# WEAK SOLVABILITY OF ONE ALPHA–MODEL FOR VOIGT–TYPE FLUID WITH INFINITE MEMORY

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**Abstract:** In this paper the weak solvability of one Voigt–$\alpha$ model with infinite memory is investigated. The topological approximation method for studying hydrodynamic problems is used to prove the weak solvability of this model. Also the theory of regular Lagrangian flow is used in the study of weak solvability. The existence of a weak solution of the problem is proved in the paper. Also the convergence of solutions of the alpha–model to solutions of the original model as the parameter $\alpha$ tends to zero is established.

**Keywords:** Existence theorem, Voigt–$\alpha$ model, weak solution.

## 1 Introduction

In $Q = (-\infty, T] \times \Omega$, where $T > 0$ and $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega \subset C^2$ the following problem is considered

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} - \mu_0 \Delta v-$$

$$-\frac{\mu_1}{\Gamma(1-\zeta)}\text{Div}\int_{-\infty}^{t}e^{\frac{-(t-s)}{\lambda}}(t-s)^{-\zeta}\mathcal{E}(v)(s,z(s;t,x))ds+\nabla p=f; \quad (1)$$

$$u=(I-\alpha^2\Delta)^{-1}v; \quad (2)$$

$$z(\tau;t,x)=x+\int_t^{\tau}v(s,z(s;t,x))ds, \qquad t,\tau\in(-\infty,T],\ x\in\Omega; \quad (3)$$

$$\text{div }v=0; \quad (4)$$

$$v(t,x)\mid_{(t,x)\in(-\infty,T]\times\partial\Omega}=0. \quad (5)$$

Here $v(t,x)$ and $p(t,x)$ are unknown velocity and pressure of the considered fluid, $u(t,x)$ is the modified vector–valued velocity function as defined in (2), $\mathcal{E}(v)=\{\mathcal{E}_{ij}\}_{i,j=1}^n$ is a strain rate tensor with components $\mathcal{E}_{ij}=\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})$, $\mu_0>0$, $\mu_1\geq 0$, $0<\zeta<1$, $\lambda>0$, $\alpha>0$ are constants and $z(\tau;t,x)$ is a trajectory of the fluid particles. The sign Div denotes the divergence of a matrix, i. e. the vector whose coordinates are the divergences of the matrix column vectors.

The boundary value problem (1)–(3) for $\alpha=0$ describes the motion of Voigt–type fluids (linearly elastically retarded fluids) with memory. The study of such mathematical models is devoted papers, for example, [1, 2, 3, 4, 5, 6]. In this paper we study the alpha–model for corresponding problem. Alpha models are a kind of regularized approximate systems that depend on some positive parameter $\alpha$. The regularization is accomplished by some filtering of the velocity vector, which appears in the argument of the nonlinear term (see [7]).

Interest in studying alpha models is primarily related to their application to the study of turbulence effects in fluid flows, as well as their numerical results. It should be noted that one of the defining characteristics of turbulent fluid flow is the wide range of spatial and temporal scales. This characteristic property is a source of difficulties in both theoretical research and practical calculations. Moreover, in many practical applications, physically significant flow characteristics are often concentrated on large spatial scales, for example, in numerical hydrodynamic weather forecasting. Therefore, considerable effort has been devoted to modeling the large–scale dynamics of turbulent flow by filtering out smaller scales.

Typically, such filtering occurs by applying the inverse Helmholtz operator to the first or second argument of the bilinear operator of the system of equations of fluid motion (or to the entire operator). The alpha parameter has the dimensions of the square of the length and determines the scale at which high–frequency (in space) modes will be filtered out. The corresponding regularized systems are commonly called alpha models.

In theoretical studies, the idea of using this kind of approximations first appeared in the work of J. Leray [8] (in this work, J. Leray used the general form of the filtration kernel) to prove the existence of a weak solution to the Navier–Stokes equations. Later, various alpha models were constructed based on this idea for the Euler [9, 10], Navier–Stokes [11, 12], Leray [13, 14],

Jeffreys–Oldroyd [15], fractional Voigt [16, 17] equations, and others. This paper continues the study of alpha models and considers the solvability of the boundary value problem (1)–(5) for the alpha–Voigt–type fluid motion.

## 2 Preliminaries and statement of main result

Let us consider the following functional spaces. Let $L_p(\Omega)$, $1 \le p < \infty$ is set of measurable vector functions $v : \Omega \to \mathbb{R}^n$, summable with $p$–th degree. Let $W_p^m(\Omega)$, $m \ge 1$, $p \ge 1$, are Sobolev spaces. Consider the space $C_0^\infty(\Omega)$ of infinitely differentiable vector functions from $\Omega$ to $\mathbb{R}^n$ with compact support in $\Omega$. Denote by $\mathcal{V}$ the set $\{v \in C_0^\infty(\Omega), \operatorname{div} v = 0\}$. We denote by $V^0$ the closure of $\mathcal{V}$ with respect to the norm of $L_2(\Omega)$, by $V^1$ the closure of $\mathcal{V}$ with respect to the norm of $W_2^1(\Omega)$, and by $V^2$ the space $W_2^2(\Omega) \cap V^1$.

Let us introduce a scale of spaces $V^\beta$, $\beta \in \mathbb{R}$ (see [18]). Consider the Lere projector $P : L_2(\Omega) \to V^0$ and the operator $A = -P\Delta$ defined on $D(A) = V^2$. This operator can be continued in $V^0$ to a closed operator, which is a self–adjoint positive operator with a compact inverse. Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \ldots$ are eigenvalues of the operator $A$. By virtue of Hilbert's theorem on the spectral decomposition of compact operators the eigenfunctions $\{e_j\}$ of the operator $A$ form an orthonormal basis in $V^0$. Denote by $E_\infty = \{v = \sum_{j=1}^N v_j e_j : v_j \in \mathbb{R}, N \in \mathbb{N}\}$, the set of finite linear combinations composed of $e_j$ and define the space $V^\beta$, $\beta \in \mathbb{R}$, as the closure of $E_\infty$ with respect to the norm

$$\|v\|_{V^\beta} = (\sum_{k=1}^\infty \lambda_k^\beta |v_k|^2)^{\frac{1}{2}}, \quad \text{where } v = \sum_{k=1}^\infty v_k e_k. \tag{6}$$

By $V^{-\beta} = (V^\beta)^{-1}$, $\beta \in N$, we denote the conjugate space to $V^\beta$. On the space $V^\beta$, $\beta \in \mathbb{N}$, the norm (6) is equivalent to the usual norm $\|\cdot\|_{W_2^\beta(\Omega)}$ of the space $W_2^\beta(\Omega)$. Moreover, norms in spaces $V^1$ and $V^3$ can be defined as follows:

$$\|v\|_{V^1} = (\int_\Omega \nabla v(x) : \nabla v(x) \, dx)^{\frac{1}{2}}, \quad \|v\|_{V^3} = (\int_\Omega \Delta \nabla v(x) : \Delta \nabla v(x) \, dx)^{\frac{1}{2}}.$$

Here the symbol " : " denotes the componentwise matrix product.

Let us introduce the space in which the solvability of the problem under study will be proved:

$$W_1 = \{v \in L_2(-\infty, T; V^1) \cap L_\infty(-\infty, T; V^0), v' \in L_{4/3,loc}(-\infty, T; V^{-1})\}$$

with the norm

$$\|v\|_{W_1} = \|v\|_{L_2(-\infty,T;V^1)} + \|v\|_{L_\infty(-\infty,T;V^0)} + \|v'\|_{L_{4/3,loc}(-\infty,T;V^{-1})}.$$

Here $L_{4/3,loc}(-\infty, T; V^{-1})$ is the space consisting of functions $v$, defined almost everywhere on $(-\infty, T]$ and taking the value to $V^{-1}$, whose restriction to any segment $[r, T] \in (-\infty, T]$ belongs to $L_{4/3}(r, T; V^{-1})$.

Let us denote by $\Delta_\alpha : V^\beta \to V^{\beta-2}$, $\beta \geq 0$ the operator $\Delta_\alpha = (J + \alpha^2 A)$, where $J = PI$, $I$ is the identity operator. The operator $\Delta_\alpha$ is invertible (see [19]). Applying the Leray projector $P$ to both sides of the equality $v = (I - \alpha^2 \Delta)u$ for $\beta = 3$ and expressing from the last equality $u$: $u = (J + \alpha^2 A)^{-1} v = \Delta_\alpha^{-1} v$. Since $v(t) \in V^1$ we obtain that $u(t) \in V^3$ for almost all $t \in (-\infty, T]$.

Note that for the boundary value problem to be well posed, it is necessary that the trajectories $z$ be uniquely determined by the velocity field $v$. In other words, the equation (3) must have a unique solution for the given velocity field $v$. However, the existence of solutions of (3) for a fixed $v$ is known only when $v \in L_1(0, T; C(\Omega))$, and this solution is unique if $v \in L_1(0, T; C^1(\Omega))$ and $v|_{(0,T) \times \partial\Omega} = 0$ (see, for example, [20]). Hence the trajectories of motion are not uniquely determined even for strong solutions whose partial derivatives occurring in (3) belong to $L_2(0, T; L_2(\Omega))$. The solvability of the integral Cauchy problem (3) has been studied comparatively recently (see, for example, [21, 22, 23, 24]) in the case when $v$ belongs to the Sobolev space, and the existence, uniqueness and stability of regular Lagrangian flows have been established.

**Definition 1.** *The Regular Lagrangian Flow (RLF) associated to $v$ is called the function $z(\tau; t, x)$, $(\tau; t, x) \in [0, T] \times [0, T] \times \overline{\Omega}$ satisfying the following conditions:*

*1. for a.e. $x$ and any $t \in [0, T]$ the function $z(\tau; t, x)$ is absolutely continuous and satisfies the equation (3);*

*2. $m(z(\tau; t, B)) = m(B)$ for any $t, \tau \in [0, T]$ and every Borel set $B \subset \overline{\Omega}$, here $m$ is the Lebesgue measure in $\mathbb{R}^n$;*

*3. for $t_i \in [0, T], i = \overline{1, 3}$, and a.e. $x \in \overline{\Omega}$ the following relation is valid:*

$$z(t_3; t_1, x) = z(t_3; t_2, z(t_2; t_1, x)).$$

For RLF the following results were obtained

**Theorem 1.** *Let $v \in L_1(0, T; W_p^1(\Omega)), 1 \leq p \leq +\infty$, $\operatorname{div} v(t, x) = 0$ and $v|_{[0,T] \times \partial\Omega} = 0$. Then there exists a unique RLF $z \in C(D; L)$, associated to $v$,*

$$z(\tau; t, \overline{\Omega}) \subset \overline{\Omega}, \qquad \frac{\partial}{\partial\tau} z(\tau; t, x) = v(\tau, z(\tau; t, x)), \qquad \tau \in [0, T], \quad x \in \Omega,$$

*here $C(D, L)$ — Banach space of continuous functions on $D = [0, T] \times [0, T]$ with values in $L$, where $L$ is metric space of vector functions measurable on $\Omega$.*

**Theorem 2.** *Let $v, v^m \in L_1(0, T; W_1^p(\Omega)), m = 1, 2, ...,$ for some $p > 1$. Let $\operatorname{div} v(t, x) = 0, \operatorname{div} v^m(t, x) = 0, v|_{[0,T] \times \partial\Omega} = v^m|_{[0,T] \times \partial\Omega} = 0$. Let the following inequalities are hold*

$$\|v_x\|_{L_1(0,T;L_p(\Omega))} + \|v\|_{L_1(0,T;L_1(\Omega))} \leq C_1,$$
$$\|v_x^m\|_{L_1(0,T;L_p(\Omega))} + \|v^m\|_{L_1(0,T;L_1(\Omega))} \leq C_2.$$

*Also let the sequence $v^m$ converges to $v$ in $L_1(Q_T)$ as $m \to +\infty$. And let $z(\tau; t, x)$ and $z^m(\tau; t, x)$ be RLF associated to $v$ and $v^m$ respectively. Then the sequence of $z^m$ converges to $z$ in Lebesgue measure in $[0, T] \times \Omega$ for $t \in [0, T]$.*

Here $v_x$ is the Jacobian matrix of a vector–function $v$.

Thus, we are ready to formulate the definition of a weak solution to the boundary value problem under study.

**Definition 2.** *Let $f \in L_2(-\infty, T; V^{-1})$. A weak solution of the problem (1)–(5) is a function $v \in W_1$, satisfying for any $\varphi \in V^1$ and a.e. $t \in (-\infty, T]$ the identity*

$$\langle v', \varphi \rangle - \int_\Omega \sum_{i=1}^n (\Delta_\alpha^{-1} v)_i v_j \frac{\partial \varphi}{\partial x_i} dx + \mu_0 \int_\Omega \nabla v : \nabla \varphi \, dx +$$

$$+ \frac{\mu_1}{\Gamma(1-\zeta)} \int_\Omega \int_{-\infty}^t e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v)(s, z(s; t, x)) \, ds \, \mathcal{E}(\varphi) \, dx =$$

$$\langle f, \varphi \rangle, \tag{7}$$

*here $z$ is the RLF associated to $v$.*

Main results of the paper are following theorems.

**Theorem 3.** *Let $f \in L_2(-\infty, T; V^{-1})$. Then the problem (1)–(5) has at least one weak solution $v \in W_1$.*

**Theorem 4.** *Let $f \in L_2(-\infty, T; V^{-1})$. Consider the family (1)–(5) of alpha–models depending on the parameter $\alpha_m$. Then there is a sequence $v_m$ of solutions of the family (1)–(5) converging to the weak solution $v \in W_1$ of the origian initial–boundary value problem as $\alpha_m$ tends to zero.*

The proof of this results consists of several parts. First, we prove the existence of weak solutions of the alpha–model on the basis of the topological approximation approach to the study of the mathematical problems of hydrodynamics (see [19, 25]). To do this, we introduce a family of auxiliary problems depending on a small parameter, prove a priori estimates for the solutions and use the topological degree theory for condensing vector fields to prove the existence of weak solutions of the auxiliary problem. Furthermore, to prove the solvability of the alpha–model, we pass to the limit using appropriate estimates. In conclusion we show that the sequence of solutions of the alpha–model converges to a solution of the original model that describes the motion of a viscoelastic fluid with memory.

## 3    Approximative problem

Consider the following auxiliary family ($0 \le \xi \le 1$):

$$\frac{\partial v^m}{\partial t} + \xi \sum_{i,j=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i \frac{\partial v^m}{\partial x_i}}{(1 + \frac{|v^m|^2}{\varepsilon})} - \mu_0 \Delta v^m -$$

$$-\frac{\xi\mu_1}{\Gamma(1-\zeta)}\text{Div}\int_{-m}^{t}e^{\frac{-(t-s)}{\lambda}}(t-s)^{-\zeta}\mathcal{E}(v^m)(s,z^m(s;t,x))ds+$$

$$+\nabla p = \xi f^m; \tag{8}$$

$$\text{div } v^m(t,x) = 0, \qquad t \in [-m,T], \quad x \in \Omega; \tag{9}$$

$$z^m(\tau;t,x) = x + \int_{t}^{\tau}\tilde{v}^m(s,z^m(s;t,x))ds, \qquad t,\tau \in [-m,T], \quad x \in \Omega; \tag{10}$$

$$v^m(t,x)\mid_{(t,x)\in[-m,T]\times\partial\Omega}= 0, \qquad v^m(-m,x) = 0, \qquad x \in \Omega, \tag{11}$$

here $f^m$ is the restriction of $f$ on $[-m,T] \times \Omega$, $m = 1,2,...$ and $\varepsilon > 0$.

The solvability of this family will be proven in the following function space

$$W_2 = \{v^m \in L_2(-m,T;V^1), (v^m)' \in L_2(-m,T;V^{-1})\}.$$

The regularization of the velocity field in (10) is caused by the fact that the study of Cauchy problem (10) for $v^m \in W_2$ runs into difficulty because in this case the velocity field, generally speaking, does not determine the trajectory of fluid particles. One possible way to avoid this situation is regularization $\tilde{v}^m = S_{1/\varepsilon}v^m$ of the velocity field $v^m \in W_2$, using the regularization operator (see [25]): $S_{1/\varepsilon} : V^0 \to C^1(\overline{\Omega})$.

Consider the construction and properties of regularization operator. Let $\rho$ be a function of the class $C^\infty$ with a compact support in the ball of radius 1 centered at the origin, such that $\rho \geq 0$, $\int_{\mathbb{R}^n}\rho(x)dx = \int_{B_1(0)}\rho(x)dx = 1$. Denote by $\rho_\delta$ the function $\frac{2^n}{\delta^n}\rho(\frac{2x}{\delta})$. As $\delta \to 0$ the functions $\rho_\delta$ converge in the sense of distributions to Dirac function and $\rho_\delta * v \to v$ in $L_q(\mathbb{R}^n)$ for any function $v \in L_q(\mathbb{R}^n)$, where " $*$ " is the convolution of functions. In this work we use $\delta = \frac{1}{\varepsilon}$. Namely, $S_{1/\varepsilon} : V^0 \to V^0$ is continuous. Since the construction of $S_{1/\varepsilon}$ does not depend on $t$. It is easy to see that the maps $S_{1/\varepsilon} : V^0 \to C^1(\overline{\Omega})$ are correctly defined and continuous (see [25], section 7.7).

**Lemma 1.** *The following properties of the regularization operator $S_{1/\varepsilon}$ hold:*

(1) *The following inequalities hold:*

$$\|S_{1/\varepsilon}\|_{V^0\to V^0} \leq C_3, \qquad \|S_{1/\varepsilon}\|_{V^0\to C^1(\bar\Omega)} \leq C_3. \tag{12}$$

(2) *For any $v \in V^0$ as $m \to +\infty$ the following convergence is satisfied*

$$\|S_{1/\varepsilon}v^m - v^m\|_{L_2(-\infty,T;L_2(\Omega))} \to 0. \tag{13}$$

Let formulate the definition of a weak solution to the initial–boundary value problem (8)–(11).

**Definition 3.** *Let $f^m \in L_2(-m,T;V^{-1})$. A weak solution to the problem (8)–(11) is a function $v^m \in W_2$, satisfying for any $\varphi \in V^1$ and a.e. $t \in$*

$(-m, T)$ *identity:*

$$\langle (v^m)', \varphi \rangle - \xi \int_\Omega \sum_{i=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \frac{\partial \varphi_j}{\partial x_i} dx + \mu_0 \int_\Omega \nabla v^m : \nabla \varphi dx +$$

$$+ \frac{\xi \mu_1}{\Gamma(1-\zeta)} \int_\Omega \int_{-m}^t e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v^m)(s, z^m(s; t, x)) ds \, \mathcal{E}(\varphi) dx =$$

$$= \xi \langle f^m, \varphi \rangle, \qquad (14)$$

*and initial condition $v^m(-m, \cdot) = 0$. Here $z^m$ is the RLF associated to $v^m$.*

Let us move on to the operator interpretation of the problem (8)–(11):

$$J : V^1 \to V^{-1}, \quad \langle J v^m, \varphi \rangle = \int_\Omega v^m \varphi dx, \quad v \in V^1, \quad \varphi \in V^1;$$

$$A : V^1 \to V^{-1}, \quad \langle A v^m, \varphi \rangle = \int_\Omega \nabla v^m : \nabla \varphi dx, \quad v^m \in V^1, \quad \varphi \in V^1;$$

$$B : V^1 \times [-m, T] \times [-m, T] \times \overline{\Omega} \to V^{-1},$$

$$(B(v^m, z^m), \varphi) = (\int_{-m}^t e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v^m)(s, z^m(s; t, x)) ds, \mathcal{E}(\varphi)),$$

$$v^m \in V^1, \quad z^m \in [-m, T] \times [-m, T] \times \overline{\Omega}, \quad \varphi \in V^1, \quad t \in (-m, T);$$

$$K : L_4(\Omega) \to V^{-1}, \quad \langle K(v^m), \varphi \rangle = \int_\Omega \sum_{i,j=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \frac{\partial \varphi_j}{\partial x_i} dx,$$

$$v^m \in L_4(\Omega), \varphi \in V^1.$$

We get the operator equation:

$$J(v^m)' + \frac{\mu_1 \xi}{\Gamma(1-\zeta)} B(v^m, z^m) - \xi K(v^m) + \mu_0 A v^m = \xi f^m, \qquad (15)$$

satisfying the initial condition $v^m(-m, \cdot) = 0$.

We also define operators using the following equalities:

$$L : W_2 \to L_2(-m, T; V^{-1}) \times V^1, \quad L(v^m) = ((J(v^m)' + \mu_0 A v^m, v^m|_{t=-m});$$

$$C : W_2 \to L_2(-m, T; V^{-1}) \times V^1, \quad C(v^m) = (K(v^m), 0).$$

$$G : W_2 \to L_2(-m, T; V^{-1}) \times V^1, \quad G(v^m) = (\frac{\mu_1}{\Gamma(1-\zeta)} B(v^m, z^m), 0).$$

Then the problem of finding a solution to the operator equation (15) for a fixed $0 \leqslant \xi \leqslant 1$, satisfying the initial condition $v^m(-m, \cdot) = 0$, is equivalent the problem of finding a solution for a fixed $0 \leqslant \xi \leqslant 1$ operator equation

$$L(v^m) = \xi(C(v^m) - G(v^m) + (f^m, 0)).$$

To study this operator equality we first study the properties of the incoming operators.

# 4 Operator properties

**Lemma 2.** *1) For any $v^m \in L_2(-m, T; V^1)$ the function $Av^m$ belongs to $L_2(-m, T; V^{-1})$, the operator $A : L_2(-m, T; V^1) \rightarrow L_2(-m, T; V^{-1})$ is continuous and following estimates hold:*

$$\|Av^m\|_{V^{-1}} \leqslant \|v^m\|_{V^1}; \qquad \|Av^m\|_{L_2(-m,T;V^{-1})} \leqslant \|v^m\|_{L_2(-m,T;V^1)}.$$

*2) For any $v^m \in V^1$ the operator $(\mu_0 A + J) : V^1 \rightarrow V^{-1}$ is linear, continuous and invertible, and the estimate is true for it:*

$$\mu_0 \|v^m\|_{V^1} \leqslant \|(\mu_0 A + J)v^m\|_{V^{-1}} \leqslant C_4 \|v^m\|_{V^1}.$$

*The inverse operator $(\mu_0 A + J)^{-1} : V^{-1} \rightarrow V^1$ is continuous and following estimate holds:*

$$\|(\mu_0 A + J)^{-1} f^m\|_{V^1} \leqslant \frac{1}{\mu_0} \|f^m\|_{V^{-1}}.$$

*3) For any $v^m \in L_p(-m, T; V^1), 1 \leqslant p < \infty$ the function $(\mu_0 A + J)v^m \in L_p(-m, T; V^{-1})$ and operator $(\mu_0 A + J) : L_p(-m, T; V^1) \rightarrow L_p(-m, T; V^{-1})$ is continuous and invertible and following estimates hold:*

$$\mu_0 \|v^m\|_{L_p(-m,T;V^1)} \leqslant \|(\mu_0 A + J)v^m\|_{L_p(-m,T;V^{-1})} \leqslant C_5 \|v^m\|_{L_p(-m,T;V^1)}.$$

*The inverse operator $(\mu_0 A + J)^{-1} : L_p(-m, T; V^{-1}) \rightarrow L_p(-m, T; V^1)$ is continuous and for any $\omega \in L_p(-m, T; V^1)$ we have the estimate:*

$$\|(\mu_0 A + J)^{-1} \omega\|_{L_p(-m,T;V^1)} \leqslant \frac{1}{\mu_0} \|\omega\|_{L_p(-m,T;V^{-1})}.$$

The proof of this Lemma is carried out in a standard way (see, for example, [25], Section 7.2.)

**Lemma 3.** *1) The mapping $K : L_4(\Omega) \rightarrow V^{-1}$ is continuous and the following estimate hold:*

$$\|K(v^m)\|_{V^{-1}} \leqslant C_6 \|v^m\|_{L_4(\Omega)}^2. \tag{16}$$

*2) For any $v^m \in L_4(-m, T; L_4(\Omega))$ the function $K(v^m) \in L_2(-m, T; V^{-1})$ and the mapping $K : L_4(-m, T; L_4(\Omega)) \rightarrow L_2(-m, T; V^{-1})$ is continuous.*

*3) For any function $v^m \in W_2$ the function $K(v^m) \in L_2(-m, T; V^{-1})$ and the mapping $K : W_2 \rightarrow L_2(-m, T; V^{-1})$ is completely continuous.*

*Proof.* 1) For any $v^m \in L_4(\Omega), \varphi \in V^1, \varepsilon \geq 0$, using Schwarz inequality, get

$$|\langle K(v^m), \varphi \rangle| = \left| \sum_{i,j=1}^{n} \int_{\Omega} \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \frac{\partial \varphi_j}{\partial x_i} \, dx \right| \leqslant$$

$$\leqslant \sum_{i,j=1}^{n} \left( \int_{\Omega} |((I - \alpha^2 \Delta)^{-1} v^m)_i v_j^m|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \frac{\partial \varphi_j}{\partial x_i} \right|^2 \, dx \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \sum_{i,j=1}^{n} \left( \int_{\Omega} |((I - \alpha^2 \Delta)^{-1} v^m)_i|^4 \, dx \right)^{\frac{1}{4}} \left( \int_{\Omega} |v_j^m|^4 \, dx \right)^{\frac{1}{4}} \|\varphi\|_{V^1} \leqslant$$

$$\leqslant C_7 \|(I - \alpha^2 \Delta)^{-1} v^m\|_{L_4(\Omega)} \|v^m\|_{L_4(\Omega)} \|\varphi\|_{V^1} \leqslant$$

$$\leqslant C_7 C_8 \|v^m\|_{L_4(\Omega)}^2 \|\varphi\|_{V^1} = C_6 \|v^m\|_{L_4(\Omega)}^2 \|\varphi\|_{V^1}.$$

We got inequality (16). Note that here we used the following well–known estimate (см. [26], [27]):

$$\|\Delta_\alpha^{-1} v^m\|_{L_p(\Omega)} = \|(I - \alpha^2 \Delta)^{-1} v^m\|_{L_p(\Omega)} \leqslant C_9 \|v^m\|_{L_p(\Omega)}, \quad p > 1. \quad (17)$$

To prove the continuity, it suffices to prove the continuity of the mappings $\psi_{ij} : L_4(\Omega) \to L_2(\Omega), \psi_{ij}(v^m) = \dfrac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})}, i, j = 1, 2, ..., n$. The continuity property of this mapping follows from the M.A. Krasnoselsky Theorem on the continuity of the superposition operator and from the estimate (17) for any $v^m$.

2) To prove it, we use the last estimate and repeat the proof from [19].

3) To prove this point, we use the Aubin–Simon Theorem:

**Theorem 5.** *(see [19]) Let $X \subset E \subset Y$ are Banach spaces, the embedding $X \subset E$ is compact and the embedding $E \subset Y$ is continuous. Let $F \subset L_p(0, T; X)$, $1 \leqslant p \leqslant \infty$. We will assume that for any $f \in F$ its generalized derivative belongs to $L_r(0, T; Y)$, $1 \leqslant r \leqslant \infty$. Next let 1) the set $F$ bounded in $L_p(0, T; X)$, 2) the set $\{f' : f \in F\}$ bounded in $L_r(0, T; Y)$. Then for $p < \infty$ the set $F$ is relatively compact in $L_p(0, T; E)$, and for $p = \infty$ and $r > 1$ the set $F$ relatively compact in $C([0, T]; E)$.*

Consider the set

$$F = \{v^m \in L_4(-m, T; V^1), (v^m)' \in L_2(-m, T; L_2(\Omega))\}.$$

Since the embedding $V^1 \subset L_4(\Omega)$ is compact, the embedding $F \subset L_4(-m, T; L_4(\Omega))$ is compact too. From continuous of embeddings $C([-m, T]; V^1) \subset L_4(-m, T; V^1)$, $L_2(-m, T; V^1) \subset L_2(-m, T; L_2(\Omega))$ follows a continuous embedding $W_2 \subset F$. In addition, from the second point of this Lemma we have that the operator $K : L_4(-m, T; L_4(\Omega)) \to L_2(-m, T; V^{-1})$ is continuous. Thus, we have the following superposition of embeddings:

$$W_2 \subset F \subset L_4(-m, T; L_4(\Omega)) \xrightarrow{K} L_2(-m, T; V^{-1}),$$

where the first embedding is continuous, the second is compact and the mapping $K$ is continuous. Therefore, for any function $v^m \in W_2$ we obtain that the function $K(v^m) \in L_2(-m, T; V^{-1})$, and the mapping $K : W_2 \to L_2(-m, T; V^{-1})$ is compact. $\square$

**Lemma 4.** *For any $v^m \in L_2(-m, T; V^1)$, $z^m \in [-m, T] \times [-m, T] \times \bar{\Omega}$ we have $B(v^m, z^m) \in L_2(-m, T; V^{-1})$ and mapping $B : L_2(-m, T; V^1) \times [-m, T] \times [-m, T] \times \bar{\Omega} \to L_2(-m, T; V^{-1})$ is continuous and bounded.*

*Moreover, for any fixed $z^m \in [-m, T] \times [-m, T] \times \bar{\Omega}$ and for any $u$, $v^m \in L_2(-m, T; V^1)$ estimate holds*

$$\|B(v^m, z^m) - B(u, z^m)\|_{k, L_2(-m, T; V^{-1})}$$

$$\leqslant C_{10} T^{\frac{1}{2} - \zeta} \sqrt{\frac{\lambda(T + m)}{2 + 2k\lambda}} \|v^m - u\|_{k, L_2(-m, T; V^1)}. \tag{18}$$

The first part of this Lemma is proved similarly [28] (Lemma 2.2), the second part was proven in [2] (Lemma 3).

Let us define several notions concerning the measure of non–compactness and $L$ — condensing operators (see [29], [30]).

**Definition 4.** *A non–negative real function $\psi$ defined on a subset of a Banach space $F$ is called a measure of non–compactness if for any subset $\mathcal{M}$ of this space the following properties hold: 1) $\psi(\overline{co}\,\mathcal{M}) = \psi(\mathcal{M})$; 2) for any two sets $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \subset \mathcal{M}_2$ it follows that $\psi(\mathcal{M}_1) \leq \psi(\mathcal{M}_2)$.*

Here $\overline{co}\,\mathcal{M}$ denotes the convex closure of the set $\mathcal{M}$. As an example of a measure of non-compactness, we present the measure of non–compactness of Kuratowski: the exact lower bound $d > 0$ for which the set $\mathcal{M}$ allows splitting into a finite number of subsets whose diameters are less than $d$. The measure of non–compactness of Kuratowski has several important properties: 3) $\psi(\mathcal{M}) = 0$, if $\mathcal{M}$ is a relatively compact subset; 4) $\psi(\mathcal{M} \cup K) = \psi(\mathcal{M})$, if $K$ is a relatively compact set.

**Definition 5.** *Let $X$ is bounded subset of a Banach space and $L : X \to F$ is mapping from $X$ to a Banach space $F$. A mapping $g : X \to F$ is called $L$–condensing if $\psi(g(\mathcal{M})) < \psi(L(\mathcal{M}))$ for any set $\mathcal{M} \subseteq X$ such that $\psi(g(\mathcal{M})) \neq 0$.*

In what follows, we will use the $\gamma_k$–Kuratowski non–compactness measure in the space $L_2(-m, T; V^{-1})$ with the norm $\|v\|_{k, L_2(-m, T; V^{-1})}$ given by the integral $(\int_{-m}^{T} \|v^m\|_{V^{-1}}^2 e^{-kt} dt)^{\frac{1}{2}}$. Then, the following Lemma holds.

**Lemma 5.** *The mapping $G : W_2 \to L_2(-m, T; V^{-1}) \times V^1$ is $L$ — condensing with respect to the measure of non-compactness of the Kuratowski $\gamma_k$.*

The proof of this Lemma is carried out similarly to [2] (Lemma 4), using estimate (18).

**Lemma 6.** *The operator $L : W_2 \to L_2(-m, T; V^{-1}) \times V^1$ is invertible and the inverse $L^{-1} : L_2(-m, T; V^{-1}) \times V^1 \to W_2$ is a continuous operator.*

The proof of this Lemma is carried out similarly to [25] (Lemma 7.7.6).

## 5   A priori estimates

To prove the solvability of an operator equality (15), it is necessary to obtain a priori estimates.

**Lemma 7.** *Let $f^m \in L_2(-m, T; V^{-1})$. Then for any solution $v \in W_2$ of the operator equation* (15) *the estimates hold:*

$$\|v^m\|_{L_2(-m,T;V^1)} \leqslant C_{11}\|f^m\|_{L_2(-m,T;V^{-1})}; \tag{19}$$

$$\|v^m\|_{C([-m,T];V^0)} \leqslant C_{12}\|f^m\|_{L_2(-m,T;V^{-1})}, \tag{20}$$

*where $C_{11}$, $C_{12}$ do not depend on $\varepsilon$ and $m$.*

*Proof.* Let $v^m \in W_2$ is solution of the operator equation (15). Then for any $\varphi \in V^1$ and a.e. $t \in (-m, T)$ there is an equality (14). Since it is valid for all $\varphi \in V^1$, we take $\varphi = \overline{v^m}$, where $\overline{v^m}(t) = e^{-kt}v^m$. Then

$$\int_\Omega (v^m)' \overline{v^m}\, dx - \xi \int_\Omega \sum_{i,j=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \frac{\partial \overline{v_j^m}}{\partial x_i}\, dx + \mu_0 \int_\Omega \nabla(v^m) : \nabla(\overline{v^m}) dx +$$

$$+ \frac{\xi \mu_1}{\Gamma(1-\zeta)} \int_{-m}^t e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} (\mathcal{E}(v^m)(s, z^m(s;t,x)) ds, \mathcal{E}(\overline{v^m})) =$$

$$= \xi \langle f^m, \overline{v^m} \rangle.$$

Let us change the variable $v^m = e^{kt}\overline{v^m}$ and separately transform the terms on the left side:

$$\int_\Omega (v^m)' \overline{v^m}\, dx = \int_\Omega (e^{kt}\overline{v^m})' \overline{v^m}\, dx = e^{kt} \int_\Omega \overline{v^m}' \overline{v^m}\, dx + k e^{kt} \int_\Omega \overline{v^m}\, \overline{v^m}\, dx =$$

$$= \frac{e^{kt}}{2} \int_\Omega \frac{\partial(\overline{v^m}\, \overline{v^m})}{\partial t}\, dx + k e^{kt} \|\overline{v^m}\|_{V^0}^2 = \frac{e^{kt}}{2} \frac{d}{dt} \|\overline{v^m}\|_{V^0}^2 + k e^{kt} \|\overline{v^m}\|_{V^0}^2;$$

$$\int_\Omega \sum_{i,j=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \frac{\partial \overline{v_j^m}}{\partial x_i}\, dx = e^{kt} \int_\Omega \sum_{i,j=1}^n \frac{(\Delta_\alpha^{-1} v^m)_i}{(1 + \frac{|v_j^m|^2}{\varepsilon})} \overline{v_j^m} \frac{\partial \overline{v_j^m}}{\partial x_i}\, dx =$$

$$= \frac{e^{kt}}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial(\Delta_\alpha^{-1} v^m)_i}{\partial x_i} \frac{\overline{v_j^m}^2}{(1 + \frac{|v_j^m|^2}{\varepsilon})}\, dx;$$

$$e^{kt} \mu_0 \int_\Omega \nabla(\overline{v^m}) : \nabla(\overline{v^m}) dx = e^{kt} \mu_0 \|\overline{v^m}\|_{V^1}^2.$$

As a result, we get:

$$\frac{e^{kt}}{2} \frac{d}{dt} \|\overline{v^m}\|_{V^0}^2 + k e^{kt} \|\overline{v^m}\|_{V^0}^2 + e^{kt} \mu_0 \|\overline{v^m}\|_{V^1}^2 =$$

$$= -\frac{\xi \mu_1}{\Gamma(1-\zeta)} \int_{-m}^t e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} (\mathcal{E}(\overline{v^m})(s, z^m(s;t,x)) ds, \mathcal{E}(\overline{v^m})) =$$

$$= \xi e^{kt} \langle f^m, \overline{v^m} \rangle.$$

Let us estimate the right side of the obtained equality. Using the Cauchy inequality $bc \leqslant \frac{\delta b^2}{2} + \frac{c^2}{2\delta}$ for $\delta = 1/\mu_0$, we get:

$$\xi e^{kt} \langle f^m, \overline{v^m} \rangle \leqslant e^{kt} \|f^m\|_{V^{-1}} \|\overline{v^m}\|_{V^1} \leqslant \frac{e^{kt}}{2\mu_0} \|f^m\|_{V^{-1}}^2 + \frac{\mu_0 e^{kt}}{2} \|\overline{v^m}\|_{V^1}^2.$$

Multiplying both sides of the equality by $e^{-kt}$, for almost all $t \in (-m, T)$ we have

$$\frac{1}{2}\frac{d}{dt}\|\overline{v^m}(t)\|^2_{V^0} + k\|\overline{v^m}(t)\|^2_{V^0} + \frac{\mu_0}{2}\|\overline{v^m}(t)\|^2_{V^1} \leqslant$$

$$\leqslant -\frac{\mu_1}{\Gamma(1-\zeta)}|(e^{-kt}\int_{-m}^{t} e^{\frac{-(t-s)}{\zeta}}(t-s)^{-\alpha}(\mathcal{E}(e^{-kt}\overline{v^m})(s, z^m(s;t,x))ds, \mathcal{E}(\overline{v^m})))|+$$

$$+\frac{1}{2\mu_0}\|f^m\|^2_{V^{-1}}.$$

Let us integrate the last inequality over $t$ from $-m$ to $\tau$, where $\tau \in [-m, T]$. Then

$$\frac{1}{2}\|\overline{v^m}(t)\|^2_{V^0} + k\int_0^\tau \|\overline{v^m}(t)\|^2_{V^0}\, dt+$$

$$+\frac{\mu_0}{2}\int_0^\tau \|\overline{v^m}(t)\|^2_{V^1}\, dt \leqslant \frac{1}{2}\|v^m(-m)\|^2_{V^0} + \frac{1}{2\mu_0}\int_0^\tau \|f^m(t)\|^2_{V^{-1}}\, dt+$$

$$+\frac{\mu_1}{\Gamma(1-\zeta)}\int_{-m}^\tau |(e^{-kt}\int_{-m}^{t} e^{\frac{-(t-s)}{\lambda}}(t-s)^{-\zeta}(\mathcal{E}(e^{-kt}\overline{v^m})(s, z^m(s;t,x))ds, \mathcal{E}(\overline{v^m})))|dt.$$

Using estimate (18) for $u = 0$, we get:

$$\frac{1}{2}\|\overline{v^m}(t)\|^2_{V^0} + k\int_0^\tau \|\overline{v^m}(t)\|^2_{V^0}\, dt + \frac{\mu_0}{2}\int_0^\tau \|\overline{v^m}(t)\|^2_{V^1}\, dt \leqslant \frac{1}{2}\|v^m(-m)\|^2_{V^0}+$$

$$+\frac{\mu_1 C_{10} T^{\frac{1}{2}-\zeta}\sqrt{\frac{\lambda(T+m)}{2+2k\lambda}}}{\Gamma(1-\zeta)}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)} + \frac{1}{2\mu_0}\|f^m\|^2_{L_2(-m,T;V^{-1})}.$$

Take $k$ large enough that $\dfrac{\mu_1 C_{10} T^{\frac{1}{2}-\zeta}\sqrt{\frac{\lambda(T+m)}{2+2k\lambda}}}{\Gamma(1-\zeta)} \leqslant \mu_0/4$. Let us estimate each summand of the left–hand side:

$$\frac{\mu_0}{2}\int_0^\tau \|\overline{v^m}(t)\|^2_{V^1}\, dt \leqslant \frac{1}{2}\|v^m(-m)\|^2_{V^0}+$$

$$+\frac{1}{2\mu_0}\|f^m\|^2_{L_2(-m,T;V^{-1})} + \frac{\mu_0}{4}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)},$$

$$\frac{1}{2}\|\overline{v^m}(t)\|^2_{V^0} \leqslant \frac{1}{2}\|v_0\|^2_{V^0} + \frac{1}{2\mu_0}\|f^m\|^2_{L_2(-m,T;V^{-1})} + \frac{\mu_0}{4}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)}.$$

Since the right side in all the above inequalities does not depend on $\tau$, then on the left side we take maximum over $\tau \in [-m, T]$, and also use the fact that $v^m(-m) = 0$, we get

$$\frac{\mu_0}{2}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)} \leqslant \frac{1}{2\mu_0}\|f^m\|^2_{L_2(0,T;V^{-1})} + \frac{\mu_0}{4}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)},$$

$$\frac{1}{2}\|\overline{v^m}\|^2_{C([-m,T];V^0)} \leqslant \frac{1}{2\mu_0}\|f^m\|^2_{L_2(-m,T;V^{-1})} + \frac{\mu_0}{4}\|\overline{v^m}\|^2_{L_2(-m,T;V^1)}.$$

This implies the required estimates (19)–(20). $\square$

**Lemma 8.** *Let $f \in L_2(-m, T; V^{-1})$. Then for any solution $v \in W_2$ of the operator equation* (15) *the estimates hold:*

$$\|(v^m)'\|_{L_2(-m,T;V^{-1})} \leqslant C_{13}\|f^m\|^2_{L_2(-m,T;V^{-1})}; \tag{21}$$

$$\|(v^m)'\|_{L_{4/3}(-m,T;V^{-1})} \leqslant C_{14}(\|f^m\|^2_{L_2(-m,T;V^{-1})} + 1), \tag{22}$$

*where constants $C_{13}$, $C_{14}$ do not depend on $v^m$ and $m$.*

*Proof.* Let $v^m \in W_2$ is solution of (15). Then it satisfies the following operator equation

$$J(v^m)' + \mu_0 A v^m + \frac{\xi\mu_1}{\Gamma(1-\zeta)}B(v^m, z^m) - \xi K(v^m) = \xi f^m.$$

Hence,

$$\|Jv^{m\prime}\|_{L_2(-m,T;V^{-1})} =$$

$$= \|\xi f^m - \mu_0 A v^m - \frac{\xi\mu_1}{\Gamma(1-\zeta)}B(v^m, z^m) + \xi K^m(v)\|_{L_2(-m,T;V^{-1})}.$$

Let us estimate the right side. By virtue of estimates (16) and (18), we obtain:

$$\|\xi f^m - \mu_0 A v^m - \frac{\xi\mu_1}{\Gamma(1-\zeta)}B(v^m, z^m) + \xi K(v^m)\|_{L_2(-m,T;V^{-1})} \leqslant$$

$$\leqslant \|f^m\|_{L_2(-m,T;V^{-1})} + \frac{\mu_1 C_{10} T^{\frac{1}{2}-\zeta}}{\Gamma(1-\zeta)}\|v^m\|_{L_2(-m,T;V^1)} +$$

$$+ \mu_0\|v^m\|_{L_2(-m,T;V^1)} + C_6\|v^m\|_{L_2(-m,T;V^1)}. \tag{23}$$

Let us separately estimate the value of $\|K(v^m)\|_{L_2(-m,T;V^{-1})}$. Using continuity embedding $V^1 \subset L_4(\Omega)$, we get:

$$\|K(v^m)\|_{L_2(-m,T;V^{-1})} = \left(\int_{-m}^{T}\|K(v^m)\|^2_{V^{-1}}\,dt\right)^{\frac{1}{2}} \leqslant C_6\left(\int_{-m}^{T}\|v^m(t)\|^4_{L_4(\Omega)}\,dt\right)^{\frac{1}{2}} \leqslant$$

$$\leqslant C_{15}\left(\int_{-m}^{T}\|v^m(t)\|^4_{V^1}\,dt\right)^{\frac{1}{2}} \leqslant C_{16}\max_{t\in[-m,T]}\|v^m(t)\|^2_{V^1} = C_{16}\|v^m\|^2_{C([-m,T];V^1)}.$$

Let us rewrite (23) in the form:

$$\|\xi f^m - \mu_0 A v^m - \frac{\xi\mu_1}{\Gamma(1-\zeta)}B(v^m, z^m) + \xi K^m(v)\|_{L_2(-m,T;V^{-1})} \leqslant$$

$$\leqslant C_{18}(\|f^m\|_{L_2(-m,T;V^{-1})} + \|v^m\|_{L_2(-m,T;V^1)}).$$

From a priori estimates (19) and (20) it follows that

$$\|J(v^m)'\|_{L_2(-m,T;V^{-1})} \leqslant C_{19}\|f^m\|^2_{L_2(-m,T;V^{-1})}.$$

Therefore, inequality (21) is proved.

Now we will prove (22). As before, $v^m \in W_2$ is the solution of the operator equation (15). Then

$$\|(v^m)'\|_{L_{4/3}(-m,T;V^{-1})} \leqslant \|f^m - \mu_0 A v^m - \frac{\mu_1}{\Gamma(1-\zeta)} B(v^m, z^m) +$$

$$+ K(v^m)\|_{L_{4/3}(-m,T;V^{-1})} \|f^m\|_{L_{4/3}(-m,T;V^{-1})} + \mu_0 \|A v^m\|_{L_{4/3}(-m,T;V^{-1})} +$$

$$+ \|K(v^m)\|_{L_{4/3}(-m,T;V^{-1})} + \frac{\mu_1}{\Gamma(1-\zeta)} \|B(v^m, z^m)\|_{L_{4/3}(-m,T;V^{-1})}.$$

Let us separately consider the summand on the right–hand side of the last inequality. Using Holder's inequality:

$$\|A v^m\|_{L_{4/3}(-m,T;V^{-1})} = \Big( \int_{-m}^{T} \|A v^m\|_{V^{-1}}^{\frac{4}{3}} dt \Big)^{\frac{3}{4}} \leqslant \Big( \int_{-m}^{T} \|v^m\|_{V^1}^{\frac{4}{3}} dt \Big)^{\frac{3}{4}} \leqslant$$

$$\leqslant T^{\frac{1}{4}} \Big( \int_{-m}^{T} \|v^m\|_{V^1}^{2} dt \Big)^{\frac{1}{2}} = T^{\frac{1}{4}} \|v^m\|_{L_2(-m,T;V^1)}.$$

Similarly, using the Holder inequality and estimate (18) for $u = 0$, we obtain:

$$\|B(v^m, z^m)\|_{L_{4/3}(-m,T;V^{-1})} = \Big( \int_{-m}^{T} \|B(v^m, z^m)\|_{V^{-1}}^{\frac{4}{3}} dt \Big)^{\frac{3}{4}} \leqslant$$

$$\leqslant T^{\frac{1}{4}} \Big( \int_{-m}^{T} \|B(v^m, z^m)\|_{V^{-1}}^{2} dt \Big)^{\frac{1}{2}} \leqslant$$

$$= T^{\frac{1}{4}} \|B(v^m, z^m)\|_{L_2(-m,T;V^{-1})} \leqslant T^{\frac{1}{4}} T^{\frac{1}{2}-\zeta} C_{10} \|v^m\|_{L_2(-m,T;V)}.$$

Let us set the estimate on $\|K(v^m)\|_{L_{4/3}(-m,T;V^{-1})}$. Taking into account the well–known inequality for $n = 3$

$$\|u\|_{L_4(\Omega)} \leqslant 2^{\frac{1}{2}} \|u\|_{L_2(\Omega)}^{\frac{1}{4}} \|\nabla u\|_{L_2(\Omega)}^{\frac{3}{4}}, \quad u \in V,$$

and estimate (16), we obtain (for the case $n = 2$ the proof is similar):

$$\|K(v^m)\|_{L_{4/3}(-m,T;V^{-1})} = \Big( \int_{-m}^{T} \|K(v^m)\|_{V^{-1}}^{\frac{4}{3}} dt \Big)^{\frac{3}{4}} \leqslant C_6 \Big( \int_{-m}^{T} \|v^m\|_{V^1}^{\frac{8}{3}} dt \Big)^{\frac{3}{4}} \leqslant$$

$$\leqslant 2 C_6 \Big( \int_{-m}^{T} \|v^m\|_{L_2(\Omega)}^{\frac{2}{3}} \|\nabla v^m\|_{L_2(\Omega)}^{2} dt \Big)^{\frac{3}{4}} \leqslant C_{18} \Big( \int_{-m}^{T} \|v^m\|_{V^0}^{\frac{2}{3}} \|v^m\|_{V^0}^{2} dt \Big)^{\frac{3}{4}} \leqslant$$

$$\leqslant C_{18} \|v^m\|_{C([-m,T];V^0)}^{\frac{1}{2}} \Big( \int_{-m}^{T} \|v^m\|_{V^1}^{2} dt \Big)^{\frac{3}{4}} =$$

$$= C_{18} \|v^m\|_{C([-m,T];V^0)}^{\frac{1}{2}} \|v^m\|_{L_2(-m,T;V^1)}^{\frac{3}{2}}.$$

Let us estimate the right side. To do this, we use the left side of estimate (20) for $p = 4/3$.

$$\|(v^m)'\|_{L_{4/3}(-m,T;V^1)} \leqslant C_{18}(\|f^m\|_{L_2(-m,T;V^{-1})} + \|v^m\|_{L_2(-m,T;V^1)} +$$

$$+\|v^m\|_{C([-m,T];V^0)}^{\frac{1}{2}}\|v^m\|_{L_2(-m,T;V^1)}^{\frac{3}{2}}) \leqslant C_{19}(\|f^m\|_{L_2(-m,T;V^{-1})} + 1)^2 \leqslant$$

$$\leqslant 4C_{19}(\|f^m\|_{L_2(-m,T;V^{-1})}^2 + 1).$$

Which proves (22), where $C_{14} = 4C_{19}$. $\square$

**Lemma 9.** *Let $f^m \in L_2(-m, T; V^{-1})$. Then for any solution $v^m \in W_2$ of the operator equation (15) with the initial condition $v^m(-m, \cdot) = 0$ there is estimate:*

$$\|v\|_{W_2} \leqslant C_{20},$$

*where $C_{20}$ does not depend on $m$.*

## 6 Solvability of approximative problem

**Theorem 6.** *Let $f^m \in L_2(-m, T; V^{-1})$. Then the problem (8)–(11) for $\xi = 1$ has at least one solution $v^m \in W_2$.*

*Proof.* The proof of this Theorem is based on the theory of topological degree of condensing vector fields. Consider the operator equation (15):

$$L(v^m) - \xi C(v^m) + \xi G(v^m) = \xi(f^m, 0). \tag{24}$$

Lemma 9 implies that the solutions of equation (24) lie in the ball $B_R \subset W_2$ with center at zero and radius $R = C_{20} + 1$. According to Lemma 6, the operator $L : W_2 \to L_2(-m, T; V^{-1}) \times V^1$ is invertible. Then there is no solution

$$v^m = \xi L^{-1}(C(v^m) - G(v^m) + (f^m, 0))$$

which belongs to the boundary of the ball $B_R$.

By virtue of Lemma 6, the operator $L^{-1} : L_2(-m, T; V^{-1}) \times V^1 \to W_2$ is continuous. According to Lemmas 3 and 5, the mapping $(C(v^m) - G(v^m) + (f^m, 0)) : W_2 \to L_2(-m, T; V^{-1}) \times V^1$ is $L$—condensing with respect to the measure of non–compactness of Kuratovskii $\gamma_k$. Therefore, the operator $L^{-1}(C(v^m) - G(v^m) + (f^m, 0)) : W_2 \to W_2$ is condensing with respect to the measure of non–compactness Kuratovsky $\gamma_k$.

Thus, the vector field $v^m - \xi(L)^{-1}(C(v^m) - G(v^m) + (f^m, 0))$ is non–degenerate on the boundary of the ball $B_R$, and hence for this vector field the topological degree $\deg(I - \xi((L)^{-1}(C - G + f), B_R, 0)$ exists. By the properties of homotopy invariance and normalization of the degree, we obtain that

$$\deg(I - \xi L^{-1}(C - G + f), B_R, 0) = \deg(I, B_R, 0) = 1.$$

The nonzero degree of the mapping ensures the existence of at least one solution $v^m \in W_2$ of equation (15), and, consequently, the approximative problem (8)–(11). $\square$

## 7   Passage to the limit

Next, we obtain a solution to problem (1)–(5) by passing to the limit for the obtained solutions $v^m$ of the problem (8)–(11) as $m \to +\infty$. To do this, we need estimates of $v^m$ uniform in $m$ on the semiaxis.

**Lemma 10.** *Let* $f^m \in L_2(-\infty, T; V^{-1})$. *Then the following estimates hold for the function* $v^m$:

$$\sup_{-\infty < t \leq T} \|v^m(t,.)\|_{V^0} + \|v^m(t,.)\|_{L_2(-\infty,T;V^1)} \leq C_{21}\|f^m\|_{L_2(-\infty,T;V^0)} \quad (25)$$

*with a constant* $C_{21}$ *independent of* $m$.

This Lemma is carried out in the same way as in [2] (Lemma 10).

Estimate (25) means that the sequence $v^m$ is bounded in $L_2(-\infty, T; V^1)$. This allows us to assert that there exists a function $v \in L_2(-\infty, T; V^1)$ such that $v^m$ (up to a subsequence) converges to $v$ weakly in $L_2(-\infty, T; V^1)$. In addition, estimate (25) entails the convergence of $v^m$ to $v$ (up to a subsequence) a.e. to $[-k, T] \times \Omega$ for any $k > 0$.

**Lemma 11.** *Let* $k < m$, $f^m \in L_2(-\infty, T; V^{-1})$. *Then for functions* $v^m$ *the estimate*

$$\|\frac{dv^m}{dt}\|_{L_1(-k,T;V^{-1})} \leq C_{22}(k)(1 + \|f\|_{L_1(-k,T;V^{-1})} + \|f\|^2_{L_1(-k,T;V^{-1})}).$$

*hold with independent on* $m$, *but dependent on* $k$ *constant* $C_{22}(k)$

This Lemma is carried out in the same way as in [31] (Lemma 3.5).

Next, a number of lemmas will be considered.

**Lemma 12.** *The sequence* $\tilde{v}^m$ *converges in* $L_2(-\infty, T; V^1)$ *to* $v$.

*Proof.* It is easy to see that

$$\tilde{v}^m - v = I_1(m) + I_2(m),$$

here $I_1(m) = S_{1/\varepsilon}(v^m - v)$, $I_2(m) = S_{1/\varepsilon}v - v$. Consider

$$\|\tilde{v}^m - v\|_{L_2(-\infty,T;V^1)} \leq \|\rho_{1/\varepsilon} * v^m - \rho_{1/\varepsilon} * v\|_{L_2(-\infty,T;V^1)} +$$
$$+ \|\rho_{1/\varepsilon} * v - v\|_{L_2(-\infty,T;V^1)} \to 0.$$

For the first term we have the estimate

$$\|\rho_{1/\varepsilon} * v^m - \rho_{1/\varepsilon} * v\|_{L_2(-\infty,T;V^1)} \leq C_3\|v^m - v\|_{L_2(-\infty,T;V^1)},$$

$C_3$ does not depend on $1/\varepsilon$.

Here we have used estimate (12) of Lemma 1. Second term tends to zero at $\varepsilon \to +\infty$ by estimate (13) of Lemma 1. $\square$

Consider the Cauchy problem (10) for the limit function $v^m$. Since $v^m \in W_2$, then $v^m$ satisfies the conditions of Theorem 1 on any finite interval $[-k, T]$ for any $-\infty < k < T$. Then Theorem 1 implies the existence of the RLP $z(\tau; t, x), -\infty < \tau, t \leq T, x \in \bar{\Omega}$, associated to $v$.

**Lemma 13.** *For $t \in [-k, T]$ and for any $k \geq 0$ the sequence $z^m(\tau; t, x)$ converges to $z(\tau; t, x)$ in measure $[-k, T] \times \Omega$.*

Lemma 13 implies that the sequence $z^m(\tau; t, x)$ converges to $z(\tau; t, x)$ a.e. on $Q(k, T) = [-k, T] \times \Omega$ as a function of the variables $(\tau, x) \in Q(k, T) = [-k, T] \times \Omega$ for any $-k \in (-\infty, T)$ for $t \in [-k, T]$ (see [32] Lemma VI.5.1).

**Lemma 14.** *The limit function $v^m(t, x)$ satisfies the identity*

$$\int_{-\infty}^{T} (v^m(t, x), \varphi) \psi'(t) dt - \sum_{i=1}^{n} \int_{-\infty}^{T} \left( \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})}, \frac{\partial \varphi}{\partial x_i} \right) \psi(t) dt +$$

$$+ \mu_0 \int_{-\infty}^{T} (\nabla v^m(t, x), \nabla \varphi(x)) \psi(t) dt +$$

$$+ \frac{\mu_1}{\Gamma(1 - \zeta)} \int_{-\infty}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t - s)^{-\zeta} (\mathcal{E}(v^m)(s, z^m(s; t, x)), \mathcal{E}(\varphi)(x)) ds \right) \times$$

$$\times \psi(t) dt = \int_{-\infty}^{T} \langle f^m(t, x), \varphi(x) \rangle \psi(t) dt \qquad (26)$$

*for any $\varphi \in V^1$ and $\psi \in C_0^\infty(-\infty, T)$.*

*Proof.* First let $\varphi \in V^1$ be smooth. Let supp $\psi \subset [k, T]$, where $k > 0$. Let us introduce the notation for the terms on the left side of (26):

$$J_1^m = \int_{-\infty}^{T} (v^m(t, x), \varphi) \psi'(t) dt; \quad J_2^m = \sum_{i=1}^{n} \int_{-\infty}^{T} \left( \frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})}, \frac{\partial \varphi}{\partial x_i} \right) \psi(t) dt;$$

$$J_3^m = \mu_0 \int_{-\infty}^{T} (\nabla v^m(t, x), \nabla \varphi(x)) \psi(t) dt;$$

$$J_4^m = \mu_1 \int_{-\infty}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t - s)^{-\zeta} (\mathcal{E}(v^m)(s, z^m(s; t, x)), \mathcal{E}(\varphi)(x)) ds \right) \psi(t) dt.$$

We denote the corresponding terms on the left side by $J_i$, $i = \overline{1, 4}$.

The weak convergence of $v^m$ to $v$ in $L_2(-\infty, T; V^1)$ implies that $J_i^m$ converge to $J_i, i = 1, 3$. It is easy to see that the sequence $f^m$ converges to $f$ in $L_2(-\infty, T; V^0)$, strongly in $L_2(Q)^n$, a. e. to $Q = (-\infty, T] \times \Omega$, while the sequence $\frac{dv^m}{dt}$ is bounded in the norm of the space $L_1(-k, T; V^{-1})$ and converges to $\frac{dv}{dt}$ in the sense distributions on $[-k, T]$ for any $-\infty < k < T$. The sequence $\frac{(\Delta_\alpha^{-1} v^m)_i v_j^m}{(1 + \frac{|v_j^m|^2}{\varepsilon})}$ weakly convergence to $(\Delta_\alpha^{-1} v)_i v_j$ in $L_2([t_1, t_2] \times \Omega)$ as $\varepsilon \to +\infty$ on a finite interval $[t_1, t_2]$. Since the integration in $J_2^m$ and $J_2$ are drawn on a finite interval supp $\psi \subset [k, T]$, then $J_2^m$ converges to $J_2$ as $\varepsilon \to +\infty$.

Consider now $J_4^m$. Obviously,

$$J_4^m = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \times \right.$$

$$\times \int_{\Omega} (\mathcal{E}(v^m)(s, z^m(s; t, x)) : \mathcal{E}(\varphi)(x)) dx\, ds) \psi(t) dt$$

Let us show that $\lim_{m \to \infty} J_4^m = J_4$, where

$$J_4 = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} (\mathcal{E}(v)(s, z(s; t, x)) : \mathcal{E}(\varphi)(x)) dx\, ds) \psi(t) dt$$

It is easy to see that $J_4^m - J_4 = Z_1^m + Z_2^m$, where

$$Z_1^m = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s, z^m(s; t, x)) - \right.$$
$$- \mathcal{E}(v)(s, z^m(s; t, x))] : \mathcal{E}(\varphi)(x)) dx ds) \psi(t) dt,$$

$$Z_2^m = \mu_1 \int_{-k}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v)(s, z^m(s; t, x)) - \right.$$
$$- \mathcal{E}(v)(s, z(s; t, x))] : \mathcal{E}(\varphi)(x)) dx ds) \psi(t) dt.$$

Let us show that $\lim_{m \to \infty} Z_1^m = 0$. Denote the integral over $\Omega$ in $Z_1$ by

$$I = \int_{\Omega} [(\mathcal{E}(v^m)(s, z^m(s; t, x)) - \mathcal{E}(v)(s, z^m(s; t, x))] : \mathcal{E}(\varphi)(x) dx.$$

In the integral $I$ given above, we make the change of variable $x = z^m(t; s, y)$. Then $I = \int_{\Omega} [(\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)] : \mathcal{E}(\varphi)(z^m(t; s, y))) dy$. Using this relation and changing the order of integration, we have

$$Z_1^m = \int_{-k}^{T} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)] : \right.$$

$$: \mathcal{E}(\varphi)(z^m(t; s, y)) dy\, ds \Big) \psi(t) dt =$$

$$= e^{\frac{-(t-s)}{\lambda}} \int_{-\infty}^{-k} \left( \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [\mathcal{E}(v^m)(s, y) - \mathcal{E}(v)(s, y)] : \right.$$

$$: \mathcal{E}(\varphi)(z^m(t; s, y)) dy\, ds \Big) \psi(t) dt +$$

$$+ \int_{-k}^{T} \left( \int_{s}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [\mathcal{E}(v^m)(s, y) - \right.$$

$$- \mathcal{E}(v)(s, y)] : \mathcal{E}(\varphi)(z^m(t; s, y)) dy\, ds \Big) \psi(t) dt = Z_{11}^m + Z_{12}^m.$$

It's obvious that

$$Z_{12}^m = \int_{-k}^{T} \Big( \int_{s}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] :$$

$$: [\mathcal{E}(\varphi)(z^m(t;s,y)) - \mathcal{E}(\varphi)(z(t;s,y))] dy\, ds \Big) \psi(t) dt +$$

$$+ \int_{-k}^{T} \Big( \int_{s}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [\mathcal{E}(v^m)(s,y) -$$

$$- \mathcal{E}(v)(s,y)] : \mathcal{E}(\varphi)(z(t;s,y)) dy\, ds \Big) \psi(t) dt = Z_{121}^m + Z_{122}^m.$$

From the weak convergence of $v^m$ to $v$ in $L_2(-\infty, T; V^1)$ we get that $Z_{122}^m \to 0$ as $m \to +\infty$. Using the boundedness of the function $\psi$ and $e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\alpha}$ and applying the Cauchy–Bunyakovsky and Holder inequalities, we obtain that

$$|Z_{121}^m|^2 \le M \Big( \int_{-k}^{T} \int_{s}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \times$$

$$\times \|v^m(s,y) - v(s,y)\|_{V^1} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} \psi(t) dt\, ds \Big)^2 \le$$

$$\le M \Big( \int_{-k}^{T} \|v^m(s,y) - v(s,y)\|_{V^1} \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) -$$

$$- \varphi_x(z(t;s,y))\|_{V^0}) dt\, ds \Big)^2 \le M \int_{-k}^{T} \|v^m(s,y) - v(s,y)\|_{V^1}^2 ds \times$$

$$\times \int_{-k}^{T} \Big( \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} dt \Big)^2 ds \le M \|v^m(s,y) -$$

$$- v(s,y)\|_{L_2(-k,T;V^1)}^2 \int_{-k}^{T} \Big( \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} dt \Big)^2 ds. \tag{27}$$

Let us show that $Z_{121}^m \to 0$ as $m \to +\infty$. Let us denote the last term in (27) as

$$\Psi(s) = \int_{-k}^{T} \Big( \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} dt \Big)^2 ds.$$

Let us write it in the form

$$\Psi(s) = \int_{-k}^{T} g_m(s) ds, g_m(s) = \Big( \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} dt \Big)^2.$$

Let get the convergence of $g_m(s) \to 0$ as $m \to +\infty$ for all $s \in [-k, T]$. It is easy to see that

$$g_m(s) = \Big( \int_{s}^{T} \|\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))\|_{V^0} dt \Big)^2 =$$

$$= \int_{-k}^{T} \Big( \int_{s}^{T} \int_{\Omega} |\varphi_x(z^m(t;s,y)) - \varphi_x(z(t;s,y))| dy\, dt \Big)^2 ds. \tag{28}$$

Let $\varepsilon > 0$. Since the function $\varphi_x$ is continuous on $\overline{\Omega}$, there exists $\delta_1(\theta) > 0$, what if $|x'' - x'| \leq \delta_1(\theta)$, then

$$|\varphi_x(x'') - \varphi_x(x')| \leq \theta. \tag{29}$$

Since the sequence $z^m(t; s, y)$ converges to $z(t; s, y)$ in measure $(t, y)$ on $[s, T] \times \Omega$ for $s \in [-k, T]$, then for $\delta_1(\theta)$ one can specify $N = N(\delta_1(\theta))$ such that for $m \geq N$ the inequality

$$m(\{(t, y) : |z(t; s, y) - z^m(t; s, y)| \geq \delta_1(\theta)\}) \leq \theta. \tag{30}$$

Let

$$Q(> \delta_1(\theta)) = \{(t, y) \in Q : |z(t; s, y) - z^m(t; s, y)| > \delta_1(\theta)\};$$
$$Q(\leq \delta_1(\theta)) = \{(t, y) \in Q : |z(t; s, y) - z^m(t; s, y)| \leq \delta_1(\theta)\}.$$

Then from (28) it follows that

$$g_m(s) \leq M_1 \Big( \int_{Q(>\delta_1(\theta))} |\varphi_x(z(t; s, y)) - \varphi_x(z^m(t; s, y))|^2 dy\, dt +$$

$$+ \int_{Q(\leq\delta_1(\theta))} |\varphi_x(z(t; s, y)) - \varphi_x(z^m(t; s, y))|^2 dy\, dt = C_{23}(G_1 + G_2). \tag{31}$$

For $G_2$, due to (29) we have $|z(t; s, y) - z^m(t; s, y)| \leq \delta_1(\theta)$, and therefore

$$G_2 \leq \int_{Q(\leq\delta_1(\theta))} \theta^2 dy dt \leq C_{24}\theta^2. \tag{32}$$

Since $m(Q(> \delta_1(\theta))) \leq \theta$ due to (30), then

$$G_1 \leq 2\|\varphi_x\|_{C(\Omega)} \int_{Q(>\delta_1(\theta))} dy\, dt \leq 2\|\varphi_x\|_{C(\Omega)}\theta. \tag{33}$$

Estimates (31)–(33) imply that for $m \geq N(\delta_1(\theta))$ the inequality $g_m(s) \leq C_{24}^{1/2}$. The convergence of $g_m(s) \to 0$ as $m \to +\infty$ for all $s \in [-k, T]$ is established. In addition, $g_m(s)$ is bounded due to the smoothness of $\varphi_x(s)$. Therefore $\Psi_m \to 0$.

The first term on the right in (27) is bounded in $m$ due to the uniform boundedness of $\|v^m\|_{L_2(-k,T;V^1)}$, and the second tends to zero ($\Psi_m \to 0$).

Thus, from (27) and (28) it follows that $Z_{121}^m \to 0$ as $m \to +\infty$.

The convergence of $Z_{122}^m \to 0$ as $m \to +\infty$ follows from the weak convergence of $v^m \to v$ in $L_2(-\infty, T; V^1)$. From $Z_{121}^m \to 0$ and $Z_{122}^m \to 0$ for $m \to +\infty$ we get $Z_{12}^m \to 0$ for $m \to +\infty$.

Now consider $Z_{11}^m$. It is easy to see that for arbitrary $-\infty < R < -k$

$$Z_{11}^m = \int_{-\infty}^{R} \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] :$$

$$: \mathcal{E}(\varphi)(z^m(t;s,y)) dy \psi(t) dt \, ds +$$

$$\int_{R}^{T} \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] :$$

$$: \mathcal{E}(\varphi)(z^m(t;s,y)) dy \psi(t) dt \, ds = Z_{111}^m + Z_{112}^m.$$

Consider the $Z_{111}$. For $Z_{111}^m$ due to the boundedness of $\varphi_x$ and $\psi$ we get:

$$Z_{111} = \int_{-\infty}^{R} \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] :$$

$$: \mathcal{E}(\varphi)(z^m(t;s,y)) dy \psi(t) dt \, ds =$$

$$= \int_{-\infty}^{R} \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] : \mathcal{E}(\varphi)(z^m(t;s,y)) +$$

$$+ \mathcal{E}(\varphi)(z(t;s,y)) - \mathcal{E}(\varphi)(z(t;s,y)) dy \psi(t) dt \, ds =$$

$$= \int_{-\infty}^{R} \int_{-k}^{T} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \int_{\Omega} [(\mathcal{E}(v^m)(s,y) - \mathcal{E}(v)(s,y)] : \mathcal{E}(\varphi)(z(t;s,y)) +$$

$$+ [(\mathcal{E}(\varphi)(z^m(t;s,y)) - \mathcal{E}(\varphi)(z(t;s,y))] : \mathcal{E}(\varphi)(z^m(t;s,y)) dy \psi(t) dt \, ds =$$

$$= \int_{-k}^{T} e^{\frac{-t}{\lambda}} \psi(t) dt \int_{-\infty}^{R} e^{\frac{s}{\lambda}} (t-s)^{-\zeta} ds \int_{\Omega} [(\mathcal{E}(v^m)(s,y) -$$

$$- \mathcal{E}(v)(s,y)] : \mathcal{E}(\varphi)(z(t;s,y)) + [(\mathcal{E}(\varphi)(z^m(t;s,y)) -$$

$$- \mathcal{E}(\varphi)(z(t;s,y))] : \mathcal{E}(\varphi)(z^m(t;s,y)) dy = \int_{-k}^{T} e^{\frac{-t}{\lambda}} \psi(t) dt G_3 \int_{\Omega} [(\mathcal{E}(v^m)(s,y) -$$

$$- \mathcal{E}(v)(s,y)] : \mathcal{E}(\varphi)(z(t;s,y)) + [(\mathcal{E}(\varphi)(z^m(t;s,y)) -$$

$$- \mathcal{E}(\varphi)(z(t;s,y))] : \mathcal{E}(\varphi)(z^m(t;s,y)) dy \to 0. \tag{34}$$

Let us calculate $G_3$ separately:

$$G_3 = \int_{-\infty}^{R} e^{\frac{s}{\lambda}} (t-s)^{-\zeta} ds.$$

Make a change of variables: $u = (t-s)^{\zeta+1}$, $du = -(\zeta+1)(t-s)^{\zeta} ds$, $s = -\infty$, $u = +\infty$, $s = R$, $u = (t-R)^{\zeta+1}$. Then we get:

$$G_3 = -\frac{e^{\frac{t}{\lambda}}}{\zeta+1} \int_{(t-R)^{\zeta+1}}^{+\infty} e^{-\frac{u^{\frac{1}{\zeta+1}}}{\lambda}} du.$$

Let us make a change of variables: $w = \lambda^{-\zeta-1} u$, $dw = \lambda^{\zeta+1} du$, $u = +\infty$, $w = +\infty$, $u = (t-R)^{\zeta+1}$, $w = \lambda^{-\zeta-1}(t-R)^{\zeta+1}$. We get:

$$G_3 = e^{\frac{t}{\lambda}} \lambda^{\zeta+1} \Gamma(\zeta+1, w^{\frac{1}{\zeta+1}}).$$

Due to the boundedness of the first term in (34), the weak convergence of $v^m$ to $v$, the convergence of $z^m$ to $z$, we get that $|Z_{111}^m|$ tends to zero. $|Z_{112}^m|$ tends to zero as $m \to +\infty$ is established similarly to the case $|Z_{12}^m|$.

Thus, for $m \to +\infty$ it was found that

$$|Z_1^m| \to 0. \tag{35}$$

Let us show for $m \to +\infty$ that

$$|Z_2^m| \to 0. \tag{36}$$

We will estimate $|Z_2^m|$. Consider the approximation of $v(t,x)$ by a smooth finite function $\widetilde{v}$ on $(-\infty, T] \times \Omega$ so that $\|v - \widetilde{v}\|_{L_2(-\infty,T;V^1)} \le \theta_2$ and $\widetilde{v} \equiv 0$ at $t < k_1$, where $k_1 < T$, $\theta_2 > 0$ is arbitrary small number. Then $|Z_2^m| \le M(Z_{21}^m + Z_{22}^m + Z_{23}^m)$, where

$$Z_{21}^m = \int_{-k}^{T} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \|v(s, z^m(s;t,x)) - \widetilde{v}(s, z^m(s;t,x))\|_{V^1} ds\, dt;$$

$$Z_{22}^m = \int_{-k}^{T} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \|\widetilde{v}(s, z^m(s;t,x)) - \widetilde{v}(s, z(s;t,x))\|_{V^1} ds\, dt;$$

$$Z_{23}^m = \int_{-k}^{T} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \|\widetilde{v}(s, z(s;t,x)) - v(s, z(s;t,x))\|_{V^1} ds\, dt.$$

Making a change of variables in the integral in $Z_{21}^m$ $x = z^m(t;s,y), y = z^m(s;t,x)$, we have:

$$\|v(s, z^m(s;t,x)) - \widetilde{v}(s, z^m(s;t,x))\|_{V^1} = \|v(s,y) - \widetilde{v}(s,y)\|_{V^1}. \tag{37}$$

Similarly, for $Z_{23}^m$, using the replacement $y = z(s;t,x)$, we obtain

$$\|\widetilde{v}(s, z(s;t,x)) - v(s, z(s;t,x))\|_{V^1} = \|\widetilde{v}(s,y) - v(s,y)\|_{V^1}. \tag{38}$$

From inequalities (37) and (38) follows that

$$Z_{21}^m + Z_{23}^m \le C_{25} \int_{-k}^{T} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \|v(s,y) - \widetilde{v}(s,y)\|_{V^1} ds\, dt \le C_{25}\theta_2$$

Further, since $\widetilde{v}$ is finite, it follows that

$$Z_{22}^m \le C_{26} \int_{-k}^{T} \left( \int_{k_1}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \times \right.$$

$$\left. \times \int_{\Omega} \|\widetilde{v}_x(s, z^m(s;t,x)) - \widetilde{v}_x(s, z(s;t,x))\|_{V^1} dx\, ds \right) dt.$$

Since $z^m(s;t,x)$ converges a.e. to $z(s;t,x)$ uniformly in $t$, and the function $\widetilde{v}_x(t,x)$ is smooth and bounded, then by Lebesgue's theorem for $m \to +\infty$ we obtain (36). From (35) and (11) it follows that $J_4^m \to J_4$.

Thus, in each term (26) we can pass to the limit, which gives identity (26) for any smooth $\varphi$. Let us establish identity (26) for any $\varphi \in V^1$ and $\psi \in C_0^\infty(-\infty, T)$. We write (26) for smooth $\varphi$ as

$$[G_1, \varphi] - [G_2, \varphi] = 0, \tag{39}$$

where

$$[G_1, \varphi] = \int_{-\infty}^{T} (v, \varphi) \psi'(t)\, dt - \int_{-\infty}^{T} \sum_{i,j=1}^{n} v_i v_j \frac{\partial \varphi_j}{\partial x_i} \psi(t)\, dt +$$

$$+ \mu_0 \int_{-\infty}^{T} (\nabla v : \nabla \varphi) \psi(t)\, dt$$

$$+ \frac{\mu_1}{\Gamma(1-\zeta)} \int_{-\infty}^{T} \Big( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v)(s, z(s; t, x))\, ds, \mathcal{E}(\varphi) \Big) \psi(t)\, dt;$$

$$[G_2, \varphi] = \int_{-\infty}^{T} (f, \varphi) \psi(t)\, dt.$$

We need the following Lemma.

**Lemma 15.** *Let function $\varphi$ be smooth. Then*

$$|[G_1, \varphi]| \leqslant C_{27} \|\varphi\|_{V^1}, \quad |[G_2, \varphi]| \leqslant C_{28} \|\varphi\|_{V^1}. \tag{40}$$

Since the set of smooth functions is dense in $V^1$, for $\varphi \in V^1$ there exists a sequence of smooth functions $\varphi^l \in V^1$ such that $|\varphi^l - \varphi|_{V^1} \to 0$ for $l \to \infty$. Due to (39), we get:

$$[G_1, \varphi] - [G_2, \varphi] = [G_1, \varphi - \varphi^l] - [G_2, \varphi - \varphi^l] + [G_1, \varphi^l] - [G_2, \varphi^l] =$$
$$= [G_1, \varphi - \varphi^l] - [G_2, \varphi - \varphi^l].$$

From the last equality and estimates (40) we obtain $|[G_1, \varphi] - [G_2, \varphi]| \leqslant C_{29} \|\varphi - \varphi^l\|_{V^1}$.

Taking into account the last inequality and passing to the limit as $l \to \infty$ for $\varphi = \varphi^l$ we obtain equality (26) for an arbitrary $\varphi \in V^1$. $\square$

In order to prove that $v$ is a weak solution to problem (1)–(5) it is now enough to prove that $v$ satisfies the identity (7).

**Lemma 16.** *The limit function $v$ satisfies the identity* (7).

The proof of this Lemma is similar to the proof of Lemma 1.1 from [32].

Thus, we have proved the existence of at least one weak solution to problem (1)–(5), which describes the motion of a viscoelastic fluid.

## 8   Convergence of solutions as $\alpha \to 0$

In this section we will establish the convergence of solutions of the $\alpha$–model (1)–(5). Before proceeding directly to the proof, we formulate the definition

of a weak solution to the following initial–boundary value problem:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mu_0 \triangle v -$$

$$-\frac{\mu_1}{\Gamma(1-\zeta)} \text{Div} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v)(s, z(s; t, x)) ds +$$

$$+\nabla p = f(t, x); \tag{41}$$

$$z(\tau; t, x) = x + \int_{t}^{\tau} v(s, z(s; t, x)) ds, \qquad t, \tau \in (-\infty, T], \ x \in \Omega; \tag{42}$$

$$\text{div } v = 0; \qquad v(t, x) \mid_{(t,x) \in (-\infty, T] \times \partial\Omega} = 0. \tag{43}$$

**Definition 6.** *Weak solution of the problem* (41)–(43) *is the function* $v \in W_1$, *satisfying for any* $\varphi \in V^1$ *and a. e.* $t \in (-\infty, T]$ *the identity*

$$\langle v', \varphi \rangle - \int_{\Omega} \sum_{i=1}^{n} v_i v \frac{\partial \varphi}{\partial x_i} dx + \mu_0 \int_{\Omega} \nabla v : \nabla \varphi \, dx +$$

$$+\frac{\mu_1}{\Gamma(1-\zeta)} \int_{\Omega} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v)(s, z(s; t, x)) ds \, \mathcal{E}(\varphi) dx =$$

$$= \langle f, \varphi \rangle. \tag{44}$$

Consider a sequence of numbers $\alpha_k$ such that $\alpha_k \to 0$ for $k \to \infty$, and another family approximative problem depending on the parameter $\alpha_k$:

$$\frac{\partial v^k}{\partial t} + \sum_{i=1}^{n} u_i^k \frac{\partial v^k}{\partial x_i} - \mu_0 \Delta v^k -$$

$$-\frac{\mu_1}{\Gamma(1-\zeta)} \text{Div} \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v^k)(s, z^k(s; t, x)) \, ds = f, \tag{45}$$

$$z^k(\tau; t, x) = x + \int_{t}^{\tau} v^k(s, z^k(s; t, x)) \, ds, \tag{46}$$

$$u^k = (I - \alpha_k^2 \Delta)^{-1} v^k, \tag{47}$$

$$\text{div } v^k = 0; \qquad v^k(t, x) \mid_{(t,x) \in (-\infty, T] \times \partial\Omega} = 0. \tag{48}$$

In view of the Theorem 3 for each $\alpha_k$ there is a solution $v^k \in W_1$ to the approximative problem (45)–(48). Then for all $\varphi \in V^1$ and a.e. $t \in (-\infty, T)$ the equality holds

$$\langle (v^k)', \varphi \rangle - \int_{\Omega} \sum_{i,j=1}^{n} (\Delta_\alpha^{-1} v^k)_i v_j^k \frac{\partial \varphi_j}{\partial x_i} \, dx + \mu_0 \int_{\Omega} \nabla v^k : \nabla \varphi \, dx +$$

$$+\frac{\mu_1}{\Gamma(1-\zeta)} \left( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v^k)(s, z(v^k)(s; t, x)) \, ds, \mathcal{E}(\varphi) \right)$$

$$= \langle f, \varphi \rangle. \tag{49}$$

Estimates of Lemmas 8 and 9 give us the following convergences: $v^k$ (up to a subsequence) weakly converges to $v$ in $L_2(-\infty, T; V^1)$; $v^k$ converges to $v$ (up to a subsequence) a. e. on $[-l, T] \times \Omega$ for any $l > 0$; $\frac{dv^k}{dt}$ is bounded by the norm of space $L_1(-l, T; V^{-1})$ and converges to $\frac{dv}{dt}$ in the sense distributions on $[-l, T]$ for any $-\infty < l < T$.

Using these convergences, we move to the limit in the equality (49). Let us consider the second term separately.

$$|\langle K(v^k), \varphi \rangle - \langle K(v), \varphi \rangle| = \left| \int_\Omega \sum_{i,j=1}^n u_i^k v_j^k \frac{\partial \varphi_j}{\partial x_i} \, dx - \int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i} \, dx \right| =$$

$$= \left| \int_\Omega \sum_{i,j=1}^n \left( (u_i^k - v_i^k)v_j^k + (v_i^k - v_i)v_j^k + (v_j^k - v_j)v_i \right) \frac{\partial \varphi_j}{\partial x_i} \, dx \right| \leqslant$$

$$\leqslant \left| \int_\Omega \sum_{i,j=1}^n (u_i^k - u_i^k + \alpha_k^2 \Delta u_i^k)v_j^k \frac{\partial \varphi_j}{\partial x_i} \, dx \right| + \left| \int_\Omega \sum_{i,j=1}^n (v_i^k - v_i)v_j^k \frac{\partial \varphi_j}{\partial x_i} \, dx \right| +$$

$$+ \left| \int_\Omega \sum_{i,j=1}^n (v_j^k - v_j)v_i \frac{\partial \varphi_j}{\partial x_i} \, dx \right|.$$

We estimate each term separately. In the first term, using Holder's inequality, as well as the continuity of the embedding $V^1 \subset L_4(\Omega)$, for all $\varphi \in V^1$ we obtain

$$\left| \int_\Omega \sum_{i,j=1}^n \alpha_k^2 \Delta u_i^k v_j^k \frac{\partial \varphi_j}{\partial x_i} \, dx \right| \leqslant$$

$$\leqslant \alpha_k \sum_{i,j=1}^n \left( \int_\Omega |\alpha_k \Delta u_i^k|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \left| v_j^k \frac{\partial \varphi_j}{\partial x_i} \right|^2 \, dx \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \alpha_k \sum_{i,j=1}^n \|\alpha_k \Delta u_i^k\|_{L_2(\Omega)} \|v_j^k\|_{L_4(\Omega)} \left\| \frac{\partial \varphi_j}{\partial x_i} \right\|_{L_4(\Omega)} \leqslant$$

$$\leqslant C_{30} \alpha_k \sum_{i,j=1}^n \|\alpha_k \Delta u_i^k\|_{L_2(\Omega)} \|v_j^k\|_{V^1} \left\| \frac{\partial \varphi_j}{\partial x_i} \right\|_{V^1} \leqslant$$

$$\leqslant C_{31} \alpha_k \|\alpha_k \Delta u^k\|_{L_2(\Omega)} \|v^k\|_{V^1} \|\varphi\|_{V^1}.$$

The remaining terms are estimated in a similar way. Thus,

$$|\langle K(v^k), \varphi \rangle - \langle K(v), \varphi \rangle| \leqslant C_{32}(\alpha_k \|\alpha_k \Delta u^k\|_{L_2(\Omega)} \|v^k\|_{V^1} \|\varphi\|_{V^1} +$$

$$+ \|v^k - v\|_{L_4(\Omega)} \|v^k\|_{L_4(\Omega)} \|\varphi\|_{V^1} + \|v^k - v\|_{L_4(\Omega)} \|v\|_{L_4(\Omega)} \|\varphi\|_{V^1}) \leqslant$$

$$\leqslant C_{33}(\alpha_k \|\alpha_k \Delta u^k\|_{L_2(\Omega)} \|v^k\|_{V^1} + \|v^k - v\|_{L_4(\Omega)} \|v^k\|_{V^1} +$$

$$+ \|v^k - v\|_{L_4(\Omega)} \|v\|_{V^1}) \|\varphi\|_{V^1}.$$

Hence,

$$\|K(v^k) - K(v)\|_{V^{-1}} \leqslant C_{33}(\alpha_k \|\alpha_k \Delta u^k\|_{L_2(\Omega)} \|v^k\|_{V^1} +$$
$$+ \|v^k - v\|_{L_4(\Omega)} \|v^k\|_{V^1} + \|v^k - v\|_{L_4(\Omega)} \|v\|_{V^1}).$$

Let's integrate both sides of the last inequality over $t$ in the range from $-\infty$ to $T$. Applying Holder's inequality, we conclude that

$$\int_{-\infty}^{T} \|K(v^k) - K(v)\|_{V^{-1}} \, dt \leqslant \alpha_k C_{21} \Big( \int_{-\infty}^{T} \|\alpha_k \Delta u^k\|_{L_2(\Omega)}^2 \, dt \Big)^{\frac{1}{2}} \times$$
$$\times \Big( \int_{-\infty}^{T} \|v^k\|_{V^1} \, dt \Big)^{\frac{1}{2}} + C_{33} \|v^k - v\|_{L_2(-\infty,T;L_4(\Omega))} \|v^k\|_{L_2(-\infty,T;V^1)} +$$
$$+ C_{33} \|v^k - v\|_{L_2(-\infty,T;L_4(\Omega))} \|v\|_{L_2(-\infty,T;V^1)}. \tag{50}$$

The convergences obtained earlier allow us to conclude that the last two terms in the inequality (49) tend to zero.

Let us recall that

$$\|v\|_{V^1}^2 = \|u - \alpha^2 \Delta u\|_{V^1}^2 = \|u\|_{V^1}^2 + 2\|\alpha \Delta u\|_{L_2(\Omega)}^2 + \alpha^4 \|u\|_{V^3}^2.$$

Therefore, by estimate (25) it follows that

$$\int_{-\infty}^{T} \|\alpha \Delta u\|_{L_2(\Omega)}^2 \, dt \leqslant \frac{1}{2} \int_{-\infty}^{T} \|v\|_{V^1}^2 \, dt \leqslant C_{21} \|f\|_{L_2(-\infty,T;V^{-1})}.$$

For $\alpha_k \to 0$ we get

$$\int_{-\infty}^{T} \|K(v^k) - K(v)\|_{V^{-1}} \, dt \leqslant \alpha_k C_{33} C_{21} \|f\|_{L_2(-\infty,T;V^{-1})} \to 0.$$

Thus, passing in equality (49) to the limit as $k \to \infty$ we obtain that the limit function $v$ satisfies the equality

$$\langle (v)', \varphi \rangle - \int_{\Omega} \sum_{i,j=1}^{n} v_i v_j \frac{\partial \varphi_j}{\partial x_i} \, dx + \mu_0 \int_{\Omega} \nabla v : \nabla \varphi \, dx +$$
$$+ \frac{\mu_1}{\Gamma(1-\zeta)} \Big( \int_{-\infty}^{t} e^{\frac{-(t-s)}{\lambda}} (t-s)^{-\zeta} \mathcal{E}(v)(s, z(v)(s; t, x)) \, ds, \mathcal{E}(\varphi) \Big) = \langle f, \varphi \rangle.$$

Hence, $v$, according to the Definition 6, is a weak solution to the initial–boundary value problem (41)–(43) for the function $\varphi \in V^1$. However, we note that the function $v$, due to the obtained convergences, satisfies the estimates obtained above. Hence, each term of the last equality holds for an arbitrary function $\varphi \in V^1$. Thus, we proof the convergence of weak solutions of the alpha model (1)–(5) to weak solutions of the initial–boundary value problem (41)–(43).

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