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AN OPERATOR APPROACH TO STATISTICAL CONVERGENCE IN TOPOLOGICAL FRAMEWORK FOR SEQUENCE OF FUNCTION

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Communicated by ……

Abstract: In this study, a novel approach is developed to analyze the statistical convergence of sequences of functions that are derived under the context of topological spaces. The study introduces two new operators, $\diamondsuit^-_{f(x)}$ and $\diamondsuit^+_{f(x)}$, which are designed to explore the local behavior of sequences of functions $\langle f_n \rangle$ that converges to a function f within a neighborhood structure. In addition, some important and useful inclusion properties of these operators are analyzed. Using these properties, an approach to deal with statistical convergence (both point-wise and uniform) for sequences of functions in topological spaces is presented. With some new theorems on the properties of statistical limit, uniqueness of statistical limit and counterexamples, the importance of the study has been validated.

Keywords: Sequence of functions; Statistical convergence; Pointwise convergence; Uniform convergence.

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1 Introduction

In 1951, Fast [5] introduced statistical convergence for real sequences as a more flexible form of convergence that allows a small number of terms to differ from the limit. In 1959, Schoenberg [13] also studied the concept independently and highlighted it connection to summability theory. In their study the concept of natural density plays the key role. For any $T \subseteq \mathbb{N}$, the natural density (also known as asymptotic density) is defined as,

$$\delta(T) = \lim_{n\to\infty} \frac{1}{n} \times |\{k \leq n : k \in T\}|$$

if the limit exists. This concept was extended later to topological spaces by Maio and Kočinac [11] in 2008, enabling the study of convergence in more general settings. In the same way, many mathematicians contributed their deeper insights on statistical convergence in different fields through the works [1, 3, 6, 8, 9].

The concept of statistical convergence was extended beyond real sequences to function spaces in the year 2002 by Gökhan and Güngör [7]. They introduced the notions of pointwise statistical convergence and statistical Cauchy sequences for real-valued functions, showing that a sequence is pointwise statistically convergent if and only if it is statistically Cauchy. This foundational work opened the door for deeper analysis of statistical behavior in sequences of functions. Later, in the year 2007, Balcerzak, Dems, and Komisarski [2] studied statistical and ideal convergence for sequences of functions in metric spaces. Also, Sarabadan and Talebi [12] investigated the concepts of statistical convergence and ideal convergence for sequences of functions within the framework of two-normed spaces. Further developments were made by Yegül and Dündar [14] in the year 2017 to investigate statistical convergence for sequences of functions in the context of two normed spaces. Their study extended the applicability of statistical convergence to a broader class of sequences of functions and provided new insights into convergence properties in two-normed settings.

Using the operators $N_x^-(A_n)$ and $N_x^+(A_n)$, Inan et al. [10] defined the statistical convergence for the sequence of sets. But we do not find the neighborhood any where in his discussions, where as in a topological setting the neighborhood plays the most important role in any type of convergence. We take an alternative approach to use operators that involve the neighborhood termed $\diamondsuit^-_{f(x)}$ and $\diamondsuit^+_{f(x)}$. These operators will not be helpful for the study of sequence of sets, but have been found to be helpful in exploring more authentic aspects of such convergence criteria for sequence of functions. In this work, we build upon the previous literature [11, 10, 14] to extend the theory of statistical convergence of sequences of functions into more general and abstract topological settings, which are explained thoroughly through theorems and counterexamples.

2 Preliminaries

Before delving into the new variant of statistical convergence, some prerequisite definitions, terms, and notations are added for the reader's convenience. For common symbols, nomenclature, and concepts we can follow the article [4].

Definition 1. [5] *For any $T \subseteq \mathbb{N}$, the natural density (also known as asymptotic density) is defined as,*

$$\delta(T) = \lim_{n\to\infty} \frac{1}{n} \times |\{k \leq n : k \in T\}|$$

if the limit exists.

It is known that $\delta(T^{\complement}) = 1 - \delta(T)$. This property of natural density has been frequently used in this paper for the establishment of several key features and theorems.

Definition 2. [10] *Let $X \neq \emptyset$ be a set and $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. Let*

$$N_x^-(A_n) := \{n \in \mathbb{N} : x \notin A_n\},$$

and

$$N_x^+(A_n) := \{n \in \mathbb{N} : x \in A_n\},$$

for $x \in X$.

Definition 3. [11] *A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to converge statistically (or shortly, s-converge) to $x \in X$, if for every neighborhood U of x, $\delta(\{n \in \mathbb{N} : x \notin U\}) = 0$. In this case, we write $x = s - \lim x_n$.*

Definition 4. [14] *Two sequences of functions $\{f_n : (X,\tau) \to (Y,\sigma)\}$ and $\{g_n : (X,\tau) \to (Y,\sigma)\}$ are said to be equal for almost all $n \in \mathbb{N}$ (in short a.a.n.) if $\delta(\{n \in \mathbb{N} : f_n(x) \neq g_n(x)\}) = 0$ for each $x \in X$.*

3 Construction of $\diamondsuit^-_{f(x)}(f_n(x))$ and $\diamondsuit^+_{f(x)}(f_n(x))$ operators

Although for a function $f : (X,\tau) \to (Y,\sigma)$, $f(a) \in V$ is the standard notion to represent belonging of $f(a)$ to V, we rewrite it as $\{f(a)\} \cap V = \emptyset$ which makes it easier to handle. Similarly, we represent $f(a) \notin V$ as $\{f(a)\} \cap V = \emptyset$. Our first step is to make a collection of natural numbers that index a sequence $\{f_n(a)\}$ which fails to enter every neighborhood of the point $f(a)$ where $f_n, f : (X,\tau) \longrightarrow (Y,\sigma)$ and $a \in X$.

Definition 5. *Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of functions where $f_n, f : (X,\tau) \longrightarrow (Y,\sigma)$ for all $n \in \mathbb{N}$. Then for an arbitrary $a \in X$,*

$$\diamondsuit^-_{f(a)}(f_n(a)) = \{n \in \mathbb{N} : \{f_n(a)\} \cap V = \emptyset \text{ for at least one neighborhood } V \text{ of } f(a)\}$$

.

Alternatively, we make another collection of indices of sequence $\{f_n(a)\}$ which is a possible complement of $\diamondsuit^-_{f(a)}(f_n(a))$.

Definition 6. *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions where $f_n, f : (X,\tau) \longrightarrow (Y,\sigma)$ for all $n \in \mathbb{N}$. Then for an arbitrary $a \in X$,*

$$\diamondplus_{f(a)}(f_n(a)) = \{n \in \mathbb{N} : \{f_n(a)\} \cap V \neq \emptyset \text{ for every neighborhood } V \text{ of } f(a)\}.$$

In the next step, we will explore different algebraic and combinatorial properties of these operators, and then we will use them to define point-wise convergence and uniform convergence of sequence of functions.

Theorem 1. *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of function where $f_n, f : (X,\tau) \longrightarrow (Y,\sigma)$. Then*

(i) $\Diamond^-_{f(x)}(f_n(x)) \cap \Diamond^+_{f(x)}(f_n(x)) = \emptyset$.

(ii) $\Diamond^-_{f(x)}(f_n(x)) \cup \Diamond^+_{f(x)}(f_n(x)) = \mathbb{N}$.

holds for every element $x \in X$.

Proof. (i) Let $p \in \Diamond^-_{f(a)}(f_n(a))$ for an arbitrary element $a \in X$. Therefore, there exists at least one neighborhood V of $f(a)$ such that $f_p(a) \cap V = \emptyset$. So, $f_p(a) \cap V \neq \emptyset$ cannot be true for every neighborhood V of $f(a)$. Thus, $p \notin \Diamond^+_{f(a)}(f_n(a))$ which implies $\Diamond^-_{f(a)}(f_p(a)) \cap \Diamond^+_{f(a)}(f_p(a)) = \emptyset$.
Hence, $\Diamond^-_{f(x)}(f_n(x)) \cap \Diamond^+_{f(x)}(f_n(x)) = \emptyset$ is true for every element $x \in X$.

(ii) Let $k \in \mathbb{N}$ be arbitrary and $k \notin \Diamond^+_{f(a)}(f_n(a))$ for an arbitrary $a \in X$. Therefore, for at least one neighborhood V of $f(a)$ such that $f_k(a) \cap V = \emptyset$. Thus, $k \in \Diamond^-_{f(a)}(f_n(a))$. So, $k \in \Diamond^-_{f(a)}(f_n(a)) \cup \Diamond^+_{f(a)}(f_n(a))$ that implies $\mathbb{N} \subseteq \Diamond^-_{f(a)}(f_n(a)) \cup \Diamond^+_{f(a)}(f_n(a))$. But, $\Diamond^-_{f(a)}(f_n(a)) \cup \Diamond^+_{f(a)}(f_n(a)) \subseteq \mathbb{N}$. Thus, $\mathbb{N} = \Diamond^-_{f(a)}(f_n(a)) \cup \Diamond^+_{f(a)}(f_n(a))$ holds for an arbitrary $a \in X$. Hence, $\Diamond^-_{f(x)}(f_n(x)) \cup \Diamond^+_{f(x)}(f_n(x)) = \mathbb{N}$ is true for every element $x \in X$. □

Note: Thus we can conclude $\left(\Diamond^-_{f(a)}(f_n(a))\right)^{\complement} = \Diamond^+_{f(a)}(f_n(a))$.

Theorem 2. *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ and $\langle g_n \rangle_{n\in\mathbb{N}}$ be two sequence of functions where $f_n, g_n, f : (X,\tau) \longrightarrow (Y,\sigma)$. Then the following properties holds,*

(i) $\Diamond^-_{f(x)}(f_n(x) \cup g_n(x)) = \Diamond^-_{f(x)}(f_n(x)) \cap \Diamond^-_{f(x)}(g_n(x))$.

(ii) $\Diamond^+_{f(x)}(f_n(x) \cup g_n(x)) = \Diamond^+_{f(x)}(f_n(x)) \cup \Diamond^+_{f(x)}(g_n(x))$.

for every element $x \in X$.

Proof. (i) For an arbitrary element $a \in X$, we have
$\Diamond^-_{f(a)}(f_n(a) \cup g_n(a)) = \{n \in \mathbb{N} : \{f_n(a) \cup g_n(a)\} \cap V = \emptyset$ for at least one neighborhood V of $f(a)\}$.
$= \{n \in \mathbb{N} : \{\{f_n(a) \cap V\} \cup \{g_n(a) \cap V\}\} = \emptyset$ for at least one neighborhood V of $f(a)\}$.
$= \{n \in \mathbb{N} : f_n(a) \cap V = \emptyset$ for at least one neighborhood V of $f(a)\} \cap$
$\{n \in \mathbb{N} : g_n(a) \cap V = \emptyset$ for at least one neighborhood V of $f(a)\}$.
$= \Diamond^-_{f(a)}(f_n(a)) \cap \Diamond^-_{f(a)}(g_n(a))$.
Hence, $\Diamond^-_{f(x)}(f_n(x) \cup g_n(x)) = \Diamond^-_{f(x)}(f_n(x)) \cap \Diamond^-_{f(x)}(g_n(x))$ is true for every $x \in X$.

(ii) Let for an arbitrary $a \in X$, we have
$\diamondplus_{f(a)}^{+}(f_n(a) \cup g_n(a)) = \{n \in \mathbb{N} : \{f_n(a) \cup g_n(a)\} \cap V \neq \emptyset$ for every neighborhood V of $f(a)\}$
$= \{n \in \mathbb{N} : \{\{f_n(a) \cap V\} \cup \{g_n(a) \cap V\}\} \neq \emptyset$ for every neighborhood V of $f(a)\}$
$= \{n \in \mathbb{N} : f_n(a) \cap V \neq \emptyset$ for every neighborhood V of $f(a)$ $\} \cup \{n \in \mathbb{N} :$ $g_n(a) \cap V \neq \emptyset$ for every neighborhood V of $f(a)\}$.
$= \Diamond_{f(a)}^{+}(f_n(a)) \cup \Diamond_{f(a)}^{+}(g_n(a))$.
Hence, $\Diamond_{f(x)}^{+}(f_n(x) \cup g_n(x)) = \Diamond_{f(x)}^{+}(f_n(x)) \cup \Diamond_{f(x)}^{+}(g_n(x))$ holds for every $x \in X$. □

Theorem 3. *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ and $\langle g_n \rangle_{n\in\mathbb{N}}$ be two sequence of functions where $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g_n, f : (X, \tau) \longrightarrow (Y, \sigma)$. Then the following properties hold:*

(i) $\Diamond_{f(x)}^{-}(f_n(x) \cap g_n(x)) \supseteq \Diamond_{f(x)}^{-}(f_n(x)) \cup \Diamond_{f(x)}^{-}(g_n(x))$
(ii) $\Diamond_{f(x)}^{+}(f_n(x) \cap g_n(x)) \subseteq \Diamond_{f(x)}^{+}(f_n(x)) \cap \Diamond_{f(x)}^{+}(g_n(x))$

for every element $x \in X$.

Proof. (i) Let for an arbitrary element $a \in X$, $p \in \Diamond_{f(a)}^{-}(f_n(a)) \cup \Diamond_{f(a)}^{-}(g_n(a))$.
So, $p \in \{n \in \mathbb{N} : \{f_n(a)\} \cap V\} = \emptyset$ for at least one neighborhood V of $f(a)\}$ or $p \in \{n \in \mathbb{N} : \{g_n(a)\} \cap U\} = \emptyset$ for at least one neighborhood U of $f(a)\}$.
Thus, $\{f_p(a)\} \cap V = \emptyset$ or $\{g_p(a)\} \cap U = \emptyset$. So, $\{f_p(a) \cap g_p(a)\} \cap (V \cap U) = \emptyset$, where $V \cap U$ is a neighborhood of $f(a)$.
So, $p \in \{n \in \mathbb{N} : \{f_n(a) \cap g_n(a)\} \cap W = \emptyset$ for at least one neighborhood W of $f(a)\} =$ $\Diamond_{f(a)}^{-}(f_n(a) \cap g_n(a))$.
Thus, $\Diamond_{f(a)}^{-}(f_n(a) \cap g_n(a)) \supseteq \Diamond_{f(a)}^{-}(f_n(a)) \cup \Diamond_{f(a)}^{-}(g_n(a))$.
Hence, $\Diamond_{f(x)}^{-}(f_n(x) \cap g_n(x)) \supseteq \Diamond_{f(x)}^{-}(f_n(x)) \cup \Diamond_{f(x)}^{-}(g_n(x))$ is true for every $x \in X$.

(ii) Let for an arbitrary $a \in X$, $p \in \Diamond_{f(a)}^{+}(f_n(a) \cap g_n(a)) = \{n \in \mathbb{N} :$ $\{f_n(a) \cap g_n(a)\} \cap V \neq \emptyset$ for every neighborhood V of $f(a)\}$.
So, $\{f_p(a) \cap g_p(a)\} \cap V \neq \emptyset$ for every neighborhood V of $f(a)$.
This implies $\{f_p(a)\} \cap V \neq \emptyset$ and $\{g_p(a)\} \cap V \neq \emptyset$ for every neighborhood V of $f(a)$.
Thus, $p \in \Diamond_{f(a)}^{+}(f_n(a))$ and $p \in \Diamond_{f(a)}^{+}(g_n(a))$ i.e., $p \in \Diamond_{f(a)}^{+}(f_n(a)) \cap \Diamond_{f(a)}^{+}(g_n(a))$.
So, $\Diamond_{f(a)}^{+}(f_n(a) \cap g_n(a)) \subseteq \Diamond_{f(a)}^{+}(f_n(a)) \cap \Diamond_{f(a)}^{+}(g_n(a))$.
Hence, $\Diamond_{f(x)}^{+}(f_n(x) \cap g_n(x)) \subseteq \Diamond_{f(x)}^{+}(f_n(x)) \cap \Diamond_{f(x)}^{+}(g_n(x))$ holds for every $x \in X$. □

Theorem 4. *Let $\langle f_n^j \rangle_{n\in\mathbb{N}}$; $(j = 1, 2, \cdots, r)$ be sequences of functions where $f_n^j, f^j : (X, \tau) \longrightarrow (Y, \sigma)$. Then the following properties hold:*

$$\text{(i)} \quad \Diamond_{f^j(x)}^{-}\Big(\prod_{j=1}^{r} f_n^j(x)\Big) = \bigcup_{j=1}^{r} \Diamond_{f^j(x)}^{-}(f_n^j(x))$$

$$\text{(ii)} \quad \Diamond_{f^j(x)}^{+}\Big(\prod_{j=1}^{r} f_n^j(x)\Big) = \bigcap_{j=1}^{r} \Diamond_{f^j(x)}^{+}(f_n^j(x))$$

for every element $x \in X$.

Proof.

(i) $\diamondsuit^{-}_{f^j(x)}(\prod_{j=1}^{r} f_n^j(x)) = \{n \in \mathbb{N} : \{\prod_{j=1}^{r} f_n^j(x)\} \cap V^j = \emptyset,$ for at least one neighborhood V^j of $f^j(x)\}$.

$= \{n \in \mathbb{N} : (f_n^1(x), f_n^2(x), \cdots, f_n^r(x)) \cap V^j = \emptyset,$ for at least one neighborhood $V^j = V^1 \times V^2 \times \cdots \times V^r$ of $(f^1(x), f^2(x), \cdots, f^r(x))$.

$= \{n \in \mathbb{N} : f_n^j(x) \notin V^j$ for at least one $j = 1, 2, \cdots, r$; where V^j is a neighborhood of $f^j(x)\}$.

$= \bigcup_{j=1}^{r} \{n \in \mathbb{N} : \{f_n^j(x)\} \cap V^j = \emptyset$ for at least one neighborhood V^j of $f^j(x)\}$.

$= \bigcup_{j=1}^{r} \diamondsuit^{-}_{f^j(x)}(f_n^j(x))$.

Hence, $\diamondsuit^{-}_{f^j(x)}(\prod_{j=1}^{r} f_n^j(x)) = \bigcup_{j=1}^{r} \diamondsuit^{-}_{f^j(x)}(f_n^j(x))$ holds for every $x \in X$.

(ii) $\diamondsuit^{+}_{f^j(x)}(\prod_{j=1}^{r} f_n^j(x)) = \{n \in \mathbb{N} : \{\prod_{j=1}^{r} f_n^j(x)\} \cap V^j \neq \emptyset,$ for every open neighborhood V^j of $f^j(x)$.

$= \{n \in \mathbb{N} : (f_n^1(x), f_n^2(x), \cdots, f_n^r(x)) \cap V^j \neq \emptyset$ for every neighborhood $V^j = V^1 \times V^2 \times \cdots \times V^r$ of $(f^1(x), f^2(x), \cdots, f^r(x))\}$

$= \{n \in \mathbb{N} : f_n^j(x) \in V^j$ for every $j = 1, 2, \cdots, r$; where V^j is a neighborhood of $f^j(x)\}$.

$= \bigcap_{j=1}^{r} \{n \in \mathbb{N} : \{f_n^j(x)\} \cap V^j \neq \emptyset$ for every neighborhood V^j of $f^j(x)\} = \bigcap_{j=1}^{r} \diamondsuit^{+}_{f^j(x)}(f_n^j(x))$.

Hence, $\diamondsuit^{+}_{f^j(x)}(\prod_{j=1}^{r} f_n^j(x)) = \bigcap_{j=1}^{r} \diamondsuit^{+}_{f^j(x)}(f_n^j(x))$ holds for every $x \in X$.

□

Theorem 5. *Let $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ be a sequence of functions $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ and f is open continuous. Then the following holds for every $x \in X$,*

(i) $\diamondsuit^{-}_{f(x)}(f_n(x)) = \diamondsuit^{-}_{x}(f^{-1}(f_n(x)))$.

(ii) $\diamondsuit^{+}_{f(x)}(f_n(x)) = \diamondsuit^{+}_{x}(f^{-1}(f_n(x)))$.

Proof. (i) Let $q \in \diamondsuit^{-}_{f(a)}(f_n(a))$ for any $a \in X$. Therefore, atleast one neighborhood V of $f(a)$, $f_q(a) \cap V = \emptyset$. Since, the mapping f is continuous, for at least one neighborhood $f^{-1}(V)$ of $f^{-1}(f(a))$, $f^{-1}(f_q(a)) \cap V = \emptyset$. That implies $q \in \diamondsuit^{-}_{f^{-1}(f(a))}(f^{-1}(f_n(x)))$. So, $\diamondsuit^{-}_{f(a)}(f_n(a)) \subseteq \diamondsuit^{-}_{a}(f^{-1}(f_n(a)))$ is obtained. Likewise, $\diamondsuit^{-}_{a}(f^{-1}(f_n(a))) \subseteq \diamondsuit^{-}_{f(a)}(f_n(a))$ can also be obtained.

Thus, $\diamondsuit^{-}_{f(a)}(f_n(a)) = \diamondsuit^{-}_{a}(f^{-1}(f_n(a)))$. Hence (i) holds for every $x \in X$.

(ii) Let $l \in \diamondsuit^{+}_{f(a)}(f_n(a))$ for any $a \in X$. Therefore for every neighborhood V of $f(a)$, $f(a) \cap V \neq \emptyset$. Since the mapping f is continuous, for every neighborhood $f^{-1}(V)$ of $f^{-1}(f(a))$, $f^{-1}(f_l(a)) \cap V \neq \emptyset$. That gives $l \in \diamondsuit^{+}_{f^{-1}(f(a))}(f^{-1}(f_n(a)))$. So, $\diamondsuit^{+}_{f(a)}(f_n(a)) \subseteq \diamondsuit^{+}_{a}(f^{-1}(f_n(a)))$ is obtained. Similarly, $\diamondsuit^{+}_{a}(f^{-1}(f_n(a))) \subseteq \diamondsuit^{+}_{f(a)}(f_n(a))$ can be obtained.

Thus, $\diamondsuit^+_{f(a)}(f_n(a)) = \diamondsuit^+_{f^{-1}(f(a))}(f^{-1}(f_n(a)))$. Hence, (ii) satisfied for every $x \in X$. $\square$

4 Statistical limit of the sequence of functions

Now, using the operators discussed in section 3, we represent the concept of point-wise and uniform statistical convergence of the sequence of functions.

Definition 7. *A sequence of function $\langle f_n \rangle_{n\in\mathbb{N}}$ such that $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be pointwise statistically convergent to f, if*

$$\delta(\diamondsuit^-_{f(x)}(f_n(x))) = 0 \text{ for all } x \in X.$$

From Proposition 1, an equivalent criteria for the statistical convergence of the sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of functions to f is

$$\delta(\diamondsuit^+_{f(x)}(f_n(x))) = 1 \text{ for all } x \in X.$$

Mathematically it is expressed as, $f_n \overset{Pt-s-\lim}{\longrightarrow} f$.

Definition 8. *A sequence of function $\langle f_n(x) \rangle_{n\in\mathbb{N}}$ such that $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be uniform statistically convergent to $f(x)$, if $\delta(\bigcup_{x\in X} \diamondsuit^-_{f(x)}(f_n(x))) = 0$.*
In this case, it is mathematically indicated as, $f_n \overset{U-s-\lim}{\longrightarrow} f$.

Theorem 6. *Every uniform statistically convergent sequence of function is pointwise statistically convergent.*

Proof. The proof follows directly, hence omitted. $\square$

Example:
The converse implication of Theorem 6 does not necessarily holds. i.e., there exist sequence of functions which is uniformly statistically convergent, but not point-wise statistical convergent.
Let us consider a function $f : (\mathbb{N}, \tau) \longrightarrow (Y, \sigma)$ where $Y = \{0, 1\}$ and $\sigma = \{\emptyset, \{0\}, Y\}$. Define for all $x \in X$, for each $n \in \mathbb{N}$, $f_n : (\mathbb{N}, \tau) \longrightarrow (Y, \sigma)$ defined by,

$$f_n(x) = \begin{cases} 1, & \text{if } x = n^2 \\ 0, & otherwise. \end{cases}$$

Now, the neighborhoods of $f(x) = 0$ are $V_1 = \{0\}$ and $V_2 = Y$. For the neighborhood V_1 of $f(x)$, $\delta(\{n \in \mathbb{N} : f_n(x) \cap V_1 = \emptyset$ for the neighborhood $\{0\}$ of $0\}) = \delta(\{\{x\}\}) = 0$ for all $x \in X$ and for the neighborhood V_2 of $f(x)$, $\delta(\{n \in \mathbb{N} : f_n(x) \cap V_2 = \emptyset$ for the neighborhood Y of $0\}) = \delta(\emptyset) = 0$ for all $x \in X$. Then $f_n(x) \overset{Pt-s-\lim}{\longrightarrow} f$.
But for all $x \in X$, $\delta(\bigcup_{x\in X}\{n \in \mathbb{N} : f_n(x) \cap V = \emptyset$ for every neighborhood V of $f(x)\}) = \delta(\{x : x \in \mathbb{N}\}) = \delta(\mathbb{N}) = 1 \neq 0$. Thus, $f_n(x) \overset{U-s-\lim}{\not\longrightarrow} f$.

Theorem 7. *A sequence of point-wise statistical convergent functions, where co-domain space is Haudörff, possess a unique limit.*

Proof. Let $f_n, f, g : (X,\tau) \to (Y,\sigma)$ be such that (Y,σ) is a Hausdörff space. Let $f_n \overset{Pt-s-\lim}{\longrightarrow} f$ and $f_n \overset{Pt-s-\lim}{\longrightarrow} g$ with $f(a) \neq g(a)$ for at least one $a \in X$. Then there exists open neighborhoods $V_1, V_2 \in \sigma$ such that $f(a) \in V_1$ and $g(a) \in V_2$ with $V_1 \cap V_2 = \emptyset$. So, $V_1 \subseteq V_2^{\complement}$.

Since $f_n \overset{Pt-s-\lim}{\longrightarrow} f$ and $f_n \overset{Pt-s-\lim}{\longrightarrow} g$,

$$\delta(\diamondsuit^{+}_{f(x)}(f_n(x))) = 1 \text{ and } \delta(\diamondsuit^{+}_{g(x)}(f_n(x))) = 1 \text{ for all } x \in X.$$

$$\text{i.e., } \delta(\diamondsuit^{+}_{f(a)}(f_n(a))) = 1 \text{ and } \delta(\diamondsuit^{+}_{g(a)}(f_n(a))) = 1.$$

So,

$$\delta(\{n \in \mathbb{N} : \{f_n(a)\} \cap V \neq \emptyset \text{ for every neighborhood } V \text{ of } f(a)\}) = 1.$$

and

$$\delta(\{n \in \mathbb{N} : \{g_n(a)\} \cap U \neq \emptyset \text{ for every neighborhood } U \text{ of } g(a)\}) = 1.$$

But V_1 is also a neighborhood of $f(a)$. Therefore, $\delta(\{n \in \mathbb{N} : \{f_n(a)\} \cap V_1 \neq \emptyset\}) = 1$. Since $V_1 \subseteq V_2^{\complement}$, $\{n \in \mathbb{N} : \{f_n(a)\} \cap V_2^{\complement} \neq \emptyset\} \subseteq \{n \in \mathbb{N} : \{f_n(a)\} \cap V_1 \neq \emptyset\}$. Therefore, $\delta(\{n \in \mathbb{N} : \{f_n(a)\} \cap V_2^{\complement} \neq \emptyset\}) = 1$. So, $\delta(\{n \in \mathbb{N} : \{f_n(a)\} \cap V_2 = \emptyset\}) = 1$ (Since $\{f(a)\}$ is a singleton set).

$$\text{Thus, } \delta(\{n \in \mathbb{N} : \{f_n(a)\} \cap V_2 \neq \emptyset\}) = 0.$$

But this is a contradiction since V_2 is a neighborhood of $g(a)$ and

$$\delta(\{n \in \mathbb{N} : \{g_n(a)\} \cap U \neq \emptyset \text{ for every neighborhood } U \text{ of } g(a)\}) = 1.$$

So, $f(a) = g(a)$ for all $a \in X$. i.e., $f = g$. Thus the limit of the sequence $\{f_n\}$ is unique. □

Theorem 8. *If $f_n, f : (X,\tau) \longrightarrow (Y,\sigma)$ and $g_n, g : (Y,\sigma) \longrightarrow (Z,\eta)$ are functions such that $g_n \overset{Pt-s-\lim}{\longrightarrow} g$. Then $g_n \circ f_n \overset{Pt-s-\lim}{\longrightarrow} g \circ f$.*

Proof. Since $g_n \overset{Pt-s-\lim}{\longrightarrow} g$ so, $\delta(\diamondsuit^{-}_{g(y)} g_n(y)) = 0$ for all $y \in Y$.
Let $a \in X$ be arbitrary. Therefore, $f(a) \in Y$ that gives $\delta(\diamondsuit^{-}_{g(f(a))} g_n(f_n(a))) = 0$.
As a result $\delta(\diamondsuit^{-}_{g\circ f(a)} g_n \circ f_n(a)) = 0$ for all $a \in X$.
Hence, $g_n \circ f_n \overset{Pt-s-\lim}{\longrightarrow} g \circ f$. □

Theorem 9. *If $f_n, f : (X,\tau) \to (Y,\sigma)$ and $g_n : (X,\tau) \to (Y,\sigma)$ be such that $f_n \overset{Pt-s-\lim}{\longrightarrow} f$ and $f_n = g_n$ for a.a.n. then $g_n \overset{Pt-s-\lim}{\longrightarrow} f$.*

Proof. Since $f_n \overset{Pt-s-\lim}{\longrightarrow} f$, $\delta(\diamondsuit^{-}_{f(x)}(f_n(x))) = 0$ for all $x \in X$ and since $f_n = g_n$ for a.a.n., $\delta(\{n \in \mathbb{N} : f_n(x) \neq g_n(x)\}) = 0$ for all $x \in X$.

Now, $\diamondsuit^{-}_{f(x)}(g_n(x)) \subseteq \diamondsuit^{-}_{f(x)}(f_n(x)) \cup \{n \in \mathbb{N} : f_n(x) \neq g_n(x)\}$ for all $x \in X$.

So, $\delta(\diamondsuit^{-}_{f(x)}(g_n(x))) \leq \delta(\diamondsuit^{-}_{f(x)}(f_n(x))) + \delta(\{n \in \mathbb{N} : f_n(x) \neq g_n(x)\}) = 0+0 = 0$ for all $x \in X$.

Therefore, $\delta(\diamondsuit^{-}_{f(x)}(g_n(x))) = 0$ for all $x \in X$. Hence, $g_n \overset{Pt-s-\lim}{\longrightarrow} f$. □

5 Conclusion

In this work, two new operators $\diamondsuit^{-}_{f(x)}$ and $\diamondsuit^{+}_{f(x)}$ are introduced to examine the local behavior of sequences of functions in the topological spaces. Using these operators, we have developed a new framework for statistical convergence. Important characteristics such as the connection between statistical pointwise and statistical uniform convergence and the statistical pointwise convergent sequences of functions that do not have more than one limit in Hausdörff space are established. Furthermore, a theorem demonstrated that the composition of two statistically pointwise convergent sequences of functions also results in the statistical pointwise convergence of sequences of functions.

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