Perturbations of Unbounded Fredholm Linear Operators in non-archimedean Banach Spaces

Abdelkhalek El Amrani

Sidi Mohamed Ben Abdellah University, Morocco

Youness Hadder

Sidi Mohamed Ben Abdellah University, Morocco

Yassine Mouzour

Sidi Mohamed Ben Abdellah University, Morocco

Abstract. *In the classical setting, T. Diagana provided in his paper [1] some sufficient conditions under which the algebraic sum of three (possibly unbounded) linear operators $A + B + C$ is Fredholm with A being Fredholm. The main objective of this paper is to generalize this result to n (possibly unbounded) linear operators in the non-archimedean context with $n \geq 3$.*

Keywords: Non-archimedean Banach space, spherically complete field, closed operator, unbounded Fredholm operator.

Introduction

The theory of Fredholm operators plays a central role in functional analysis and its applications to partial differential equations, index theory, and mathematical physics.

The study of perturbations of p-adic linear operators has progressed through several stages. The initial investigation was undertaken by J. P. Serre in [6], who examined compact perturbations of the identity in Banach spaces with orthogonal bases. This was later extended by L. Gruson in [5], to a more general class of Banach spaces, still focusing on perturbations of the identity. A comprehensive treatment of the subject was ultimately provided by W. H. Schikhof in [7]. A complete study of compact perturbations of Fredholm operators and stability of the index on non-archimedean Banach spaces was provided a few years later by Perez-Garcia, J. Araujo and S. Vega in [11]. This result was genralized later in [12] by S. Sliwa to Fréchet spaces. An interesting work on the Fredholm theory for p-adic locally convex spaces was previously presented by N. De Grande-De Kimpe and J. Martinez-Maurica in [16]. For additional details on locally convex spaces, the reader is directed to consult [17].

While the perturbation theory for bounded Fredholm operators is well established, the unbounded case presents deeper analytical challenges due to the lack of global continuity. Understanding how perturbations affect the Fredholmness and index of unbounded operators is crucial, particularly in settings where operators naturally arise as unbounded, such as differential operators on function spaces. For more details on this class of operators, we refer to [8], [9] and [10].

In [1, Theorem 1], some sufficient conditions are given so that if A, B and C are three unbounded linear operators with A being a Fredholm operator, then their algebraic sum $A + B + C$ is also a Fredholm operator. The main goal of this note is to generalize this result to n (possibly unbounded) linear operators with $n \geq 3$. Namely, we investigate conditions under which the algebraic sum $\sum_{k=1}^{n} A_k$ is still Fredholm and how the index behaves under such perturbations. The algebraic sum $\sum_{k=1}^{n} A_k$ is defined as follows: If $A_1 : D(A_1) \subset X \to Y$, $A_2 : D(A_2) \subset X \to Y, ...,$ and $A_n : D(A_n) \subset X \to Y$ are n unbounded linear operators. Then the algebraic sum $\sum_{k=1}^{n} A_k$ is defined on $D = \bigcap_{k=1}^{n} D(A_k)$ by $\Big(\sum_{k=1}^{n} A_k\Big)(x) = \sum_{k=1}^{n} A_k(x)$ for all $x \in D$. Recall that this algebraic sum may not be closed even if each A_k is closed. The validity of this result requires additional conditions that will be provided later. For more information on closed linear operators, we refer the reader to [14].

Preliminaries and Notations

In this paper, $(X, \| \,.\, \|_X)$ and $(Y, \| \,.\, \|_Y)$ stand for two non-archimedean infinite-dimensional Banach spaces over a spherically complete valued field $(\mathbb{K}, | \,.\, |)$. When it is clear from the context, we denote by $\| \,.\, \|$ for both $\| \,.\, \|_X$ and $\| \,.\, \|_Y$. Additional details about this class of spaces can be found in [15].

Definition 1 *We say that the field $\mathbb{K}$ is* ***spherically complete****, if every decreasing sequence of balls has a nonempty intersection.*

Examples 1 *i) Complete discretely valued fields are spherically complete.*

ii) If $\mathbb{K}$ is locally compact, then $\mathbb{K}$ is spherically complete.

For more details on spherically complete fields, we refer to [13].

Definition 2 [4, *Definition* 6.4] *An unbounded operator $A : D(A) \subset X \to Y$ is said to be closed, if its graph $G(A) = \big\{(x, Ax)/x \in D(A)\big\}$ is a closed subset of $X \times Y$.*

Definition 3 [4, *Definition* 6.15] *Let $A : D(A) \subset X \to Y$ be a (possibly unbounded) operator. Then A is called Fredholm if the following conditions are satisfied*

- *A is closed*
- *$\alpha(A) = \dim(N(A))$ and $\beta(A) = codim(R(A))$ are finite.*

In this case, we define the index of A to be the integer $ind(A) = \alpha(A) - \beta(A)$.

Definition 4 *A subspace M of X is said to be complemented, if there exists a closed subspace N of X such that $X = M \oplus N$.*

Let $A : D(A) \subset X \to Y$ be an unbounded linear operator where $D(A)$ stands for the domain of A. As usual, we denote by $N(A)$ and $R(A)$ the kernel and the range of A respectively.

If A is closed, then its graph norm is denoted by $\|\,.\,\|_A$ and defined by $\|x\|_A := \max(\|x\|, \|Ax\|)$ for all $x \in D(A)$ $\big($see [4, p.128]$\big)$. It is clear that $D_A := \big(D(A), \|\,.\|_A\big)$ is a non archimedean Banach space as A is closed.

Let $T : X \to Y$ be a Fredholm operator. Since $\mathbb{K}$ is spherically complete, then $N(A)$ and $R(A)$ are topologically complemented $\big($see [3, proposition 2.9]$\big)$, that is, there exist two closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that $X = N(A) \oplus X_0$ and $Y = R(A) \oplus Y_0$. Define $\hat{T} : X_0 \times Y_0 \to Y$ by $\hat{T}(x_0, y_0) = Tx_0 + y_0$. It is easy to see that $\hat{T}$ is bijective, this operator is called the bijection associated with T.

In particular, if $A : D(A) \subset X \to Y$ is Fredholm, then its restriction A_1 to D_A is clearly bounded since $\|A_1 x\| \leq \|x\|_A$ for all $x \in D(A)$ and the bijection associated with it is denoted by $\hat{A}$. The operator $\hat{A}$ is also called the bijection associated with A.

Main results

Before studying the perturbation of n operators, we will first examine the case of three operators. The following lemma is of great importance for what follows.

Lemma 1 *Let $A : D(A) \subset X \to Y$ be a closed linear operator. If $B : D(B) \subset X \to Y$ and $C : D(C) \subset X \to Y$ are two linear operators satisfying the following conditions:*

- $D(A) \subset D(B) \subset D(C)$;
- *There exists a positive constant γ_1 such that $\|Bx\| \leq \gamma_1 \|x\|_A$ for all $x \in D(A)$;*
- *There exists a positive constant γ_2 such that $\|Cx\| \leq \gamma_2 \|x\|_B$ for all $x \in D(B)$, with*

$$\gamma_1(1 + \gamma_2) < 1.$$

Then the algebraic sum $A + B + C$ is closed.

Proof. As a starting point, the operator $A+B+C$ is well defined since $D(A+B+C) = D(A) \cap D(B) \cap D(C) = D(A) \neq \emptyset$. Let x be an arbitrary element in $D(A)$, we have:

$$\begin{aligned} \|(A+B+C)x\| &\leq \|Ax\| + \|Bx\| + \|Cx\| \\ &\leq \|Ax\| + \gamma_1\|x\| + \gamma_1\|Ax\| + \gamma_2\|x\| + \gamma_2\|Bx\| \\ &= (\gamma_1 + \gamma_2)\|x\| + (1 + \gamma_1)\|Ax\| + \gamma_2(\gamma_1\|x\| + \gamma_1\|Ax\|) \end{aligned}$$

Therefore,

$$\|(A+B+C)x\| \leq (\gamma_1 + \gamma_2 + \gamma_1\gamma_2)\|x\| + (1 + \gamma_1 + \gamma_1\gamma_2)\|Ax\|. \tag{1}$$

By the same reasoning, we have

$$\|(B+C)x\| \leq (\gamma_1 + \gamma_2 + \gamma_1\gamma_2)\|x\| + \gamma_1(1 + \gamma_2)\|Ax\|$$

and hence,

$$\begin{aligned} \|(A+B+C)x\| &\geq \|Ax\| - \|(B+C)x\| \\ &\geq \|Ax\| - \Big((\gamma_1 + \gamma_2 + \gamma_1\gamma_2)\|x\| + \gamma_1(1 + \gamma_2)\|Ax\|\Big) \\ &= -(\gamma_1 + \gamma_2 + \gamma_1\gamma_2)\|x\| + (1 - \gamma_1(1 + \gamma_2))\|Ax\| \\ &= -\delta\|x\| + \lambda\|Ax\| \end{aligned}$$

where $\lambda = 1 - \gamma_1(1+\gamma_2)$ and $\delta = \gamma_1 + \gamma_2 + \gamma_1\gamma_2$. By hypothesis, we have $0 < \lambda < 1$ and $\delta > 0$. Therefore

$$\|Ax\| \leq \lambda^{-1}\Big(\|(A+B+C)x\| + \delta\|x\|\Big). \tag{2}$$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(A)$ that converges to some element $x_0 \in X$ such that $\Big((A+B+C)x_n\Big)_{n\in\mathbb{N}}$ converges to some $y_0 \in Y$. By means the equations (2), one can easily check that $\big(Ax_n\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y which is a Banach space. Hence there exists $z_0 \in Y$ such that $\big(Ax_n\big)_{n\in\mathbb{N}}$ converges to z_0 as $n \to +\infty$. The closedness of the operator A yields $x_0 \in D(A)$ and therefore $Ax_0 = z_0$. Using the equation (1), we get

$$\|(A+B+C)(x_n - x_0)\| \leq (\gamma_1 + \gamma_2 + \gamma_1\gamma_2)\|x_n - x_0\| + (1 + \gamma_1 + \gamma_1\gamma_2)\|A(x_n - x_0)\|$$

and hence $\Big((A+B+C)x_n\Big)_{n\in\mathbb{N}}$ converges to $y_0 = (A+B+C)x_0$ as $n \to +\infty$. Consequently, the operator $A+B+C$ is closed. □

Remark 1 *The proof of the previous lemma is similar to the one given in the classical case [1]. The particularity of the non-archimedean context will appear in Theorems 2 and 3.*

Theorem 1 *Let $A, B : X \to Y$ be two bounded linear operators. If A is Fredholm and $\|B\| < \|\hat{A}^{-1}\|^{-1}$. Then $A+B$ is also Fredholm and we have the following properties:*

i) $ind(A+B) = ind(A)$;

ii) $\alpha(A+B) \leq \alpha(A)$;

iii) $\beta(A+B) \leq \beta(A)$.

Proof. For *i)* we refer to [4, Theorem 3.43] and for *ii)* and *iii)*, we follow the same reasoning as in [2, Theorem 4.1]. □

Based on the previous lemma, we give in the following theorem sufficient conditions under which the algebraic sum of three (possibly unbounded) linear operators is Fredholm.

Theorem 2 *Let $A : D(A) \subset X \to Y$ be a Fredholm operator. If $B : D(B) \subset X \to Y$ and $C : D(C) \subset X \to Y$ are two linear operators satisfying the following conditions: $D(A) \subset D(B) \subset D(C)$ and there exist two positive constants γ_1 and γ_2 such that $\|Bx\| \leq \gamma_1\|x\|_A$ and $\|Cx\| \leq \gamma_2\|x\|_B$ for all $x \in D(A)$ with $\gamma_1(1+\gamma_2) < 1$ and $\max(1,\gamma_2) < \|\hat{A}^{-1}\|^{-1}$. Then $A+B+C$ is a Fredholm operator satisfying the following properties:*

i) $ind(A+B+C) = ind(A)$;

ii) $\alpha(A+B+C) \leq \alpha(A)$;

iii) $\beta(A+B+C) \leq \beta(A)$.

Proof. As a consequence of Lemma 1, the operator $A+B+C$ is closed. Denote by A_1, B_1 and C_1 for the restrictions of A, B and C to D_A, respectively. It is clear that A_1 is Fredholm. On the other hand, let $x \in D(A)$ with $\|x\|_A \leq 1$, then we have

$$\begin{array}{rcl} \|B_1 x\| & \leq & \gamma_1 \max(\|x\|, \|Ax\|) \\ & = & \gamma_1 \|x\|_A \\ & \leq & \gamma_1 \end{array}$$

and hence $\|B_1\| \leq \gamma_1$. Following the same reasoning, we have

$$\begin{array}{rcl} \|C_1 x\| & \leq & \gamma_2 \max\Big(\|x\|, \|Bx\|\Big) \\ & \leq & \gamma_2 \max\Big(\|x\|_A, \gamma_1 \|x\|_A\Big) \\ & \leq & \gamma_2 \max(1, \gamma_1) \|x\|_A \\ & \leq & \gamma_2 \max(1, \gamma_1). \end{array}$$

Since $\gamma_1(1+\gamma_2) < 1$, then $\gamma_1 < 1$, and hence $\|C_1\| \leq \gamma_2$. It follows that $\|B_1 + C_1\| \leq \max(1, \gamma_2) < \|\hat{A}^{-1}\|^{-1}$ and based on Theorem 1, we get the desired results. $\square$

Here we give the main theorem of this paper which is a generalization of Theorem 2.

Theorem 3 *If $A_1 : D(A_1) \subset X \to Y$, $A_2 : D(A_2) \subset X \to Y$,... and $A_n : D(A_n) \subset X \to Y$ are n (possibly unbounded) linear operators (with $n \geq 3$) such that:*

- *A_1 is a Fredholm operator.*
- *$D(A_1) \subset D(A_2) \subset ... \subset D(A_n)$.*
- *There exists positive constants $\gamma_1, \gamma_2, ..., \gamma_{n-1}$ such that $\|A_{k+1} x\| \leq \gamma_k \max(\|x\|, \|A_k x\|)$ for all $x \in D(A_1)$.*
- *$\gamma_1 < \frac{1}{n-1}$ and $\gamma_k < 1$ for all $k \in \{2, ..., n-1\}$.*
- *$\|\hat{A_1}^{-1}\| < 1$.*

Then the algebraic sum $A_1 + A_2 + ... + A_n$ is Fredholm and we have the following properties:

i) $ind(A_1 + A_2 + ... + A_n) = ind(A_1)$;

ii) $\alpha(A_1 + A_2 + ... + A_n) \leq \alpha(A_1)$;

iii) $\beta(A_1 + A_2 + ... + A_n) \leq \beta(A_1)$.

Proof.

Step1: In this step we will prove that $T = \sum_{k=1}^{n} A_k$ is closed. Indeed, this operator is well defined since $\bigcap_{k=1}^{n} D(A_k) = D(A_1) \neq \emptyset$. Let x be an arbitrary element of $D(A_1)$, and $k \in \{2, 3, ..., n\}$. We have

$$\begin{array}{rcl} \|A_k x\| & \leq & \max\Big(\big(\gamma_{k-1} + \gamma_{k-2}\gamma_{k-1} + ... + \gamma_1\gamma_2...\gamma_{k-1}\big)\|x\|, \gamma_1\gamma_2...\gamma_{k-1}\|A_1 x\|\Big) \\ & \leq & \big(\gamma_{k-1} + \gamma_{k-2}\gamma_{k-1} + ... + \gamma_1\gamma_2...\gamma_{k-1}\big)\|x\| + \gamma_1\gamma_2...\gamma_{k-1}\|A_1 x\|. \end{array}$$

Hence

$$\begin{aligned}\|Tx\| &\leq \sum_{k=1}^{n}\|A_k x\| \\ &\leq \delta_n\|x\| + \lambda_n\|A_1x\|. \qquad (1)\end{aligned}$$

where $\delta_n = \sum_{k=2}^{n}\big(\gamma_{k-1}+\gamma_{k-2}\gamma_{k-1}+...+\gamma_1\gamma_2...\gamma_{k-1}\big) > 0$ and $\lambda_n = 1 + \sum_{k=2}^{n}\gamma_1\gamma_2...\gamma_{k-1}$.
Similarly, we have

$$\|(T-A_1)x\| \leq \delta_n\|x\| + (\lambda_n - 1)\|A_1x\|. \qquad (2)$$

From equation (2), it follows that for all $x \in D(A_1)$, we have

$$\begin{aligned}\|Tx\| &\geq \|A_1x\| - \|(T-A_1)x\| \\ &\geq -\delta_n\|x\| + (2-\lambda_n)\|A_1x\|,\end{aligned}$$

Hence

$$(2-\lambda_n)\|A_1x\| \leq \delta_n\|x\| + \|Tx\|. \qquad (3)$$

Now, we will prove that $0 < 2-\lambda_n < 1$. Indeed, from the fact that $\gamma_k < 1$ for all $k \in \{2,...,n-1\}$, we have

$$\begin{aligned}2-\lambda_n &= 1 - \sum_{k=2}^{n}\gamma_1\gamma_2...\gamma_{k-1} \\ &> 1-(n-1)\gamma_1\end{aligned}$$

By assumption, we have $\gamma_1 < \dfrac{1}{n-1}$. Thus $2-\lambda_n > 0$. Since all the constants γ_k are positive, it follows that $1-\lambda_n = -\sum_{k=2}^{n}\gamma_1\gamma_2...\gamma_{k-1} < 0$. Hence $2-\lambda_n < 1$.
Using equation (3), we get

$$\|A_1x\| \leq (2-\lambda_n)^{-1}\big(\delta_n\|x\| + \|Tx\|\big). \qquad (4)$$

Let $(x_m)_{m\in\mathbb{N}}$ be a sequence in $D(A_1)$ that converges to some element $x_0 \in X$ such that $(Tx_m)_{m\in\mathbb{N}}$ converges to some element $y_0 \in Y$. Using equation (1) and (4), one can easily check that $(A_1x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in Y which is a Banach space. So there exists $z_0 \in Y$ such that $(A_1x_m)_{m\in\mathbb{N}}$ converges to z_0 as $m \to +\infty$. Since A_1 is closed, the $x_0 \in D(A_1)$ and $A_1x_0 = z_0$. Using equation (1) it follows that

$$\|T(x_m - x_0) \leq \delta_n\|x_m - x_0\| + \lambda_n\|A_1(x_m - x_0)\|$$

Since $\delta_n > 0$ and $\lambda_n > 0$ for all $n \geq 3$, then $(Tx_m)_{m\in\mathbb{N}}$ converges to $y_0 = Tx_0$ as $m \to +\infty$. Consequently, the linear operator $T = \sum_{k=1}^{n} A_k$ is closed.

Step2: Denote by $A'_1, A'_2, ..., A'_n$ the restrictions of $A_1, A_2, ..., A_n$ to D_{A_1}, respectively. The operator A_1 being Fredholm implies that A'_1 is also Fredholm. Put $T' = \sum_{k=1}^{n} A'_1$, we have

$$\begin{aligned}\|T' - A'_1\| &\leq \max_{1\leq k\leq n}\gamma_k \\ &< 1 \\ &< \|\hat{A}_1^{-1}\|^{-1}.\end{aligned}$$

It follows by Theorem 1 that $T' = \sum_{k=1}^{n} A'_1$ is a linear Fredholm operator and the properties $i), ii)$ and $iii)$ hold. □

Applications to $n \times n$ matrices of linear operators

Inspired by the approach presented in [18], this section is devoted to the study of certain properties of $n \times n$-block matrices of linear operators. For this reason, let

$$M = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

be an $n \times n$-block matrices defined on $\prod_{k=1}^{n} X_i$ where each X_i is a non-archimedean Banach space and the entries A_{ij} are possibly unbounded linear operators. Recall that each A_{ij} is defined from X_j to X_i for all $i, j \in \{1, 2, ..., n\}$ and the domain of the operator M is $D(M) = \prod_{j=1}^{n} \Big(\bigcap_{i=1}^{n} D(A_{ij}) \Big)$.

To make the study of the matrix M easier, we will decompose it into a sum of n matrices in the following way. For each $k \in \{1, 2, ..., n\}$, we consider the $n \times n$-block matrix M_k defined as follows

$$M_k = (B_{ij}^{(k)}) = \begin{cases} A_{ij} & \text{if} \quad i = k + j - 1 \bmod n \\ 0 & \qquad otherwise \end{cases}$$

We can easily check that $M = \sum_{k=1}^{n} M_k$.

Lemma 2 *i) M_1 is closed if and only if A_{ii} is closed for all $i \in \{1, 2, ..., n\}$.*

ii) If A_{ii} is a Fredholm linear operator for each i, then M_1 is also Fredholm. In this case we have $ind(M_1) = \sum_{k=1}^{n} ind(A_{kk})$.

Proof. $i)$ Assume that M_1 is closed and let $(x_n)_{n\in\mathbb{N}}$ be a sequenece in $D(A_{11})$ that converges to x. Hence the sequence $(v_n)_n = \begin{pmatrix} (x_n)_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ converges to $v = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and since M_1 is closed then $(A_{11}x_n)_{n\in\mathbb{N}} = (M_1 v_n)_{n\in\mathbb{N}}$ converges to $M_1 v = A_{11}x$ and therefore A_{11} is closed. A similar argument is valid for $i \geq 2$. Conversely, suppose that the linear operators A_{ii} are closed for all $i \in \{1, 2, ..., n\}$ and let $(h_m)_m = \begin{pmatrix} (x_m^{(1)})_m \\ (x_m^{(2)})_m \\ \vdots \\ (x_m^{(n)})_m \end{pmatrix}$ be a sequence in $\prod_{k=1}^{n} D(A_{ii})$ that converges to $h = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. From the closedness of the linear operators A_{ii} we conclude that

$$(M_1 h_m)_m = \begin{pmatrix} (A_{11} x_m^{(1)})_m \\ (A_{22} x_m^{(2)})_m \\ \vdots \\ (A_{nn} x_m^{(n)})_m \end{pmatrix} \xrightarrow[n\to\infty]{} \begin{pmatrix} A_{11} x_1 \\ A_{22} x_2 \\ \vdots \\ A_{nn} x_n \end{pmatrix} = M_1 h$$

Thus M_1 is closed.

ii) To establish the validity of the second assertion, it is sufficient to utilize the following equalities:

$$N(M_1) = \bigoplus_{k=1}^{n} N(A_{kk}) \quad \text{and} \quad R(M_1) = \bigoplus_{k=1}^{n} R(A_{kk}). \quad \square$$

Lemma 3 *Assume that for all $i, j \in \{1, 2, ..., n\}$ and $k \in \{1, 2, ..., n-1\}$ with $i = j + k - 1$ mod n, there exists a positive constant γ_{ijk} such that*

$$\|B_{ij}^{(k+1)} x\| \leq \gamma_{ijk} \|B_{ij}^{(k)} x\| \quad \textit{for all } x \in D(B_{ij}^{(k)}) \subset D(B_{ij}^{(k+1)}),$$

Then $D(M_k) \subset D(M_{k+1})$ and we have

$$\|M_{k+1} X\| \leq \gamma_k \|M_k X\| \quad \textit{for all } X \in D(M_k),$$

where $\gamma_k = \max\limits_{1 \leq i,j \leq n} \gamma_{ijk}$.

Proof. For $k = 1$, let $i, j \in \{1, 2, ..., n\}$. We have

$$\|B_{ij}^{(2)} x\| \leq \gamma_{ij1} \max\big(\|x\|, \|B_{ij}^{(1)} x\|\big) \quad \text{for all } \ x \in D\big(B_{ij}^{(1)}\big) \subset D\big(B_{ij}^{(2)}\big),$$

where $i = j + 1$ mod n. It follows that

$$\|A_{ij} x\| \leq \gamma_{ij1} \max\big(\|x\|, \|A_{jj} x\|\big) \quad \text{for all } \ x \in D\big(A_{jj}\big) \subset D\big(A_{ij}\big).$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \prod\limits_{i=1}^{n} D(A_{ii})$, we have

$$\begin{aligned} \|M_2 X\| &= \max_{1 \leq i,j \leq n} \big(\|A_{ij} x_j\|\big) \\ &\leq \max_{1 \leq i,j \leq n} \Big(\gamma_{ij1} \big(\max\big(\|x_j\|, \|A_{jj} x_j\|\big)\Big) \\ &\leq \gamma_1 \max\Big(\|X\|, \|M_1 X\|\Big) \end{aligned}$$

where $\gamma_1 = \max\limits_{1 \leq i,j \leq n} (\gamma_{ij1})$. The result follows using the same reasoning for $k \geq 2$. $\square$

The following theorem establishes sufficient conditions under which the matrix M of linear operators is closed, based on the entries of its diagonal:

Theorem 4 *Assume that the following conditions are satisfied for all* $i, j \in \{1, 2, ..., n\}$ *and* $k \in \{1, 2, ..., n-1\}$ *with* $i = k + j - 1$ *mod* n.

- $D\big(B_{ij}^{(k)}\big) \subset D\big(B_{ij}^{(k+1)}\big)$;
- $\|B_{ij}^{(k+1)}x\| \leq \gamma_{ijk} \max\big(\|x\|, \|B_{ij}^{(k)}x\|\big)$ *for all* $x \in D\big(B_{ij}^{(k)}\big)$;
- $\max\limits_{1 \leq i,j \leq n} (\gamma_{ij1}) < \dfrac{1}{n-1}$ *and* $\max\limits_{1 \leq i,j \leq n} (\gamma_{ijk}) < 1$ *for all* $k \in \{2, 3, ..., n\}$;
- *The linear operator* A_{ii} *is closed for each* $i \in \{1, 2, ..., n\}$.

Then the linear operator M *is closed.*

Proof. Let $i, j, k \in \{1, 2, ..., n\}$ with $i = k + j - 1$ mod n. By Lemma 3, it follows that

$$\|M_{k+1}x\| \leq \gamma_k \max\big(\|x\|, \|M_k x\|\big) \quad \text{for all } x \in D(M_k) \subset D(M_{k+1}),$$

where $\gamma_k = \max\limits_{1 \leq i,j \leq n} (\gamma_{ijk})$. By virtue of Lemma 2($i$), we conclude that M_1 is closed and the closedness of M then follows from Theorem 3. □

Theorem 5 *Suppose that the assumptions of Theorem 4 are satisfied, and that* $\|\hat{M}^{-1}\| > 1$. *Then, assuming that* A_{ii} *is Fredholm for each* $i \in \{1, 2, \ldots, n\}$, *it follows that* M *is also Fredholm, and the following properties hold.*

i) $\alpha(M) \leq \alpha(M_1) = \sum_{i=1}^{n} \alpha(A_{ii})$;

ii) $\beta(M) \leq \beta(M_1) = \sum_{i=1}^{n} \beta(A_{ii})$;

iii) $ind(M) = ind(M_1) = \sum_{i=1}^{n} ind(A_{ii})$.

Proof. The proof is a direct consequence of Theorems 3, 4 and Lemma 2(ii).

Bibliography

[1] T. Diagana, Perturbations of Unbounded Fredholm Linear Operators in Banach Spaces, Operator Theory, Springer Basel 2015

[2] Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators Vol. I. Operator Theory: Advances and Applications, vol. 49. Birkhäuser, Basel (1990)

[3] S. Nadathur, Krishnamachari, Linear Operators between Nonarchimedean Banach Spaces (1973).

[4] Diagana T, Ramaroson F. Non-Archimedean operator theory. Springer Briefs in Mathematics. Cham: Springer; 2016.

[5] Gruson, L.: Théorie de Fredholm p-adique, Bull. Soc. Math. France 94 (1966) 67–95.

[6] Serre, J. P.: Endomorphismes complètement continus des espaces de Banach p-adiques, Inst. Hautes Etudes Sci. Publ. Math. 12 (1962) 69–85.

[7] Schikhof, W. H.: On p-adic Compact Operators, Report 8911, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, (1989) 1–28.

[8] Christol, G. and Mebkhout, Z.: Sur le théorème de l'indice des équations différentielles p-adiques I, Ann. Inst. Fourier 43(5), (1993), 1545–1574.

[9] Christol, G. and Mebkhout, Z.: Sur le théorème de l'indice des équations différentielles p-adiques II, Ann. of Math. 146 (1997), 345–410.

[10] Christol, G. and Mebkhout, Z.: Sur le théorème de l'indice des équations différentielles p-adiques III, preprint.

[11] Araujo, J., Perez-Garcia, C., and Vega, S. Preservation of the index of p-adic linear operators under compact perturbations. Compositio Math. 118, 3 (1999), 291–303.

[12] S. Sliwa, "On Fredholm operators between non-archimedean Frechet spaces", Compositio Mathematica, vol. 139, no. 1, pp. 113 – 118, (2003).

[13] Van Rooij, A.C.M, Non-Archimedean Functional Analysis, Marcel Dekker, New York, (1978).

[14] Hernan R. Henriquez, Samuel Navarro H. and Jose Aguayo G., Closed linear operators between nonarchimedean Banach spaces. Indag. Mathem., N.S., 16(2), (2005), 201 − 214.

[15] A. F. Monna, Analyse non-archimedienne, Springer-Verlag Berlin Heidelberg New York (1970).

[16] N. De Grande-De Kimpe and J. Martinez-Maurica, Fredholm Theory for p-adic locally convex spaces. Annali di Matematica pura ed applicata (IV), Vol. CLX (1991), pp. 223 − 234.

[17] J. Van Tiel, Espaces localement K-convexes, Proc. Kon. Ned. Akad. v. Wetensch. A, 68 (1965), pp. 249 − 289.

[18] A. Ammar, T. Diagana, and A. Jeribi, Perturbations of Fredholm linear relations in Banach spaces with application to 3 × 3-block matrices of linear relations. Arab J Math Sci 22(2016)59–76.

Abdelkhalek El Amrani

Laboratoire des Sciences Mathématiques et Applications (LaSMA), Dhar El Mahraz Faculty of Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco

E-mail: abdelkhalek.elamrani@usmba.ac.ma

Corresponding author

Youness Hadder

Laboratoire des Sciences Mathématiques et Applications (LaSMA), Dhar El Mahraz Faculty of Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco

E-mail: haddfsm@hotmail.com

Yassine Mouzour

Laboratoire des Sciences Mathématiques et Applications (LaSMA), Dhar El Mahraz Faculty of Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco

E-mail: yassine.mouzour@usmba.ac.ma