right),\ t\in[0,T],$$

where $C_5 = C_1(C_2(2E)^p + C_3(2E)^q + C_4)$, and

$$a_2(t) = \exp\left(\int_0^t a_1(s)ds\right),$$

$$a_3(t) = \int_0^t a_2(\tau)d\tau,$$

$$\mu(t) = \frac{a_3(t)}{a_3(T)},\quad t\in[0,T].$$

Further, set

$$H(t) = 2\exp\left(\frac{3}{2}\gamma ET + kT + \frac{1}{4}C_6\mu(t)(1-\mu(t))\right),\ t\in[0,T],$$

with

$$C_6 = \frac{3(\gamma^2 E^2 + k^2)}{2T^2} a_3^2(T) \exp\left(4\left(\frac{M}{a_0} + \frac{C_5}{C_0} + \frac{3\gamma^2 E^2}{2a_0 C_0}\right) T\right).$$

**Theorem 1.** *Suppose that the above conditions are satisfied. Let $u_1(x,t)$ and $u_2(x,t)$ be smooth solutions of (2)–(4) satisfying*

$$\max_{(x,t)\in\overline{D}} \{|u_{ix}|, |u_{ixt}|\} \leqslant E. \tag{13}$$

*Then*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \le H(t) E^{1-\mu(t)} \varepsilon^{\mu(t)},\ t \in [0,T]. \tag{14}$$

**Remark 1.** *If $a_1(t) < 0$, then $\mu(t) > \frac{t}{T},\ t \in (0,T)$.*

**Corollary 1.** *Suppose that $u_i, i = 1,2$, are two smooth solutions of the problem (2)–(4) subject to the constraint*

$$\max_{(x,t)\in\overline{D}} \{|u_{ix}|, |u_{ixt}|\} \leqslant E$$

*where*

$$\Phi(s) = s^3 - \frac{3}{2}s^2 + C_7 s + C_8,\ C_7 > \frac{9}{8}.$$

*Then there exist functions $H_2(t)$ and $\mu_2(t)$ with $0 \le \mu_2(t) \le 1,\ t \in [0,T]$ such that*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \le H_2(t) E^{1-\mu_2(t)} \varepsilon^{\mu_2(t)},\ t \in [0,T].$$

*Proof.* With $\Phi(s) = s^3 - \frac{3}{2}s^2 + C_7 s + C_8$ , we have

$$\Phi(s_1) - \Phi(s_2) = (s_1 - s_2)\varphi(s_1, s_2)$$

where

$$\varphi(s_1,s_2) = s_1^2 + s_1 s_2 + s_2^2 - \frac{3}{2}(s_1 + s_2) + C_7.$$

We have

$$\varphi(s_1,s_2) \ge \frac{1}{2}(s_1+s_2)^2 - \frac{3}{2}(s_1+s_2) + C_7 \ge C_7 - \frac{9}{8}.$$

Since $\alpha(t)$ and $\beta(t)$ are smooth functions, we have

$$\begin{aligned} |\varphi_t(\alpha,\beta)| &= |2\alpha\alpha_t + \alpha_t\beta + \alpha\beta_t + 2\beta\beta_t - \frac{3}{2}(\alpha_t + \beta_t)| \\ &\le \frac{3}{2}(|\alpha| + |\beta|)^2 + 2(|\alpha_t| + |\beta_t|)^2 + \frac{9}{8} \end{aligned}$$

Therefore, we can apply Theorem 1 with $C_0 = C_7 - \frac{9}{8}, C_1 = 1, C_2 = \frac{3}{2}, C_3 = 2, C_4 = \frac{9}{8}, p = q = 2$.

The corollary is proved. □

**Corollary 2.** *Suppose that $u_i, i = 1, 2$, are two smooth solutions of the problem (2)–(4) subject to the constraint*

$$\max_{(x,t)\in\overline{D}} \{|u_{ix}|, |u_{ixt}|\} \leqslant E$$

*with*

$$\Phi(s) = \frac{2s^3 + s}{1 + s^2}.$$

*Then there exist functions $H_3(t)$ and $\mu_3(t)$ with $0 \le \mu_3(t) \le 1,\ t \in [0, T]$ such that*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\| \le H_3(t) E^{1-\mu_3(t)} \varepsilon^{\mu_3(t)},\ t \in [0, T].$$

*Proof.* With $\Phi(s) = \dfrac{2s^3 + s}{1 + s^2}$, we have

$$\Phi(s_1) - \Phi(s_2) = (s_1 - s_2)\varphi(s_1, s_2)$$

where

$$\varphi(s_1, s_2) = \frac{2(s_1^2 + s_2^2) + 3s_1 s_2 + 2s_1^2 s_2^2 + 1}{(1 + s_1^2)(1 + s_2^2)}.$$

We have

$$\varphi(s_1, s_2) \ge \frac{1}{4},\ \forall s_1, s_2 \in \mathbb{R}.$$

Since $\alpha(t)$ and $\beta(t)$ are smooth functions, we have

$$\begin{aligned}\varphi_t(\alpha, \beta) =& \frac{4(\alpha\alpha_t + \beta\beta_t) + 3\alpha\beta_t + 3\alpha_t\beta + 4\alpha\alpha_t\beta^2 + 4\alpha^2\beta\beta_t}{(1 + \alpha^2)(1 + \beta^2)} \\ &- \frac{(2(\alpha^2 + \beta^2) + 2\alpha^2\beta^2 + 3\alpha\beta + 1)(2\alpha\alpha_t(1 + \beta^2) + 2(1 + \alpha^2)\beta\beta_t)}{(1 + \alpha^2)^2(1 + \beta^2)^2},\end{aligned}$$

or

$$\begin{aligned}|\varphi_t(\alpha, \beta)| \le\ & 2(|\alpha_t| + |\beta_t|) + \frac{3}{2}(|\alpha_t| + |\beta_t|) + |\alpha\beta_t| + |\alpha_t\beta| \\ &+ \frac{8(|\alpha\alpha_t|(1 + \beta^2) + (1 + \alpha^2)|\beta\beta_t|)}{(1 + \alpha^2)(1 + \beta^2)} \\ \le\ & 8(|\alpha_t| + |\beta_t|) + |\alpha\beta_t| + |\alpha_t\beta| \\ \le\ & 8(|\alpha_t| + |\beta_t|) + \alpha^2 + \beta^2 + \alpha_t^2 + \beta_t^2 \\ \le\ & (|\alpha| + |\beta|)^2 + 2(|\alpha_t| + |\beta_t|)^2 + 16.\end{aligned}$$

Therefore, we can apply Theorem 1 with $C_0 = \frac{1}{4}, C_1 = 1, C_2 = 1, C_3 = 2, C_4 = 16, p = q = 2$.

The corollary is proved. □

**Corollary 3.** *Suppose that $u_i, i = 1, 2$, are two smooth solutions of the problem (2)–(4) subject to the constraint*

$$\max_{(x,t)\in\overline{D}} \{|u_{ix}|, |u_{ixt}|\} \leqslant E$$

*with*

$$\Phi(s) = s\sqrt{1+s^2}.$$

*Then there exist functions* $H_4(t)$ *and* $\mu_4(t)$ *with* $0 \le \mu_4(t) \le 1,\ t \in [0,T]$ *such that*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \le H_4(t) E^{1-\mu_4(t)} \varepsilon^{\mu_4(t)},\ t \in [0,T].$$

*Proof.* With $\Phi(s) = s\sqrt{1+s^2}$, we have

$$\Phi(s_1) - \Phi(s_2) = (s_1 - s_2)\varphi(s_1,s_2)$$

where

$$\varphi(s_1,s_2) = \frac{\sqrt{1+s_1^2}+\sqrt{1+s_2^2}}{2} + \frac{1}{2}\frac{(s_1+s_2)^2}{\sqrt{1+s_1^2}+\sqrt{1+s_2^2}}.$$

Therefore, we obtain

$$\varphi(s_1,s_2) \ge 1,\ \forall s_1, s_2 \in \mathbb{R}.$$

Since $\alpha(t)$ and $\beta(t)$ are smooth functions, we have

$$\begin{aligned}\varphi_t(\alpha,\beta) =& \frac{\alpha\alpha_t}{2\sqrt{1+\alpha^2}} + \frac{\beta\beta_t}{2\sqrt{1+\beta^2}} + \frac{(\alpha+\beta)(\alpha_t+\beta_t)}{\sqrt{1+\alpha^2}+\sqrt{1+\beta^2}} \\ &- \frac{(\alpha+\beta)^2}{2} \cdot \frac{\frac{\alpha\alpha_t}{2\sqrt{1+\alpha^2}} + \frac{\beta\beta_t}{\sqrt{1+\beta^2}}}{(\sqrt{1+\alpha^2}+\sqrt{1+\beta^2})^2}.\end{aligned}$$

This implies that

$$\begin{aligned}|\varphi_t(\alpha,\beta)| &\le \frac{3}{4}\left|\frac{\alpha\alpha_t}{\sqrt{1+\alpha^2}} + \frac{\beta\beta_t}{\sqrt{1+\beta^2}}\right| + |\alpha_t + \beta_t| \\ &\le \frac{7}{4}(|\alpha_t| + |\beta_t|) < 2(|\alpha_t| + |\beta_t|).\end{aligned}$$

Therefore, we can apply Theorem 1 with $C_0 = C_1 = 1, C_2 = 0, C_3 = 2, C_4 = 0, q = 1$.

The corollary is proved. □

In case $\Phi(s) = s$ and $f \equiv 0$, we obtain stability estimates for Burgers-type equations backward in time:

**Corollary 4.** *Suppose that* $u_i, i = 1,2,$ *are two smooth solutions of the Burgers-type equation*

$$\begin{aligned}&u_t = (a(x,t)u_x)_x + \gamma u u_x, \quad (x,t) \in D,\\ &u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant T,\end{aligned}$$

*subject to the constraint*

$$\max_{(x,t)\in\overline{D}} |u_{ix}| \leqslant E$$

*with $\gamma \geq 0$ being a constant, $a(x,t)$ a smooth function satisfying the conditions $0 < a_0 \leq a(x,t)$, $|a_t(x,t)| \leq M,\ (x,t) \in \overline{D}$. Then there exist functions $H_5(t)$ and $\mu_5(t)$ with $0 \leq \mu_5(t) \leq 1,\ t \in [0,T]$ such that*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \leq H_5(t) E^{1-\mu_5(t)} \varepsilon^{\mu_5(t)},\ t \in [0,T].$$

*Proof.* Apply Theorem 1 with $C_0 = 1, C_1 = 0, k = 0$.
The corollary is proved. □

**Remark 2.** *In [4], Carasso only considered the case $\gamma = 1$ and $a(x,t) = \nu > 0$. Furthermore, he required that*

$$\max_{(x,t)\in\overline{D}} \{|u_i|,\ |u_{i,t}|,\ |u_{i,tt}|,\ |u_{i,xt}|\} \leqslant E,\ i = 1,2. \tag{15}$$

*In [25] Ponomarev only considered the case $a(x,t) = a(t) > 0$. Furthermore, he required that*

$$\|u(x,t)\|_{C^2(\overline{D})} \leq E. \tag{16}$$

*In Corollary 4, we establish stability estimates of Hölder type for more general equations and under a weaker condition. Namely, we only require that*

$$\max_{(x,t)\in\overline{D}} |u_{ix}| \leqslant E.$$

*Even if in general cases, our conditions for the solution of the equations (2) and (3)*

$$\max_{(x,t)\in\overline{D}} \{|u_{ix}|, |u_{ixt}|\} \leqslant E.$$

*are much weaker than the conditions (15) and (16).*

The proof of Theorem 1 will be presented in the next section.

## 3 Proof of the main result

Set $v = u_1 - u_2$. We have

$$\begin{cases} v_t = (a v_x \varphi(u_{1x}, u_{2x}))_x + \gamma u_1 v_x + \gamma u_{2x} v + Bv,\ (x,t) \in D, \\ v(0,t) = v(1,t) = 0,\ 0 \leq t \leq T. \end{cases} \tag{17}$$

Here, $Bv := f(t,u_1) - f(t,u_2)$ and so

$$\|Bv\| = \|f(t,u_1) - f(t,u_2)\| \leq k\|u_1 - u_2\| = k\|v\|. \tag{18}$$

Let $F(t) = \|v(\cdot,t)\|^2 = \int_0^1 v^2 dx$. We have

$$\begin{aligned} F'(t) &= 2\int_0^1 v v_t dx = 2\int_0^1 v\left((a v_x \varphi(u_{1x}, u_{2x}))_x + \gamma u_1 v_x + \gamma u_{2x} v + Bv\right) dx \\ &= -2\int_0^1 a v_x^2 \varphi(u_{1x}, u_{2x}) dx + \int_0^1 \gamma(2u_{2x} - u_{1x}) v^2 dx + \int_0^1 v Bv dx. \end{aligned} \tag{19}$$

Suppose that $\|z(\cdot,t)\| > 0,\ \forall t \in [0,T]$, we have

$$\frac{F'(t)}{F(t)} = \frac{-2\int_0^1 av_x^2\varphi(u_{1x},u_{2x})dx}{\int_0^1 v^2dx} + \frac{\int_0^1 \gamma(2u_{2x}-u_{1x})v^2dx}{\int_0^1 v^2dx} + \frac{\int_0^1 vBvdx}{\int_0^1 v^2dx}. \tag{20}$$

From (13), (18), (19) and (20), we have

$$\frac{F'(t)}{F(t)} \le -2G(t) + 3E\gamma + 2k \tag{21}$$

and

$$\frac{F'(t)}{F(t)} \ge -2G(t) - 3E\gamma - 2k, \tag{22}$$

where

$$G(t) = \frac{\int_0^1 av_x^2\varphi(u_{1x},u_{2x})dx}{\int_0^1 v^2dx}.$$

Integrating both sides of (21) from 0 to $t$, we have

$$\ln F(t) \le \ln F(0) - 2\int_0^t G(s)ds + 3\gamma Et + 2kt$$

or

$$\begin{aligned}(1-\mu(t))\ln F(t) \le{}& (1-\mu(t))\ln F(0)\\ &- 2(1-\mu(t))\int_0^t G(s)ds + 3\gamma Et + 2kt.\end{aligned} \tag{23}$$

Integrating both sides of (22) from $t$ to $T$, we obtain

$$\ln F(t) \le \ln F(T) + 2\int_t^T G(s)ds + 3\gamma E(T-t) + 2k(T-t)$$

or

$$\mu(t)\ln F(t) \le \mu(t)\ln F(T) + 2\mu(t)\int_t^T G(s)ds + 3\gamma E(T-t) + 2k(T-t). \tag{24}$$

From (23) and (24), we have

$$\begin{aligned}\ln F(t) \le{}& (1-\mu(t))\ln F(0) + \mu(t)\ln F(T) + 2\mu(t)\int_t^T G(s)ds\\ &- 2(1-\mu(t))\int_0^t G(s)ds + 3\gamma ET + 2kT\\ ={}& (1-\mu(t))\ln F(0) + \mu(t)\ln F(T) - 2\int_0^t G(s)ds\\ &+ 2\mu(t)\int_0^T G(s)ds + 3\gamma ET + 2kT.\end{aligned} \tag{25}$$

Set

$$h(t) = -\int_0^t G(s)ds, \; t \in [0,T].$$

Since $\mu(t)$ is continuous and strictly increasing on $t \in [0,T]$ and $\mu(0) = 0$, $\mu(T) = 1$, it is reasonable to set

$$g(t) := h\left(\mu^{-1}\left(t/T\right)\right), \; t \in [0,T].$$

Therefore,

$$h(t) = g(T\mu(t)),$$

$$h_t(t) = \frac{Ta_2(t)}{a_3(T)} g_\mu(T\mu(t)), \tag{26}$$

$$h_{tt}(t) = \left(\frac{Ta_2(t)}{a_3(T)}\right)^2 g_{\mu\mu}(T\mu(t)) + \frac{Ta_1(t)a_2(t)}{a_3(T)} g_\mu(T\mu(t)). \tag{27}$$

This implies that

$$\left(\frac{Ta_2(t)}{a_3(T)}\right)^2 g_{\mu\mu}(T\mu(t)) = h_{tt}(t) - a_1(t)h_t(t). \tag{28}$$

On the other hand

$$h_t(t) = -G(t) = \frac{-\int_0^1 av_x^2\varphi(u_{1x}, u_{2x})dx}{\int_0^1 v^2dx}.$$

Therefore,

$$\left(\int_0^1 v^2dx\right)^2 h_{tt}(t) =$$

$$-2\int_0^1 av_xv_{xt}\varphi(u_{1x}, u_{2x})dx \int_0^1 v^2dx + 2\int_0^1 av_x^2\varphi(u_{1x}, u_{2x})dx \int_0^1 vv_tdx$$

$$\int_0^1 (a\varphi(u_{1x}, u_{2x}))_t v_x^2dx \int_0^1 v^2dx$$

$$= 2\int_0^1 (av_x\varphi(u_{1x}, u_{2x}))_x v_tdx \int_0^1 v^2dx$$

$$-2\int_0^1 (av_x\varphi(u_{1x}, u_{2x}))_x vdx \times$$

$$\int_0^1 \Big((av_x\varphi(u_{1x}, u_{2x}))_x + \gamma u_1v_x + \gamma u_{2x}v + Bv\Big)vdx$$

$$-\int_0^1 (a_t\varphi(u_{1x}, u_{2x}) + a\varphi_t(u_{1x}, u_{2x}))v_x^2dx \int_0^1 v^2dx$$

$$
\begin{aligned}
&= 2\int_0^1 (av_x\varphi(u_{1x},u_{2x}))_x\Big((av_x\varphi(u_{1x},u_{2x}))_x + \gamma u_1 v_x + \gamma u_{2x} v + Bv\Big)dx \\
&\quad \times \int_0^1 v^2 dx \\
&\quad - 2\left(\int_0^1 (av_x\varphi(u_{1x},u_{2x}))_x v dx + \frac{1}{2}\int_0^1 (\gamma u_1 v_x + \gamma u_{2x} v + Bv) v dx\right)^2 \\
&\quad + \frac{1}{2}\left(\int_0^1 (\gamma u_1 v_x + \gamma u_{2x} v + Bv) v dx\right)^2 \\
&\quad - \int_0^1 (a_t\varphi(u_{1x},u_{2x}) + a\varphi_t(u_{1x},u_{2x}))v_x^2 dx \int_0^1 v^2 dx \\
&= 2\int_0^1 \Big((av_x\varphi(u_{1x},u_{2x}))_x + \frac{1}{2}(\gamma u_1 v_x + \gamma u_{2x} v + Bv)\Big)^2 \int_0^1 v^2 dx \\
&\quad - 2\left(\int_0^1 \Big(av_x\varphi(u_{1x},u_{2x}))_x + \frac{1}{2}(\gamma u_1 v_x + \gamma u_{2x} v + Bv)\Big) v dx\right)^2 \\
&\quad + \frac{1}{2}\left(\int_0^1 (\gamma u_1 v_x + \gamma u_{2x} v + Bv) v dx\right)^2 \\
&\quad - \frac{1}{2}\int_0^1 (\gamma u_1 v_x + \gamma u_{2x} v + Bv)^2 dx \int_0^1 v^2 dx \\
&\quad - \int_0^1 (a_t\varphi(u_{1x},u_{2x}) + a\varphi_t(u_{1x},u_{2x}))v_x^2 dx \int_0^1 v^2 dx \\
&\geq -\frac{1}{2}\int_0^1 (\gamma u_1 v_x + \gamma u_{2x} v + Bv)^2 dx \int_0^1 v^2 dx \\
&\quad - \int_0^1 (a_t\varphi(u_{1x},u_{2x}) + a\varphi_t(u_{1x},u_{2x}))v_x^2 dx \int_0^1 v^2 dx. \qquad (29)
\end{aligned}
$$

From (29) and using inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we have

$$
\begin{aligned}
\left(\int_0^1 v^2 dx\right)^2 h_{tt}(t) \geq & -\frac{3}{2}\int_0^1 \left(\gamma^2 u_1^2 v_x^2 + \gamma^2 u_{2x}^2 v^2 + (Bv)^2\right) dx \int_0^1 v^2 dx \\
& - \int_0^1 (a_t\varphi(u_{1x},u_{2x}) + a\varphi_t(u_{1x},u_{2x}))v_x^2 dx \int_0^1 v^2 dx. \qquad (30)
\end{aligned}
$$

On the other hand

$$
u_i^2(x,t) = \left(\int_0^x u_{i\eta}(\eta,t)d\eta\right)^2 \leq x\int_0^x u_{i\eta}^2(\eta,t)d\eta \leq E^2. \qquad (31)
$$

From (13), (18), (30) and (31), we have

$$\left(\int_0^1 v^2 dx\right)^2 h_{tt}(t) \geq -\frac{3}{2}\int_0^1 \left(\gamma^2 E^2 v_x^2 + \gamma^2 E^2 v^2 + k^2 v^2\right) dx \int_0^1 v^2 dx - \int_0^1 (a_t\varphi(u_{1x}, u_{2x}) + a\varphi_t(u_{1x}, u_{2x}))v_x^2 dx \int_0^1 v^2 dx,$$

or

$$\begin{aligned} h_{tt}(t) \geq & -\frac{3\gamma^2 E^2}{2}\frac{\int_0^1 v_x^2 dx}{\int_0^1 v^2 dx} - \frac{3}{2}(\gamma^2 E^2 + k^2) \\ & - \frac{\int_0^1 \left(\frac{a_t}{a} + \frac{\varphi_t(u_{1x},u_{2x})}{\varphi(u_{1x},u_{2x})}\right) a v_x^2 \varphi(u_{1x}, u_{2x}) dx}{\int_0^1 v^2 dx} \\ \geq & -\frac{\int_0^1 \left(\frac{a_t(x,t)}{a(x,t)} + \frac{\varphi_t(u_{1x},u_{2x})}{\varphi(u_{1x},u_{2x})} + \frac{3\gamma^2 E^2}{2a(x,t)\varphi(u_{1x},u_{2x})}\right) a v_x^2 \varphi(u_{1x}, u_{2x}) dx}{\int_0^1 v^2 dx} \\ & - \frac{3}{2}(\gamma^2 E^2 + k^2) \\ \geq & -\frac{\int_0^1 \left(\frac{a_t(x,t)}{a(x,t)} + \frac{\varphi_t(u_{1x},u_{2x})}{C_0} + \frac{3\gamma^2 E^2}{2a_0 C_0}\right) a v_x^2 \varphi(u_{1x}, u_{2x}) dx}{\int_0^1 v^2} dx \\ & - \frac{3}{2}(\gamma^2 E^2 + k^2). \end{aligned} \tag{32}$$

From (13), we have

$$\begin{aligned} |\varphi_t(u_{1x}, u_{2x})| &\leq C_1 \left(C_2(|u_{1x}| + |u_{2x}|)^p + C_3(|u_{1xt}| + |u_{2xt}|)^q + C_4\right) \\ &\leq C_1(C_2(2E)^p + C_3(2E)^q + C_4) = C_5. \end{aligned} \tag{33}$$

From (32) and (33), we have

$$\begin{aligned} h_{tt}(t) \geq & -\frac{\int_0^1 \left(\frac{a_t(x,t)}{a(x,t)} + \frac{C_5}{C_0} + \frac{3\gamma^2 E^2}{2a_0 C_0}\right) a v_x^2 \varphi(u_{1x}, u_{2x}) dx}{\int_0^1 v^2 dx} \\ & - \frac{3}{2}(\gamma^2 N^2 + k^2) \\ \geq & \frac{a_1(t)\int_0^1 a v_x^2 \varphi(u_{1x}, u_{2x}) dx}{\int_0^1 v^2 dx} - \frac{3}{2}(\gamma^2 E^2 + k^2) \\ \geq & a_1(t) h_t(t) - \frac{3}{2}(\gamma^2 E^2 + k^2). \end{aligned} \tag{34}$$

From (28) and (34), we obtain

$$\left(\frac{Ta_2(t)}{a_3(T)}\right)^2 g_{\mu\mu}(T\mu(t)) \geq -\frac{3}{2}(\gamma^2 E^2 + k^2),$$

or

$$g_{\mu\mu}(T\mu(t)) \geq -\frac{3}{2}(\gamma^2 E^2 + k^2)\left(\frac{Ta_2(t)}{a_3(T)}\right)^{-2}.$$

Since

$$(a_2(t))^{-2} = \exp\left(-2\int_0^t a_1(s)ds\right) \leq \exp\left(2\left(\frac{M}{a_0} + \frac{C_5}{C_0} + \frac{3\gamma^2 E^2}{2a_0 C_0}\right)T\right),$$

we have

$$g_{\mu\mu}(T\mu(t)) \geq -C_6.$$

This implies that $g(T\mu(t)) + \frac{1}{2}C_6\mu^2(t)$ is a convex function with respect to $\mu(t)$. Therefore, we have

$$g(T\mu(t)) + \frac{1}{2}C_6\mu^2(t) \leq (1-\mu(t))g(0) + \mu(t)(g(T) + \frac{1}{2}C_6).$$

Since $g(T\mu(t)) = h(t) = -\int_0^t G(s)ds, g(0) = 0$ and $g(T) = -\int_0^T G(s)ds$, we obtain

$$-\int_0^t G(s)ds \leq -\mu(t)\int_0^T G(s)ds + \frac{1}{2}C_6\mu(t)(1-\mu(t)). \tag{35}$$

From (25) and (35), we have

$$\ln F(t) \leq (1-\mu(t))\ln F(0) + \mu(t)\ln F(T) +$$
$$3\gamma ET + 2kT + \frac{1}{2}C_6\mu(t)(1-\mu(t)),$$

or

$$\|v(\cdot,t)\| \leq \|v(\cdot,0)\|^{1-\mu(t)}\|v(\cdot,T)\|^{\mu(t)}\exp\left(\frac{3}{2}\gamma ET + kT + \frac{1}{4}C_6\mu(t)(1-\mu(t))\right) \tag{36}$$

for all $t \in [0,T]$. From (31), we obtain

$$\|v(\cdot,0)\| \leq \|u_1(\cdot,0)\| + \|u_2(\cdot,0)\| \leq 2E.$$

On the other hand

$$\|v(\cdot,T)\| \leq \|u_1(\cdot,T) - \chi\| + \|u_2(\cdot,T) - \chi\| \leq 2\varepsilon.$$

Therefore, for $t \in [0,T]$ we obtain

$$\|v(\cdot,t)\| \leq 2E^{1-\mu(t)}\varepsilon^{\mu(t)}\exp\left(\frac{3}{2}\gamma ET + kT + \frac{1}{4}C_6\mu(t)(1-\mu(t))\right).$$

If $\|v(\cdot,0)\| = 0$, then $\|v(\cdot,t)\| = 0, \forall t \in [0,T]$ and the inequality (14) is obvious. If $\|v(\cdot,0)\| > 0$, then $\|v(\cdot,t)\| > 0$, $\forall t \in [0,T]$. In fact, supposing the contrary, let $t_0$ be the first point where $\|v(\cdot,t)\| = 0$. By continuity, $\|v(\cdot,t)\| > 0$ for $0 \leqslant t < t_0$. Therefore $\|v(\cdot,t)\| > 0$ for $0 \leqslant t \leqslant s < t_0$. Using the stability estimate (36) with $T$ replacing by $s < t_0$ and by letting $s \uparrow t_0$ we obtain a contradiction.

The theorem is proved.

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