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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

*Том 22, № 2, стр. 144–145 (2025)* УДК 510.67
https://doi.org/10.33048/semi.2025.20.??? MSC 03C30,
03C50

# KINDS OF PRESERVATIONS FOR PROPERTIES

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**Abstract:** We study possibilities of preservations for properties, their links and related connections between semantic and syntactic ones, both in general and as characterizations for subalgebras, congruence relations, Henkin construction, Tarski-Vaught test, some graphs, and graded structures. Traces for types and related objects are studied, too.

**Keywords:** property, preservation by a formula, preservation by a type, trace of type.

## 1 Introduction

We study various properties [1] which are preserved under given conditions. These preservations generalize the notion of $(p, q)$-preserving formula [2, 3, 4] and its variations for correspondent type-definable sets.

The paper is organized as follows. In Section 2, we introduce various kinds of preservation and describe various related properties. Traces of types are introduced in terms preservations and their general properties are found. In Section 4, subalgebras and congruence relations are characterized in terms of preservations. In Section 5, connections of model constructions, including

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Rajabov, T.E., Sudoplatov, S.V., Kinds of preservations for properties.

This work was carried out in the framework of the State Contract of Sobolev Institute of Mathematics, Project No. FWNF-2022-0012.

*Поступила 1 января 2025 г., опубликована 31 декабря 2025 г.*

Henkin construction, are described in terms of preservations of formulae. Vaught-Tarski text is characterized in terms of preservations in Section 6. Multipartite and related graphs are connected with preservations in Section 7. In Section 8, we study preservations of properties in graphs with respect to distances. Preservation properties for algebraic constructions and decompositions are discussed in Section 9.

Throughout we use the standard model-theoretic notions and notations.

## 2 Preservations of properties by types and formulae, their general properties

**Definition.** Let $\mathcal{M}$ be a structure, $P_1 \subseteq M^{k_1}, \dots, P_n \subseteq M^{k_n}$, $Q \subseteq M^m$ be properties, $\Phi = \Phi(\overline{x}_1, \dots, \overline{x}_n, \overline{y})$ be a type with $l(\overline{x}_1) = k_1, \dots, l(\overline{x}_n) = k_n$, $l(\overline{y}) = m$. We say that the tuple $(P_1, \dots, P_n, Q)$ is (*totally*) $\Phi$-*preserved*, or $\Phi$ is (*totally*) $(P_1, \dots, P_n, Q)$-*preserving*, if for any $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$,

$$\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \subseteq Q.$$

Here we also say on *universal* $\Phi$- and $(P_1, \dots, P_n, Q)$-preservation.

If $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \subseteq Q$ for some $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, then we say that $(P_1, \dots, P_n, Q)$ is *existentially* $\Phi$-*preserved*, or $\Phi$ is *existentially* $(P_1, \dots, P_n, Q)$-*preserving*.

If $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \cap Q \neq \emptyset$ for some $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$ then we say that $(P_1, \dots, P_n, Q)$ is $\exists$-*partially* $\Phi$-*preserved*, or $\Phi$ is $\exists$-*partially* $(P_1, \dots, P_n, Q)$-*preserving*. If this property holds for any $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, we say that $(P_1, \dots, P_n, Q)$ is $\forall$-*partially* $\Phi$-*preserved*, or $\Phi$ is $\forall$-*partially* $(P_1, \dots, P_n, Q)$-*preserving*.

We say that the tuple $(P_1, \dots, P_n, Q)$ is $\exists$-*partially* $\Phi$-*non-preserved*, or $\Phi$ is $\exists$-*partially* $(P_1, \dots, P_n, Q)$-*non-preserving*, if $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \cap \overline{Q} \neq \emptyset$ for some $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, where $\overline{Q} = M^m \setminus Q$. If this property holds for any $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, we say that $(P_1, \dots, P_n, Q)$ is $\forall$-*partially* $\Phi$-*non-preserved*, or $\Phi$ is $\forall$-*partially* $(P_1, \dots, P_n, Q)$-*non-preserving*. In the latter case we also say that the tuple $(P_1, \dots, P_n, Q)$ is *totally* $\Phi$-*non-preserved*, or $\Phi$ is *totally* $(P_1, \dots, P_n, Q)$-*non-preserving*.

If $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \subseteq \overline{Q}$ for some $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, then we say that $(P_1, \dots, P_n, Q)$ is *existentially* $\Phi$-*disjoint*, or $\Phi$ is *existentially* $(P_1, \dots, P_n, Q)$-*disjointing*.

If $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \subseteq \overline{Q}$ for any $\overline{a}_1 \in P_1, \dots, \overline{a}_n \in P_n$, then we say that $(P_1, \dots, P_n, Q)$ is *totally* $\Phi$-*disjoint* or *universally* $\Phi$-*disjoint*, or $\Phi$ is *totally* $(P_1, \dots, P_n, Q)$-*disjointing*, or *universally* $(P_1, \dots, P_n, Q)$-*disjointing*.

If $\Phi$ is a singleton $\{\varphi\}$ then totally/existentially/partially $\Phi$-(non-)preserved/disjoint tuples are called *totally/existentially/partially* $\varphi$-*(non-)preserved/disjoint*, respectively, and $\varphi$ is *totally/existentially/partially* $(P_1, \dots, P_n, Q)$-*(non-)preserving/disjointing*.

If $P_1 = \ldots = P_n = Q$ then $(P_1, \dots, P_n, Q)$-preserving type $\Phi$ is called $(P_1, \dots, P_n, Q)$-*idempotent* and $(P_1, \dots, P_n, Q)$ is $\Phi$-*idempotent*.

If $\Phi=\{\varphi\}$ then we replace $\Phi$ by $\varphi$ in the definitions above.

By the definition we have the following properties:

**Proposition 1.** 1. *If a type $\Phi$ is totally $(P_1,\dots,P_n,Q)$-preserving/disjointing and $P_1\times\ldots\times P_n\neq\emptyset$ then $\Phi$ is existentially $(P_1,\dots,P_n,Q)$-preserving/disjointing.*

2. *If a type $\Phi$ is $\forall$-partially $(P_1,\dots,P_n,Q)$-(non-)preserving and $P_1\times\ldots\times P_n\neq\emptyset$ then $\Phi$ is $\exists$-partially $(P_1,\dots,P_n,Q)$-(non-)preserving.*

The following example shows that the converse for the items of Proposition 1 does not hold in general.

**Example 1.** 1. Let $P_1,P_2,Q$ be nonempty unary predicates such that $P_1$ and $Q$ are connected by a bijection $f_0$, and a function $f\supset f_0$ maps $P_2$ into the complement $\overline{Q}$ of $Q$. Then the formula $f(x)\approx y$ is totally $(P_1,Q)$-preserving and totally $(P_2,Q)$-disjointing while this formula is existentially $(P_1\cup P_2,Q)$-preserving and not totally $(P_1\cup P_2,Q)$-preserving.

2. Let the previous example be extended by a constant $c\in\overline{Q}$ and the relation $f_0$ be extended by the set $\{\langle a,c\rangle\mid a\in P_1\}$. Then this extended relation $R(x,y)$ witnesses the $\forall$-partial $(P_1,Q)$-preservation. Extending $P_1$ by new element $c'$ and $R$ by the pair $\langle c',c\rangle$ we obtain the witness of $\exists$-partial $(P_1\cup\{c'\},Q)$-preservation which is not $\forall$-partial $(P_1\cup\{c'\},Q)$-preserving. Similarly there is a $\exists$-partial $(P_1\cup\{c'\},Q)$-non-preservation which is not $\forall$-partial $(P_1\cup\{c'\},Q)$-non-preserving.

**Proposition 2.** *For any type $\Phi$ and definable or non-definable relations $P_1,\dots,P_n,Q$ in a structure $\mathcal{M}$ the following conditions are equivalent:*

1) *$\Phi$ is totally/existentially $(P_1,\dots,P_n,Q)$-preserving;*

2) *$\Phi$ is totally/existentially $(P_1,\dots,P_n,\overline{Q})$-disjointing.*

Proof follows by the definition.

**Proposition 3.** *For any $(P_1,\dots,P_n,Q)$ and $\Phi$, $(P_1,\dots,P_n,Q)$ is totally/existentially/partially $\Phi$-(non-)preserved/disjoint iff $(P_1\times\ldots\times P_n,Q)$ is totally/existentially/partially $\Phi$-(non-)preserved/disjoint, where $(\overline{x}_1,\dots,\overline{x}_n,\overline{y})$ in $\Phi$ is replaced by $(\overline{x}_1\hat{}\ldots\hat{}\overline{x}_n,\overline{y})$.*

Proof follows since the condition $\overline{a}_1\in P_1,\dots,\overline{a}_n\in P_n$ is equivalent to the condition $(\overline{a}_1\hat{}\ldots\hat{}\overline{a}_n)\in P_1\times\ldots\times P_n$. □

The following assertion shows that type-definable sets can be described in terms of preservation.

**Proposition 4.** *For any type $\Phi=\Phi(\overline{x}_1,\dots,\overline{x}_n,\overline{y})$ and tuples $\overline{a}_1,\dots,\overline{a}_n$ forming the set $Q=\Phi(\overline{a}_1,\dots,\overline{a}_n,\mathcal{M})$ the set $Q$ is characterized by the following conditions: $\Phi$ is $(\{\overline{a}_1\},\dots,\{\overline{a}_n\},Q)$-preserving and $\Phi$ is not ($\forall$-) $\exists$-partially $(\{\overline{a}_1\},\dots,\{\overline{a}_n\},\overline{Q})$ for the complement $\overline{Q}=M^{l(\overline{y})}\setminus Q$. In particular, for a formula $\varphi=\varphi(\overline{x}_1,\dots,\overline{x}_n,\overline{y})$ and tuples $\overline{a}_1,\dots,\overline{a}_n$ forming the set $Q=\varphi(\overline{a}_1,\dots,\overline{a}_n,\mathcal{M})$ the set $Q$ is characterized by the following conditions: $\varphi$ is $(\{\overline{a}_1\},\dots,\{\overline{a}_n\},Q)$-preserving and $\neg\varphi$ is $(\{\overline{a}_1\},\dots,\{\overline{a}_n\},\overline{Q})$-preserving.*

Proof follows by the definition. □

**Remark 1.** Using Proposition 4 we have the following representation of a type-definable set $Q = \Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M})$:

$$Q = \{\overline{b} \mid \Phi(\overline{x}_1, \dots, \overline{x}_n, \overline{y}) \text{ is } (\forall\text{-})\ \exists\text{-partially } (\{\overline{a}_1\}, \dots, \{\overline{a}_n\}, \{\overline{b}\})\text{-preserving}\}.$$

**Definition** [2, 3, 4]. Let $T$ be a complete theory, $\mathcal{M} \models T$. Consider types $p(\overline{x}), q(\overline{y}) \in S(\emptyset)$, realized in $\mathcal{M}$, and all $(p, q)$-*preserving* formulae $\varphi(\overline{x}, \overline{y})$ of $T$, i. e., formulae for which there is $\overline{a} \in M$ such that $\models p(\overline{a})$ and $\varphi(\overline{a}, \overline{y}) \vdash q(\overline{y})$. For each such a formula $\varphi(\overline{x}, \overline{y})$, we define a relation $R_{p,\varphi,q} \rightleftharpoons \{(\overline{a}, \overline{b}) \mid \mathcal{M} \models p(\overline{a}) \wedge \varphi(\overline{a}, \overline{b})\}$. If $(\overline{a}, \overline{b}) \in R_{p,\varphi,q}$, then the pair $(\overline{a}, \overline{b})$ is called a $(p, \varphi, q)$-*arc*.

The definition above is naturally spread for types $\Phi(\overline{x}, \overline{y})$ replacing the formula $\varphi(\overline{x}, \overline{y})$.

**Proposition 5.** *For any types $p(\overline{x}), q(\overline{y}) \in S(\emptyset)$ and a formula $\varphi(\overline{x}, \overline{y})$ the following conditions are equivalent:*
1) *the formula $\varphi$ is $(p, q)$-preserving;*
2) *the pair $(p(\mathcal{M}),\ q(\mathcal{M}))$ is totally $\varphi$-preserved;*
3) *the pair $(p(\mathcal{M}),\ q(\mathcal{M}))$ is existentially $\varphi$-preserved.*

Proof. 1) $\Leftrightarrow$ 3) and 2) $\Rightarrow$ 3) follow by the definition.

3) $\Rightarrow$ 2) holds since any realizations $\overline{a}$ and $\overline{a}'$ of the complete type $p(\overline{x})$ are connected by an automorphism $f$ of an elementary extension of $\mathcal{M}$ implying that $\varphi(\overline{a}, \overline{y}) \vdash q(\overline{y})$ iff $\varphi(\overline{a}', \overline{y}) \vdash q(\overline{y})$, i.e. $(\overline{a}, \overline{b}) \in p(\mathcal{M}) \times q(\mathcal{M})$ is a $(p, \varphi, q)$-arc iff $(\overline{a}', f(\overline{b}))$ is a $(p, \varphi, q)$-arc. □

By Propositions 3 and 5 we observe a connection between syntactic and semantic conditions of preservation.

The following assertion allows to reduce the type preservation for type-definable properties till a formulaic one.

**Proposition 6.** *If some conjunction of formulae in a type $\Phi$ is totally/existentially $(P_1, \dots,\ P_n, Q)$-preserving/disjoint then $\Phi$ is totally/existentially $(P_1, \dots, P_n, Q)$-preserving/disjoint.*

Proof. If some conjunction $\varphi_0$ of formulae in $\Phi$ is totally/existentially $(P_1, \dots, P_n, Q)$-preserving/disjoint then $\Phi$ is totally/existentially $(P_1, \dots, P_n, Q)$-preserving/disjoint since the set of solutions for $\Phi$ is a subset of the set of solutions for $\varphi_0$. □

The following assertion shows that the converse of Proposition 6 can fail for type-definable properties.

**Proposition 7.** *Let $p(\overline{x})$ and $q(\overline{y})$ be nonisolated types forced by their sets of formulae $\psi_i(\overline{x})$, $\chi_i(\overline{y})$, respectively, $i \in \omega$, such that*

$$\vdash (\psi_i(\overline{x}) \to \psi_j(\overline{x})) \wedge \exists \overline{x}(\psi_j(\overline{x}) \wedge \neg\psi_i(\overline{x})),$$

$$\vdash (\chi_i(\overline{x}) \to \chi_j(\overline{x})) \wedge \exists \overline{x}(\chi_j(\overline{x}) \wedge \neg\chi_i(\overline{x})),$$

*for any $i < j \in \omega$; $\Phi(\overline{x},\overline{y})$ consist of formulae $\phi_i(\overline{x},\overline{y})$ such that for any realization $\overline{a} \models p(\overline{x})$ and any $j \in \omega$,*

$$\models \exists\overline{y}(\phi_i(\overline{x},\overline{y}) \wedge \chi_j(\overline{y}) \wedge \neg\chi_{j+1}(\overline{y}))$$

*iff $j \geq y$. Then for any saturated model $\mathcal{M}$ of the given theory, $\Phi$ is totally $(p(\mathcal{M}), q(\mathcal{M}))$-preserving whereas each finite conjunction $\theta$ of formulae in $\Phi$ is $\forall$-partially $(p(\mathcal{M}), q(\mathcal{M}))$-preserving and $\forall$-partially $(p(\mathcal{M}), q(\mathcal{M}))$-non-preserving, in particular, $\theta$ is both not totally $(p(\mathcal{M}), q(\mathcal{M}))$-preserving.*

Similarly to Proposition 6 we have the following:

**Proposition 8.** *If a type $\Phi$ is $\alpha$-partially $(P_1,\dots,P_n,Q)$-(non-)preserving, where $\alpha \in \{\forall,\exists\}$, then any conjunction of formulae in a type $\Phi$ is $\alpha$-partially $(P_1,\dots,P_n,Q)$-(non-)preserving.*

Proof follows immediately by the definition. □

**Proposition 9.** *If properties $P_1,\dots,P_n,Q$ are type-definable in a saturated structure then a type $\Phi$ is partially $(P_1,\dots,P_n,Q)$-preserving/disjoint iff some conjunction of formulae in $\Phi$ is totally/existentially/partially $(P_1,\dots,$ $P_n$, $Q)$-preserving/disjoint.*

We have the following basic properties for variations of preservation:

**Proposition 10.** (Monotony) *If $(P_1,\dots,P_n,Q)$ is $\Phi$-preserved, $P_1 \supseteq P'_1$, $\dots, P_n \supseteq P'_n$, $Q \subseteq Q'$, $\Phi \subseteq \Phi'$ then $(P'_1,\dots,P'_n,Q')$ is $\Phi'$-preserved.*

Proof follows immediately by the definition. □

For types $\Phi$ and $\Psi$ we denote by $\Phi \vee \Psi$ the type $\{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\}$, and by $\Phi \wedge \Psi$ the type $\Phi \cup \Psi$, if it is consistent. Since $\Phi \vee \Psi$ (respectively, $\Phi \wedge \Psi$) has the set of solutions represented as the union (intersection) of the type-definable sets for $\Phi$ and $\Psi$, we obtain the following:

**Proposition 11.** (Union) *If $(P_1,\dots,P_n,Q)$ is $\Phi$-preserved and $(P_1,\dots,P_n,$ $Q')$ is $\Psi$-preserved, with $Q, Q' \subseteq M^m$, then $\Phi \vee \Psi$ is $(P_1,\dots,P_n,Q \cup Q')$-preserving and $\Phi \wedge \Psi$, if it is consistent, is $(P_1,\dots,P_n,Q \cap Q')$-preserving.*

Proposition 11 immediately implies the following:

**Corollary 1.** *If there is a $(P_1,\dots,P_n,Q)$-preserving type $\Phi$ then the family $Z_\Phi(P_1,\dots,P_n,Q)$ of all sets of solutions, in a given structure $\mathcal{M}$, for $(P_1,\dots,$ $P_n,Q)$-preserving types, which are contained in $\Phi$, forms a distributive lattice $\langle Z_\Phi(P_1,\dots,P_n,Q);\cup,\cap\rangle$ with the least element $\Phi(\mathcal{M})$.*

**Remark 2.** The lattice $\langle Z_\Phi(P_1,\dots,P_n,Q);\cup,\cap\rangle$ can have or not have the greatest element $\Psi(\mathcal{M})$ depending on the existence of greatest union of type-definable subsets $\Phi(\overline{a}_1,\dots,\overline{a}_n,\mathcal{M})$ of $Q$, where $\overline{a}_1 \in P_1,\dots,\overline{a}_n \in P_n$.

Besides, if the condition for subsets of $\Phi$ is omitted then the union of considered types can be inconsistent violating that $\cap$ is an operation here.

**Remark 3.** The notions and results above can be naturally spread to the families $\Phi$ of formulae with unboundedly many free variables. So we can assume that $\Phi$ has arbitrarily many free variables. In particular, considering sets $\Phi$ of formulae whose variables belong to a countable set, one can admit countably many free variables.

## 3 Traces of types

**Definition.** (cf. [5]) Let $\Phi = \Phi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ be a type with consistent $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$, where $\overline{a}_1, \ldots, \overline{a}_n$ be tuples in a model $\mathcal{M}$ of a given theory $T$. A *trace* of $\Phi$ with respect to $(\overline{a}_1, \ldots, \overline{a}_n)$, or a $\Phi$-*trace*, is a family $\{Q_i \subseteq M^{l(\overline{y})} \mid i \in I\}$ such that $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq \bigcup\limits_{i \in I} Q_i$ and $\Phi$ is ($\forall$-) $\exists$-partially $(\{\overline{a}_1\}, \ldots, \{\overline{a}_n\}, Q_i)$-preserving for each $i \in I$.

If the sets $Q_i$ are pairwise disjoint then the $\Phi$-trace $\{Q_i \mid i \in I\}$ is *disjoint*, too.

The $\Phi$-trace $\{Q_i \mid i \in I\}$ is called $A$-(*type*-)*definable* if each $Q_i$ is a (type-)definable set, which are defined over $A$. We say on the (type-)definability of the trace is it is $A$-(*type*-)*definable* for some $A$.

Notice that any type $\Phi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ has a type-definable trace

$$\{\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M})\}$$

over the set $\cup\overline{a}_1 \cup \ldots \cup \cup\overline{a}_n$, which is a singleton. Similarly each type $\Theta(\overline{y})$ with $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \vdash \Theta(\overline{y})$ produce a singleton-trace $\{\Theta(\mathcal{M})\}$ for $\Phi$. By the definition all these traces are disjoint.

**Example 2.** Each type $\Psi = \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ has a definable trace $\{\{\overline{b}\} \mid \models \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{b})\}$. Here the trace is $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M})$-definable, and it is $\emptyset$-definable iff $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq \mathrm{dcl}(\emptyset)$.

For a type $\Psi = \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ we denote by $[\Psi]$ the set

$$\{p(\mathcal{M}) \mid p(\overline{y}) \in S^{l(\overline{y})}(\emptyset), \Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \cap p(\mathcal{M}) \neq \emptyset\}.$$

**Proposition 12.** *For any type $\Psi = \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ the family $[\Psi]$ is a disjoint $\emptyset$-type-definable $\Phi$-trace. It is definable iff each type $p(\overline{y})$ for $[\Psi]$ is isolated.*

Proof. If $\overline{b} \in \Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M})$ then $\overline{b} \in p(\mathcal{M})$, where $p(\overline{y}) = \mathrm{tp}(\overline{b})$. Therefore $\Phi$ is ($\forall$-) $\exists$-partially $(\{\overline{a}_1\}, \ldots, \{\overline{a}_n\}, Q_i)$-preserving and $\overline{b} \in \bigcup[\Psi]$. Hence $[\Psi]$ is a trace of $\Phi$ with respect to $(\overline{a}_1, \ldots, \overline{a}_n)$. Since the trace $[\Psi]$ collects sets of realizations of complete types, which are pairwise inconsistent, it is disjoint. The trace $[\Psi]$ is type-definable by the definition.

The latter characterization of the definability follows since the non-empty set of realizations of a complete type $q(\overline{y})$ is definable iff $q(\overline{y})$ is isolated by a formula $\varphi(\overline{y})$. Here $q(\mathcal{M} = \varphi(\mathcal{M})$. □

Proposition 12 immediately implies:

**Corollary 2.** *If the model $\mathcal{M}$ is atomic then each $\Phi$-trace $[\Psi]$ is definable.*

Using Ryll-Nardzewski Theorem we have:

**Corollary 3.** *If the model $\mathcal{M}$ of a countable language is saturated then each $\Phi$-trace $[\Psi]$ is definable iff $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical.*

**Proposition 13.** *For any type $\Psi = \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ with a saturated model $\mathcal{M}$ the trace $[\Psi]$ is finite iff $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ forces finitely many type in $S_{\overline{y}}(\emptyset)$.*

Proof. If $[\Psi]$ is finite then the types for $[\Psi]$ are divided by finitely many formulae $\varphi_i = \varphi_i(\overline{y})$, $i < n$, into singletons $[\Psi \cup \{\varphi_i\}]$. This singletons are consistent with unique types $p_i(\overline{y}) \in S(\emptyset)$, since $\mathcal{M}$ is saturated. Hence $\Psi \cup \{\varphi_i\}$ forces $p_i$ for each $i$ and $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ forces $\bigvee\limits_i p_i(\overline{y})$. The converse direction is obvious. □

Proposition 13 immediately implies:

**Corollary 4.** *For any type $\Psi = \Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ with a saturated model $\mathcal{M}$ the trace $[\Psi]$ is a singleton iff $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})$ forces a unique type in $S_{\overline{y}}(\emptyset)$.*

**Corollary 5.** *If a type $\Phi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ forces a $(\mathrm{tp}(\overline{a}_1\hat{}\ldots\hat{}\overline{a}_n), \mathrm{tp}(\overline{b}))$-preserving formula $\varphi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ for some/any $\overline{b} \in \Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M})$ then $[\Phi(\overline{a}_1, \ldots, \overline{a}_n, \overline{y})]$ is a singleton.*

The following example shows that Proposition 13 can fail if the given structure $\mathcal{M}$ is not saturated.

**Example 3.** Let $\mathcal{M}$ be a structure in a signature of unary predicates $P_i$, $i \in \omega$, $R_1$, $R_2$ with two nonempty signature parts $R_1$, $R_2$ such that $R_1 \cup R_2 = M$, $R_1$ is not divided by $P_i$, and the predicates $P_i$ form a substructure on $R_2$ such that these $P_i$ are independent: all $R_2(x) \wedge P_{i_1}^{\delta_1}(x) \wedge \ldots \wedge P_{i_n}^{\delta_n}(x)$ are consistent, $i_1 < \ldots < i_n < \omega$, $\delta_1, \ldots, \delta_n \in \{0, 1\}$, $n \in \omega$. Clearly the formula $R_1(x)$ forces a complete type $p_1(x)$ whereas $R_2(x)$ belongs to continuum-many 1-types $p_2^{\Delta}(x)$, $\Delta \in \{0, 1\}^{\omega}$, containing $R_2(x)$ and $P_i^{\Delta(i)}(x)$, $i \in \omega$. Additionally we assume that for some $\Delta_0 \in \{0, 1\}^{\omega}$ the complete type $p_2^{\Delta_0}(x)$ is omitted in $\mathcal{M}$.

Now we consider the type $\Phi(y)$ formed by formulae $R_1(y) \vee \varphi(y)$, $\varphi(y) \in p_2^{\Delta_0}(y)$. We have $[\Phi(y)] = \{p_1(\mathcal{M})\}$ whereas $\Phi(y)$ does not force unique type in $S_{\overline{y}}(\emptyset)$.

We also notice that $\Phi(y)$ does not force $(\mathrm{tp}(\emptyset), p_1)$-preserving formulae.

**Example 4.** Let $\Gamma = \langle M; R\rangle$ be a graph with $R \neq \emptyset$. Consider the type $\Phi = \{R(x, y)\}$ and $\Psi = \Phi(a, y)$ for an element $a \in M$. We have $[\Psi] \neq \emptyset$ iff $\Gamma \models \exists y R(a, y)$, and nonempty $[\Psi]$ is finite iff $R(a, \Gamma)$ has finitely many $a$-orbits with respect to the automorphism groups in elementary extensions of $\Gamma$. Clearly, $[\Phi(a, y)] = [\Phi(b, y)]$ for any $b$ with $\mathrm{tp}(b) = \mathrm{tp}(a)$, but if $\mathrm{tp}(b) \neq \mathrm{tp}(a)$ then the equality $[\Phi(a, y)] = [\Phi(b, y)]$ can hold or fail. Indeed, If $R(a, \Gamma) \cap$

$R(b,\Gamma)$ is a singleton $\{d\}$ then $[\Phi(a,y)]$ and $[\Phi(b,y)]$ are singletons, too, with $[\Phi(a,y)]=[\Phi(b,y)]=\{\mathrm{tp}(d)\}$. Respectively, $[\Phi(a,y)]\neq[\Phi(b,y)]$ if, for instance, $R(a,\Gamma)$ consists of elements without loops and $R(b,\Gamma)$ has elements with loops.

Now we spread the notion of trace $[\Psi]$ above for a type $\Phi=\Phi(\overline{x}_1,\ldots,\overline{x}_n,\overline{y})$ with respect to a tuple $(P_1,\ldots,P_n)$, where $P_i\subseteq M^{l(\overline{x}_i)}$, $i=1,\ldots,n$, in the following way.

We put $[\Phi]_{(P_1,\ldots,P_n)}=\bigcup\{[\Phi(\overline{a}_1,\ldots,\overline{a}_n,\overline{y})]\mid\overline{a}_1\in P_1,\ldots,\overline{a}_n\in P_n\}$. We write $[\Phi]$ instead of $[\Phi]_{(P_1,\ldots,P_n)}$ if all $P_i$ cover all admissible sets of tuples for $\Phi$, i.e. all tuples $\overline{a}_1,\ldots,\overline{a}_n$ with consistent $\Phi(\overline{a}_1,\ldots,\overline{a}_n,\overline{y})$ belong to correspondent $P_1,\ldots,P_n$.

By the definition, we have the following monotony property.

**Proposition 14.** *For any type $\Phi$ and relations $P_i\subseteq P'_i$, $i\leq n$, in a structure $\mathcal{M}$,*

$$[\Phi]_{(P_1,\ldots,P_n)}\subseteq[\Phi]_{(P'_1,\ldots,P'_n)}.$$

**Lemma 1.** *For any type $\Phi(\overline{x}_1,\ldots,\overline{x}_n,\overline{y})$ and tuples $\overline{a}_1,\ldots,\overline{a}_n$, $\overline{b}_1,\ldots,\overline{b}_n$ with $l(\overline{a}_i)=l(\overline{b}_i)=l(\overline{x}_i)$, $i=1,\ldots,n$, if $\mathrm{tp}(\overline{a}_1\hat{\ }\ldots\hat{\ }\overline{a}_n)=\mathrm{tp}(\overline{b}_1\hat{\ }\ldots\hat{\ }\overline{b}_n)$ then*

$$[\Phi(\overline{a}_1,\ldots,\overline{a}_n,\overline{y})]=[\Phi(\overline{b}_1,\ldots,\overline{b}_n,\overline{y})].$$

Proof. Since $\mathrm{tp}(\overline{a}_1\hat{\ }\ldots\hat{\ }\overline{a}_n)=\mathrm{tp}(\overline{b}_1\hat{\ }\ldots\hat{\ }\overline{b}_n)$ there is an automorphism $f$ of an elementary extension of given structure such that $f(\overline{a}_i)=\overline{b}_i$, $i=1,\ldots,n$. This automorphism moves $\Phi(\overline{a}_1,\ldots,\overline{a}_n,\overline{y})$ onto $\Phi(\overline{b}_1,\ldots,\overline{b}_n,\overline{y})$. Since all automorphisms preserves types over $\emptyset$, we have $[\Phi(\overline{a}_1,\ldots,\overline{a}_n,\overline{y})]=[\Phi(\overline{b}_1,\ldots,\overline{b}_n,\overline{y})]$. □

In view of Lemma 1 and the monotony property we have:

**Corollary 6.** *For any type $\Phi=\Phi(\overline{x},\overline{y})$, a tuple $\overline{a}$, its type $p(\overline{x})=\mathrm{tp}(\overline{a})$, and $\emptyset\neq P\subseteq p(\mathcal{M})$, $[\Phi(\overline{a},\overline{y})]=[\Phi]_P$.*

**Corollary 7.** *For any type $\Phi=\Phi(\overline{x},\overline{y})$, a tuple $\overline{a}$, its type $p(\overline{x})=\mathrm{tp}(\overline{a})$, a type $q(\overline{y})\in S(emptyset)$ and $\emptyset\neq P\subseteq p(\mathcal{M})$, where $\mathcal{M}$ is saturated, $\Phi$ is $(p,q)$-preserving iff $[\Phi(\overline{a},\overline{y})]=[\Phi]_P=\{q(\overline{y})\}$.*

**Definition.** [6] A family $p_1,\ldots,p_n$ of complete 1-types over $A$ is *weakly orthogonal* over $A$ if every $n$-tuple $\langle a_1,\ldots,a_n\rangle\in p_1(\mathcal{M})\times\ldots\times p_n(\mathcal{M})$ satisfies the same type over $A$. Here we omit $A$ if it is empty.

In view of Corollary 6 we have:

**Corollary 8.** *For any type $\Phi=\Phi(x_1,\ldots,x_n,\overline{y})$ containing $p_1(x_1)\cup\ldots\cup p_n(x_n)$ for weakly orthogonal family $p_1(x_1),\ldots,p_n(x_n)$, and for any tuple $\langle a_1,\ldots,a_n\rangle$, $[\Phi(a_1,\ldots,a_n,\overline{y})]=[\Phi]_{(p_1(\mathcal{M}),\ldots,p_n(\mathcal{M}))}$.*

For types $\Phi_i = \Phi_i(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_i)$, $i = 1, \dots, m$, $\Psi = \Psi(\overline{y}_1, \dots, \overline{y}_n, \overline{z})$ we denote the type $S(\Phi_1, \dots, \Phi_m, \Psi)$ consisting of all formulae

$$\exists \overline{y}_1, \dots, \overline{y}_m(\varphi_1(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_1) \wedge \dots \varphi_m(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_m) \wedge \psi(\overline{y}_1, \dots, \overline{y}_m, \overline{z})), \tag{1}$$

where $\varphi_i(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_i) \in \Phi_i$, $i = 1, \dots, m$, $\psi(\overline{y}_1, \dots, \overline{y}_m, \overline{z}) \in \Psi$.

Notice that if the types $\Phi_1, \dots, \Phi_m, \Psi$ define operations $f_1, \dots, f_m, g$, respectively, then the type $S(\Phi_1, \dots, \Phi_m, \Psi)$ defines the superposition

$$g(f_1(\overline{x}_1, \dots, \overline{x}_n), \dots, f_m(\overline{x}_1, \dots, \overline{x}_n), \dots).$$

It correspond to the singleton trace with respect to the arguments $\overline{a}_1, \dots, \overline{a}_n$ for tuples $\overline{x}_1, \dots, \overline{x}_n$ of variables.

In general case the type $S(\Phi_1, \dots, \Phi_m, \Psi)$ defines the trace as a superposition of traces for $\Phi_1, \dots, \Phi_m, \Psi$, i.e. the superposition for operations can be spread for type-definable relations:

**Theorem 1.** *For any types $\Phi_i = \Phi_i(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_i)$, $i = 1, \dots, m$, $\Psi = \Psi(\overline{y}_1, \dots, \overline{y}_n, \overline{z})$ and tuples $\overline{a}_1, \dots, \overline{a}_n$ for $\overline{x}_1, \dots, \overline{x}_n$, respectively, the trace $[S(\Phi_1, \dots, \Phi_m, \Psi)(\overline{a}_1, \dots, \overline{a}_n, \overline{z})]$, for a saturated structure $\mathcal{M}$, consists of all types $p(\overline{z}) \in S^{l(\overline{z})}(\emptyset)$ consistent with $\Psi(\overline{b}_1, \dots, \overline{b}_m, \overline{z})$, where $\mathrm{tp}(\overline{b}_i) \in [\Phi_i(\overline{a}_1, \dots, \overline{a}_n, \overline{y}_i)]$, $i = 1, \dots, m$.*

Proof. Let $p(\overline{z}) \in S^{l(\overline{z})}(\emptyset)$ be consistent with $\Psi(\overline{b}_1, \dots, \overline{b}_m, \overline{z})$, where $\mathrm{tp}(\overline{b}_i) \in [\Phi_i(\overline{a}_1, \dots, \overline{a}_n, \overline{y}_i)]$, $i = 1, \dots, m$, and $\overline{c} \in \Psi(\overline{b}_1, \dots, \overline{b}_n, \mathcal{M}) \cap p(\mathcal{M})$. Then the tuples $\overline{b}_1, \dots, \overline{b}_m$ witness the existences in the formulae (1). Hence $p(\overline{z}) \in [S(\Phi_1, \dots, \Phi_m, \Psi)(\overline{a}_1, \dots, \overline{a}_n, \overline{z})]$.

Conversely, if $p(\overline{z}) \in [S(\Phi_1, \dots, \Phi_m, \Psi)(\overline{a}_1, \dots, \overline{a}_n, \overline{z})]$ then by compactness there are tuples $\overline{b}_1, \dots, \overline{b}_m$ witnessing common $\overline{y}_1, \dots, \overline{y}_m$ in the formulae (1) and these tuples can be chosen in $\mathcal{M}$ since it is saturated. Hence $p(\overline{z}) \in S^{l(\overline{z})}(\emptyset)$ is consistent with $\Psi(\overline{b}_1, \dots, \overline{b}_m, \overline{z})$, where $\mathrm{tp}(\overline{b}_i) \in [\Phi_i(\overline{a}_1, \dots, \overline{a}_n, \overline{y}_i)]$, $i = 1, \dots, m$. □

**Remark 4.** In view of Corollary 8 and Theorem 1 the trace

$$[S(\Phi_1, \dots, \Phi_m, \Psi)(\overline{x}_1, \dots, \overline{x}_n, \overline{z})]$$

is represented as the union of all traces $[S(\Phi_1, \dots, \Phi_m, \Psi)(\overline{a}_1, \dots, \overline{a}_n, \overline{z})]$, which are described in terms of superpositions (1).

## 4 Preservations of properties by special formulae and types

**Remark 5.** For a type $\Phi = \Phi(\overline{y})$ a tuple $(P, \dots, P_n, Q)$ is $\Phi$-preserved iff $Q \supseteq \Phi(\mathcal{M})$, i.e. $Q$ contains that type-definable set. Similarly, by the definition, in general case, $Q$ contains type-definable sets $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M})$ whereas $Q$ may be not type-definable.

If $\Phi$ contains sentences asserting that a binary relation $Q$ is (non, ir)reflexive, (anti)symmetric, transitive then the correspondent type-definable set satisfies these properties. In particular, types $\Phi$ allow to represent equivalence relations, partial and linear orders.

**Proposition 15.** *If $\varphi$ is an atomic formula $f(x_1,\ldots,x_n)\approx y$ then a tuple $(P_1,\ldots,\ P_n,Q)=(P,\ldots,P,P)$ with $\emptyset\neq P\subseteq M$ is $\varphi$-preserved, i.e. $\varphi$-idempotent, iff $P$ is the universe of a subalgebra of a restriction of $\mathcal{M}$ till the signature symbol $f$.*

Proof follows by the definition since the restriction of $\mathcal{M}$ till the signature symbol $f$ has a subalgebra with the universe $P$ iff $P$ is closed under the operation $f$: for any $a_1,\ldots,a_n\in P$, $f(a_1,\ldots,a_n)\in P$. Since the value $f(a_1,\ldots,a_n)$ is unique, the condition $f(a_1,\ldots,a_n)\in P$ means that both $\varphi(x_1,\ldots,x_n,y)$ is totally/$\forall$-partially/$\exists$-partially $(\{a_1\},\ldots,\{a_n\},P)$-preserving, i.e. the tuple $(P,\ldots,P,P)$ is $\varphi$-idempotent. $\square$

Proposition 15 immediately implies:

**Corollary 9.** *If $\Phi$ is the family of all atomic formula $f(x_1,\ldots,x_n)\approx y$ for any functional signature symbol $f$ of a structure $\mathcal{M}$ then a tuple $(P_1,\ldots,$ $P_n,Q)=(P,\ldots,P,P)$ with $\emptyset\neq P\subseteq M$ is $\Phi$-preserved, i.e. $\Phi$-idempotent, iff $P$ is the universe of a substructure of $\mathcal{M}$.*

**Remark 6.** Considering many-sorted structures $\langle\mathcal{M}_i;\Sigma\rangle_{i\in I}$ these structures and their many-sorted substructures can be expressed in terms of preservations using the formulae $M_{i_1}(x_1)\wedge\ldots\wedge M_{i_n}(x_n)\wedge f(x_1,\ldots,x_n)\approx y\wedge M_j(y)$ for many-sorted operations $f$ with $\delta_f=M_{i_1}\times\ldots\times M_{i_n}$ and $\rho_f\subseteq M_j$. It means that in such a case each atomic formula $f(x_1,\ldots,x_n)\approx y$ is $(M_{i_1},\ldots,M_{i_n},M_j)$-preserving iff the functional symbol $f\in\Sigma$ is interpreted by a function $f$: $M_{i_1}\times\ldots\times M_{i_n}\to M_j$,

**Remark 7.** Since the formulae $f(x_1,\ldots,x_n)\approx y$ have unique solutions with respect to values $f(a_1,\ldots,a_n)$ in a structure it does not matter to distinguish between partial and total preservations for these formulae.

**Proposition 16.** *If $\varphi(x_1^1,x_1^2;\ldots;x_n^1,x_n^2;y^1,y^2)$ is a formula*

$$E(y^1,y^2)\wedge f(x_1^1,\ldots,x_n^1)\approx y^1\wedge f(x_1^2,\ldots,x_n^2)\approx y^2$$

*then a tuple $(P_1,\ldots,P_n,\ Q)=(E,\ldots,E,E)$ with an equivalence relation $E\subseteq M^2$ is $\varphi$-preserved, i.e. $\varphi$-idempotent, iff $E$ is a congruence relation of a restriction of $\mathcal{M}$ till the signature symbol $f$.*

Proof follows by the definition since the restriction of $\mathcal{M}$ till the signature symbol $f$ is coordinated with respect to the equivalence relation $E$, i.e. $E$ is a congruence relation iff the conditions $E(a_1^1,a_1^2),\ldots,E(a_n^1,a_n^2)$ imply $E(f(a_1^1,\ldots,a_n^1),f(a_1^2,\ldots,a_n^2))$, which is witnessed by the formula $\varphi$ with respect to the tuple $(P_1,\ldots,P_n,\ Q)=(E,\ldots,E,E)$. $\square$

## 5 Constructions of models of theories on a base of preservations of properties

Let $T_0$ be a consistent theory. Following Henkin construction [8, 9, 10] we extend the signature $\Sigma(T_0)$ by new constants and extend the theory $T_0$ till a complete theory $T$ such that if $\exists y\varphi(y) \in T$ then $\varphi(c) \in T$ for some constant $c$. A *canonical model* $\mathcal{M}$ of $T$ [10] is represented by all infinite equivalence classes $[t] = \{q \mid (q \approx t) \in T\}$, where terms $t, q$ do not have free variables. These equivalence classes are also represented by infinite $[c] = \{c' \mid (c' \approx c) \in T\}$, where $c, c'$ are constant symbols, and it means that each constant symbol $c' \in [c]$ is interpreted in $\mathcal{M}$ by the element $[c]$. Here for any $n$-ary functional symbol $f \in \Sigma(T_0)$, $f([c_1], \ldots, [c_n]) = [c]$ iff $f(c_1, \ldots, c_n) \approx c \in T$, and for any $n$-ary relational symbol $R \in \Sigma(T_0)$, $\models R([c_1], \ldots, [c_n])$ iff $R(c_1, \ldots, c_n) \in T$.

Now we consider a characterization for $\mathcal{M}$ to be a model of $T$ in terms of preservations of properties. Each formula $\varphi(y)$ above is represented in the form $\varphi(c_1, \ldots, c_n, y)$, where $c_1, \ldots, c_n$ are all constant symbols in the formula $\varphi$. The condition $\varphi(c_1, \ldots, c_n, c) \in T$, for appropriate $c$, means that $\varphi(x_1, \ldots, x_n, x)$ is ($\forall$-) $\exists$-partially $(\{[c_1]\}, \ldots, \{[c_n]\}, M)$-preserving.

Thus the canonical model $\mathcal{M}$ for the theory $T$, and its restriction $\mathcal{M} \upharpoonright \Sigma(T_0) \models T_0$ are characterized by the following *preserving condition*: a $\Sigma(T_0)$-formula $\varphi(x_1, \ldots, x_n, x)$ is ($\forall$-) $\exists$-partially $(\{[c_1]\}, \ldots, \{[c_n]\}, M)$-preserving whenever $\exists x\varphi(c_1, \ldots, c_n, x) \in T$.

Thus we have the following:

**Theorem 2.** *For any expansion $\mathcal{M}$ of a model $\mathcal{M}_0$ of a theory $T_0$ by naming all elements by infinitely many constants the following conditions are equivalent:*

(1) $\mathcal{M}$ *satisfies the preserving condition;*

(2) $\mathcal{M} \upharpoonright \Sigma(T_0)$ *satisfies the preserving condition;*

(3) $\mathcal{M}$ *is a canonical model of a completion of* $T_0$.

Proof. (1) $\Leftrightarrow$ (2) holds by the definition since the ()$\forall$-) $\exists$-partial $(\{[c_1]\}, \ldots, \{[c_n]\}, M)$-preservation is considered for $\Sigma(T_0)$-formulae.

(1) $\Rightarrow$ (3) is satisfied since the preserving condition $\varphi([c_1], \ldots, [c_n], \mathcal{M}) \neq \emptyset$ for a $\Sigma(T_0)$-formula $\varphi(x_1, \ldots, x_n, x)$ is transformed syntactically to the formula $\exists x\varphi(c_1, \ldots, c_n, x) \to \varphi(c_1, \ldots, c_n, c)$, where $[c] \in \varphi([c_1], \ldots, [c_n], \mathcal{M})$.

(3) $\Rightarrow$ (2) follows by the arguments above for the preserving condition. $\square$

**Remark 8.** The arguments for the representation of Henkin construction in terms of preservations can be naturally spread for generic constructions [3], both semantic and syntactic, with confirmations of consistent formulae $\exists x\varphi(x)$ by appropriate elements. These possibilities allow to realize links of tuples with respect to arbitrary admissible diagrams and to collect generic structures with given consistent lists of properties.

## 6 Preservations of properties and Tarski-Vaught test

Recall [11, 10] that a substructure $\mathcal{N} = \langle N; \Sigma\rangle$ of a structure $\mathcal{M} = \langle M; \Sigma\rangle$ is called an *elementary substructure* (denoted by $\mathcal{N} \preccurlyeq \mathcal{M}$), if for any formula $\varphi(x_1, \ldots, x_n)$ of the signature $\Sigma$ and any elements $a_1, \ldots, a_n \in N$ the condition $\mathcal{N} \models \varphi(a_1, \ldots, a_n)$ is equivalent to the condition $\mathcal{M} \models \varphi(a_1, \ldots, a_n)$. Here the structure $\mathcal{M}$ is called the *elementary extension* of

$\mathcal{N}$. If $N \neq M$, we write $\mathcal{N} \prec \mathcal{M}$ instead of $\mathcal{N} \preccurlyeq \mathcal{M}$. If $\mathcal{N} \subseteq \mathcal{M}$ and the condition $\mathcal{N} \preccurlyeq \mathcal{M}$ ($\mathcal{N} \prec \mathcal{M}$) fails, we write $\mathcal{N} \not\preccurlyeq \mathcal{M}$ (respectively, $\mathcal{N} \not\prec \mathcal{M}$).

**Theorem 3.** (Tarski-Vaught Test) [11, 10] *Let $\mathcal{N}$ be a substructure of a structure $\mathcal{M}$ in a signature $\Sigma$. Then the following conditions are equivalent:*

(1) *$\mathcal{N}$ is an elementary substructure of $\mathcal{M}$;*

(2) *for any formula $\varphi(x_1, \ldots, x_n, y)$ of the signature $\Sigma$ and any elements $a_1, \ldots, a_n \in N$ if $\mathcal{M} \models \exists y\, \varphi(a_1, \ldots, a_n, y)$ then there is an element $b \in N$ such that $\mathcal{N} \models \varphi(a_1, \ldots, a_n, b)$.*

**Corollary 10.** *Let $\mathcal{N}$ be a substructure of a structure $\mathcal{M}$ in a signature $\Sigma$. Then the following conditions are equivalent:*

(1) *$\mathcal{N}$ is an elementary substructure of $\mathcal{M}$;*

(2) *for any formula $\varphi(x_1, \ldots, x_n, y)$ of the signature $\Sigma$ and any elements $a_1, \ldots, a_n \in N$ if $\mathcal{M} \models \exists y\, \varphi(a_1, \ldots, a_n, y)$ then $\varphi(x_1, \ldots, x_n, y)$ is* ($\forall$-) $\exists$-*partially* ($\{a_1\}, \ldots, \{a_n\}, N$)*-preserving;*

(3) *for any formula $\varphi(x_1, \ldots, x_n, y)$ of the signature $\Sigma$ and any elements $a_1, \ldots, a_n \in N$ either $\varphi(x_1, \ldots, x_n, y)$ is* ($\forall$-) $\exists$-*partially* ($\{a_1\}, \ldots, \{a_n\}, M$)*-disjoint or $\varphi(x_1, \ldots, x_n, y)$ is* ($\forall$-) $\exists$-*partially* ($\{a_1\}, \ldots, \{a_n\}, N$)*-preserving;*

(4) *for any finite type $\Phi(x_1, \ldots, x_n, y)$ of the signature $\Sigma$ and any elements $a_1, \ldots, a_n \in N$ either $\Phi(x_1, \ldots, x_n, y)$ is* ($\forall$-) $\exists$-*partially* ($\{a_1\}, \ldots, \{a_n\}, M$)*-disjoint or $\Phi(x_1, \ldots, x_n, y)$ is* ($\forall$-) $\exists$-*partially* ($\{a_1\}, \ldots, \{a_n\}, N$)*-preserving.*

Proof. Since $\{a_1\}, \ldots, \{a_n\}$ are singletons, the $\forall$-conditions are equivalent to the $\exists$-conditions.

(1) $\Leftrightarrow$ (2) follows by Theorem 3 since the existence of $b \in N$ with $\mathcal{N} \models \varphi(a_1, \ldots, a_n, b)$ means that $\varphi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, N$)-preserving.

(2) $\Leftrightarrow$ (3) holds since the condition $\mathcal{M} \models \neg\exists y\, \varphi(a_1, \ldots, a_n, y)$ means that $\varphi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, M$)-disjoint.

(3) $\Leftrightarrow$ (4) is satisfied since the type $\Phi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, M$)-disjoint iff the formula $\bigwedge \Phi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, M$)-disjoint, and the type $\Phi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, N$)-preserving iff the formula $\bigwedge \Phi(x_1, \ldots, x_n, y)$ is $\exists$-partially ($\{a_1\}, \ldots, \{a_n\}, N$)-preserving. $\square$

**Remark 9.** In Corollary 10 the condition of finiteness of the type $\Phi$ is essential since infinite types can be both realized in a structure and omitted in its elementary substructure. At the same time in the condition (4) in Corollary 10 the type $\Phi$ can be replaced by infinite one if $\mathcal{N}$ is saturated.

## 7 Multipartite and related graphs with preservation properties

Below we use the standard graph-theoretic terminology [12, 13].

**Definition.** [12] For a cardinality $\varkappa$, a *$\varkappa$-partite* graph is a graph whose vertices are (or can be) partitioned into $\varkappa$ disjoint independent sets, i.e. sets without arcs connecting elements inside these sets. Equivalently, it is a graph that can be colored with $\varkappa$ colors, so that no two endpoints of an arc have the same color. When $\varkappa = 2$ these are the *bipartite* graphs, when $\varkappa = 3$ they are called the *tripartite* graphs, etc.

**Proposition 17.** *Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:*

(1) $\Gamma$ *is $\varkappa$-partite;*

(2) $M$ *is divided into disjoint parts $P_i$, $i < \varkappa$, such that the formula $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for any $i < \varkappa$.*

Proof. By the definition, if $\Gamma$ is a $\varkappa$-partite graph, the universe $M$ is divided into $\varkappa$ disjoint parts $P_i$ such that all arcs $(a, b) \in R$ with $a \in P_i$ has $b \in \overline{P_i}$. Therefore $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for any $i < \varkappa$. Conversely, if $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for any $i < \varkappa$ then all parts $P_i$ are independent implying that $\Gamma$ is $\varkappa$-partite. □

Proposition 17 immediately implies:

**Corollary 11.** *Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:*

(1) $\Gamma$ *is bipartite;*

(2) *there is $P \subseteq M$ such that the formula $R(x, y)$ is $(P, \overline{P})$-preserving and $(\overline{P}, P)$-preserving.*

**Corollary 12.** *Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:*

(1) $\Gamma$ *is tripartite;*

(2) *there are disjoint $P_0, P_1, P_2 \subseteq M$ such that $M = P_0 \cup P_1 \cup P_2$ and the formula $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for each $i < 3$.*

**Proposition 18.** *Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:*

(1) $R = \emptyset$ *(respectively, $R = M^2$);*

(2) *the formula $R(x, y)$ $(\neg R(x, y))$ is $(M, \emptyset)$-preserving;*

(3) *the formula $R(x, y)$ $(\neg R(x, y))$ is $(M, M)$-disjoint.*

Proof. Since $\overline{M^2} = \emptyset$ is suffices to consider the case $R = \emptyset$. It means that $R(a, \Gamma) = \emptyset$ for any $a \in M$, which is equivalent to $R(a, \Gamma) \subseteq \emptyset$ for any $a \in M$, i.e. the $(M, \emptyset)$-preservation of $R(x, y)$, and to $R(a, \Gamma) \cap M = \emptyset$ for any $a \in M$, i.e. the $(M, M)$-disjointness of $R(x, y)$. □

## 8 Preservations of properties in graphs via distances

In this section we consider an illustration for preservations of properties using formulae describing distances in graphs.

Let $\Gamma = \langle M; R\rangle$ be a graph, $\varphi_n(x, y)$ be a formula saying that there exists s $(x, y)$-path of length $n$, $n \in \omega$. Taking the formula $\psi_n(x, y) := \bigvee\limits_{m \leq n} \varphi_m(x, y)$, saying that there exists s $(x, y)$-path of length $\leq n$, we observe the following:

**Lemma 2.** *For any vertex $a \in M$ and a property $P \subseteq M$ the formula $\psi_n(x,y)$ is partially (totally) $(\{a\},P)$-preserving iff some (any) vertex in $\Gamma$, which is achieved from $a$ by a path of length $\leq n$, belongs to $P$.*

Proof. If $\psi_n(x,y)$ is partially (totally) $(\{a\},P)$-preserving then for some (any) $b \in M$ with $\models \psi_n(a,b)$ we have $b \in P$. Therefore some (any) vertex in $\Gamma$, which is achieved from $a$ by a path of length $\leq n$, belongs to $P$.

Conversely, let some (any) vertex in $\Gamma$, which is achieved from $a$ by a path of length $\leq n$, belongs to $P$. Then we have $\psi_n(a,\Gamma) \cap P \neq \emptyset$ $(\psi_n(a,\Gamma) \subseteq P)$ confirming that formula $\psi_n(x,y)$ is partially (totally) $(\{a\},P)$-preserving. □

For a graph $\Gamma = \langle M; R\rangle$ and an element $a \in M$ the set $\{b \mid \Gamma \models R(a,b) \vee R(b,a)\}$ is called the *R-neighbourhood* of $a$, of the length 1. Replacing $R$ by $R^n$ we obtain the *R-neighbourhood* of $a$, of the length $n$. Similarly we have the notion of $\varphi$-neighbourhood for an arbitrary binary formula $\varphi(x,y)$ in the signature $\langle R\rangle$.

**Theorem 4.** *Let $\Gamma = \langle M; R\rangle$ be a graph, $P \subseteq M$ be a nonempty property. Then the following conditions are equivalent:*

(1) *$P$ is totally preserved under the formula $R(x,y) \vee R(y,x)$;*

(2) *$P$ is totally preserved under the formula $\psi_n(x,y) \vee \psi_n(y,x)$ with some/any $n \geq 1$;*

(3) *$P$ is a union of connected components.*

Proof. (1) ⇒ (2). Let $a \in P$. By the conjecture $P$ is closed under $R$-neighbourhoods of length 1, in particular, $P$ contains all elements in $\Gamma$ connected with $a$ by $R$-edges. Continuing the process $n$ times we observe that $P$ contains all elements having the $R$-distance at most $n$. Therefore $P$ is totally preserved under the formula $\psi_n(x,y) \vee \psi_n(y,x)$ with some/any $n \geq 1$.

(2) ⇒ (1) is obvious since $\psi_n$-neighbourhoods of elements contain their $\psi_1$-neighbourhoods which are equal to $R$-neighbourhoods.

(2) ⇒ (3) follows by Lemma 2 since by the conjecture for arbitrary element $a \in P$ all elements connected with $a$ by $R$-paths belong to $P$, too.

(3) ⇒ (1). Let $a$ be an arbitrary element in $P$. Since $P$ contains the connected component $C(a)$ including $a$, then $P$ contains the $R$-neighbourhood of $a$ of length 1. Thus $P$ is totally preserved under the formula $R(x,y) \vee R(y,x)$. □

Since connected components are minimal sets among their unions, we have the following:

**Corollary 13.** *Connected components of a graph $\Gamma = \langle M; R\rangle$ are exactly minimal nonempty subsets of its universe $M$, which are totally preserved under the formula $R(x,y) \vee R(y,x)$.*

**Remark 10.** Replacing the formulae $R(x,y) \vee R(y,x)$ and $\psi_n(x,y) \vee \psi_n(y,x)$ by $R(x,y) \wedge R(y,x)$ and $\psi_n(x,y) \wedge \psi_n(y,x)$ we obtain the possibility of mutual achievements of vertices in $\Gamma$ but it is possibly not sufficient with respect to strong components since mutual connections can have distinct lengths of counter paths.

By these arguments both connected components and strong components may be not definable but it is enough if diameters of components are equal to natural numbers.

The following example shows that preservations by the formulae $\varphi_n(x, y)$ can reconstruct initial graphs after replacements of edges by mutual paths of the fixed length $n$.

**Example 5.** Let $\Gamma = \langle M; R\rangle$ be an arbitrary undirected graph, $\Gamma'\langle M'; R'\rangle$ is obtained from $\Gamma$ by adding $(a, b)$-paths of a length $n \geq 2$ with new intermediate elements of degree 2 for each $(a, b) \in R$ with removals all edges in $R$, i.e. with $R' \cap R = \emptyset$, such that these new intermediate elements are pairwise disjoint for distinct $(a, b)$-paths. Thus $M' \supseteq M$ and $M' \setminus M$ consists of new intermediate elements. We have totally $\varphi_n(x, y)$-preserved set $M$ and a family of totally $\varphi_n(x, y)$-preserved subsets $M_1, \dots, M_{n-1}$ of $M' \supseteq M$ generated by intermediate elements in fixed distances $m < n$ from elements in $M$. The reconstruction of connected components in $\Gamma$ means the choice of elements in these connected components and the step-by-step closure under solutions of the formulae $\varphi_n(b, y)$, where $b$ are these chosen elements on the initial step and obtained elements on previous steps for subsequent ones.

Recall that the *diameter* of a connected undirected graph $\Gamma$ is the supremum of shortest $(a, b)$-paths for all elements $a, b$. If the supremum is infinite it is denoted by $\infty$. It is also denoted by $\infty$ if the graph is not connected.

The following assertion characterizes the infinite diameter of a connected undirected graph in terms of partial preservation by formulae $\varphi_n(x, y)$.

**Proposition 19.** *For any connected undirected graph $\Gamma = \langle M; R\rangle$ the following conditions are equivalent:*

(1) *$\Gamma$ has the diameter $\infty$;*

(2) *for any formula $\varphi_n(x, y)$, $n \in \omega$, there is a vertex $a \in M$ such that $\varphi_n(x, y)$ is partially $(\{a\}, M)$-preserving.*

Proof. (1) $\Rightarrow$ (2). Since $\Gamma$ is connected the diameter $\infty$ means that for any $n \in \omega$ there are vertices $a_n, b_n \in M$ such that the shortest $(a_n, b_n)$-path has the length $n$. Therefore $\varphi_n(x, y)$ is partially $(\{a_n\}, M)$-preserving. Since $n$ is arbitrary, it confirms the item (2).

(2) $\Rightarrow$ (1) follows by the definition since the partial $(\{a\}, M)$-preservation by $\varphi_n(x, y)$ means that there is $b \in M$ such that the shortest $(a, b)$-path has the length $n$. As $n$ is unbounded then the diameter is equal to $\infty$. $\square$

**Remark 11.** The arguments above can be naturally transformed for directed graphs replacing arcs by edges and by replacements $\varphi'_n(x, y)$ of $\varphi_n(x, y)$ using the formulae $R(x', y') \vee R(y', x')$ instead of $R(x', y')$. Thus we obtain a characterization of the infinite diameter for directed graphs again in terms of partial preservation.

Besides the finite diameters are also characterized by the property of non-partially $(\{a\},M)$-preservation by formulae $\varphi_n(x,y)$ and $\varphi'_n(x,y)$ starting with some $n$.

## 9 Preservation properties for algebraic constructions and decompositions

Using the assertions above one can describe series of type-definable structures, in particular, classes of (ordered) semigroups, groups, rings and fields, including spherically ordered ones [14], bands of semigroups including rectangular bands of groups [15], graded algebras [16], etc., their subalgebras and quotients.

There are many kinds of graded structures $\mathcal{M}$ united by the following one: $\mathcal{M}$ contains a binary operation $\cdot$ and it is divided into parts $X_i$, $i\in I$, such that for any $i,j\in I$ there is $k\in I$ such that $X_i\cdot X_j\subseteq X_k$. In the introduced terms of preservation it means that the formulae $x_i\cdot x_j\approx x_k$ are $(X_i,X_j,X_k)$-preserved.

These kinds of preservation admit a series of natural generalizations both with respect to $n$-ary operations preserving the formulae $f(x_{i_1},\dots,x_{i_n})\approx x_j$ by tuples $(X_{i_1},\dots,X_{i_n},X_j)$ of parts, and more general natural $\Phi(\overline{x}_1,\dots,\overline{x}_m,\overline{y})$-type preservations, including total and partial ones.

These possibilities of preservations allow to decompose structures $\mathcal{M}$ into families of substructures with universes $X_i$ assuming their idempotency preservation with respect to given operations.

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