

SOME PROPERTIES OF SOME SUBRINGS OF THE RING OF FORMAL POWER SERIES

Mourad Berraho
University Ibn Tofail, Faculty of Sciences
Kenitra, Morocco
e.mail : mourad.berraho@uit.ac.ma

Abstract

It is well known thanks to the Borel's lemma that the range of the ring of smooth germs $\mathcal{E}_1$ under the Borel mapping is equal to the ring of formal power series $\mathbb{R}[[x]]$, in this paper we will show that if we consider a subring $\mathcal{C}_1$ of the ring $\mathcal{E}_1$ that contains the ring of the polynomials then the (x) -adic completion of $\mathcal{C}_1$ is the ring of formal power series $\mathbb{R}[[x]]$ and its range under the Borel mapping is dense under the (x) -adic topology, afterwards we will show the compactness of the inclusion mapping between some subrings of the ring of formal power series $\mathbb{R}[[x]]$.

Keywords: The Borel mapping, (x) -adic topology, compact mapping.

2020 Mathematics Subject Classification: 03C64, 14P99.

1. Introduction

Let $\mathcal{E}_1$ denote the ring of germs at the origin in $\mathbb{R}$ of $\mathcal{C}^\infty$ functions in a neighborhood of $0 \in \mathbb{R}$ and $\mathbb{R}[[x]]$ the ring of formal series with real coefficients. If $f \in \mathcal{E}_1$, we denote by $\hat{f} \in \mathbb{R}[[x]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_1 \ni f \rightarrow \hat{f} \in \mathbb{R}[[x]]$ is called the Borel mapping. A subring $\mathcal{C}_1 \subseteq \mathcal{E}_1$ is called quasianalytic if the restriction of the Borel mapping to $\mathcal{C}_1$ is injective.

Firstly, we know by the Borel's lemma in [2] that there is a surjection from the ring $\mathcal{E}_1$ to the ring of formal power series $\mathbb{R}[[x]]$, in this paper we will show that if we consider a subring $\mathcal{C}_1$ of the ring $\mathcal{E}_1$ that contains the ring of the polynomials then the (x) -adic completion of $\mathcal{C}_1$

is the ring of formal power series $\mathbb{R}[[x]]$ and its range under the Borel mapping is dense under the (x) -adic topology. By [4], we have the compactness for the inclusions mappings between some subrings of the ring of smooth germs $\mathcal{E}_1$. In the last section we are going to show the same for some subrings of the ring of formal power series $\mathbb{R}[[x]]$ which is the range of the ring $\mathcal{E}_1$ under the Borel mapping.

2. The (x) -adic topology over the rings $\mathcal{C}_1$.

Let's recall some facts about the m -adic topology.

Definition 2.1. Let R be a ring and m an ideal of its. The m -adic completion of the ring R is equal to the projective limit of R/m^n .

If the m -adic topology is separated (i.e., $\bigcap_{i \geq 0} m^i = \{0\}$), then this topology is metrizable. Indeed, for all $x, y \in R$, let n be the largest integer such that $x - y \in m^n$, where $m^0 = R$. Then we define the metric $d(x, y) = \frac{1}{2^n}$ when n exists and 0 otherwise.

The (x) -adic completion of the polynomials ring $\mathbb{R}[x]$ is the ring of formal power series $\mathbb{R}[[x]]$.

The interested reader will find more information about this in a very readable form in [1], Chapter 10.

If $\mathcal{C}_1$ is a ring such that $\mathbb{R}[x] \subseteq \mathcal{C}_1 \subseteq \mathcal{E}_1$, then for each $f \in \mathcal{C}_1$, let $\hat{f}$ be the power series defined by:

$$\hat{f}(x) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(0)}{n!} x^n$$

Then the map $\phi : \mathcal{C}_1 \rightarrow \mathbb{R}[[x]]$, defined by $\phi(f) = \hat{f}$ for each $f \in \mathcal{C}_1$ is a ring homomorphism.

Example: If $\mathcal{C}_1$ is the ring of real analytic functions, then $\phi(f) = f$ for all $f \in \mathcal{C}_1$.

Proposition 2.2. *The completion of the ring $\phi(\mathcal{C}_1)$ for the (x) -adic topology is the ring $\mathbb{R}[[x]]$. Therefore this ring is dense in the ring $\mathbb{R}[[x]]$ for the same topology.*

Proof. By assumption the ring $\mathbb{R}[x]$ is contained in the ring $\mathcal{C}_1$, so $\mathbb{R}[x] \subseteq \phi(\mathcal{C}_1) \subseteq \mathbb{R}[[x]]$, as $\mathbb{R}[[x]]$ is the (x) -adic completion of the ring $\mathbb{R}[x]$, the (x) -adic completion of the ring $\phi(\mathcal{C}_1)$ is also the ring of formal power series $\mathbb{R}[[x]]$. □

Lemma 2.3. *Let R be a ring and m its ideal, put $J = \bigcap_{i \geq 0} m^i$, so the m -adic completion of the ring R is exactly the m/J -adic completion of the ring R/J up to isomorphism.*

Proof. Put $J = \bigcap_{i \geq 0} m^i$. Thanks to the third isomorphism theorem, for all $i \in \mathbb{N}$ there is an isomorphism $f_i : R/m^i \rightarrow (R/J)/(m^i/J)$. Let $(p_i : \check{R} \rightarrow R/m^i)$ be the projective system corresponding to the m -adic completion $\check{R}$ of R . Composing with f_i , we get a projective system $(f_i p_i : \check{R} \rightarrow (R/J)/(m^i/J))$. Now, we can check easily that this satisfies the universal property of a projective limit ([3], chapter 5) (by using the corresponding properties of $(p_i : \check{R} \rightarrow R/m^i)$). Thus $\check{R}$ is the m/J -adic completion of R/J . □

We can now state the main result of this section.

Proposition 2.4. *The completion of the ring $\mathcal{C}_1$ for the (x) -adic topology is the ring $\mathbb{R}[[x]]$ up to isomorphism.*

Proof. Put $J = \bigcap_{i \geq 0} (x)^i$. By the previous lemma the (x) -adic completion of the ring $\mathcal{C}_1$ is equal to the $(x)/J$ -adic completion of the ring $\mathcal{C}_1/J$. We have that $\phi(\mathcal{C}_1)$ is isomorphic to the ring $\mathcal{C}_1/J$. So the (x) -adic completion of the ring $\phi(\mathcal{C}_1)$ is exactly the $(x)/J$ -adic completion of the ring $\mathcal{C}_1/J$. So, by the proposition 2.2 and the lemma 2.3 the (x) -adic completion of the ring $\mathcal{C}_1$ is the ring $\mathbb{R}[[x]]$. □

We say that the ring $\mathcal{C}_1$ is a quasianalytic ring if the Borel mapping $\phi : \mathcal{C}_1 \rightarrow \mathbb{R}[[x]]$, defined by $\phi(f) = \hat{f}$ for each $f \in \mathcal{C}_1$ is injective.

Equivalently, if the ring $\mathcal{C}_1$ satisfies the following implication for all $f \in \mathcal{C}_1$.

$$(f^{(j)}(0) = 0, j = 0, 1, \dots) \Rightarrow (f = 0).$$

Thanks to the quasianality, we will not distinguish between a germ and its range under the Borel mapping, therefore, we may assume that $\mathcal{C}_1 \subset \mathbb{R}[[x]]$.

We assume for the sequel that these rings satisfy the following property called the stability under monomial division.

Let $f \in \mathcal{C}_1$ and $f = x\hat{\varphi}$ where $\hat{\varphi} \in \mathbb{R}[[x]]$, then $\varphi \in \mathcal{C}_1$.

Remark 2.1. By the property of the stability under monomial division, the ring $\mathcal{C}_1$ is a principal domain.

Now we will give a criteria for the surjectivity of the Borel mapping $^\wedge : \mathcal{C}_1 \rightarrow \mathbb{R}[[x]]$ under the assumption that the local ring $\mathcal{C}_1$ is closed under derivation.

Proposition 2.5. *If the Borel mapping $^\wedge : \mathcal{C}_1 \rightarrow \mathbb{R}[[x]]$ is surjective, then the ring $\mathcal{C}_1$ is complete for the (x) -adic topology.*

Proof. Suppose that the mapping $^\wedge$ is surjective, and let's take $f \in \mathcal{C}_1$ such that $\hat{f} \in (x^n)\mathbb{R}[[x]]$, where $n \in \mathbb{N}$, so if $f \notin (x^n)\mathcal{C}_1$, then there exists $k_0 \in [[0, n-1]]$ such that $f^{(k_0)}(0) \neq 0$, as the ring $\mathcal{C}_1$ is a local ring and closed under derivation, $f^{(k_0)}$ is invertible, so there exist $g \in \mathcal{C}_1$ such that $f^{(k_0)}g = 1$, so $\hat{f}^{(k_0)}\hat{g} = 1$ and therefore $\hat{f}^{(k_0)}(0) \neq 0$, which is a contradiction. It is clear that if $f \in (x^n)\mathcal{C}_1$, then $\hat{f} \in (x^n)\mathbb{R}[[x]]$, consequently, we have the equivalence $f \in (x^n)\mathcal{C}_1 \Leftrightarrow \hat{f} \in (x^n)\mathbb{R}[[x]]$, for all $n \in \mathbb{N}$, therefore, the mapping ϕ is a homeomorphism for the (x) -adic topology. As the mapping $^\wedge$ is a linear homeomorphism and the ring $\mathbb{R}[[x]]$ is complete, we deduce that the ring $\mathcal{C}_1$ is also complete for the (x) -adic topology. □

3. Compactness of the inclusion mapping between some subrings of the ring of formal power series $\mathbb{R}[[x]]$.

Set $\Lambda_M := \{f = \sum_{n=0}^{\infty} a_n x^n, |a_n| \leq Ch^n M_n, \forall n \text{ pour un } C, h > 0\}$.

$\Lambda_{M,h} := \{f = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}[[x]], \|f\|_{\Lambda,h} = \sup_{n \geq 0} |a_n|/(h^n M_n) < \infty\}$.

The space $\Lambda_{M,h}$ endowed with the norm $\|f\|_{\Lambda,h} = \sup_{n \geq 0} |a_n|/(h^n M_n)$ and $f = \sum_{n \geq 0} a_n x^n$ and $(M_n) \in (0, +\infty)$ is a Banach space.

Proposition 3.1. *The application inclusion $J : \Lambda_{M,h} \rightarrow \Lambda_{M,s}$ is compact for all $s > h > 0$.*

Proof. The application inclusion $J : \Lambda_{M,h} \rightarrow \Lambda_{M,s}$ is compact whenever $s > h > 0$. We are supposing that in each $\Lambda_{M,h}$, the Banach norm $\|f\|_h = \sup_{n \geq 0} |a_n|/(h^n M_n)$ is given, where $f = \sum_{n \geq 0} a_n x^n$ and $(M_n) \in (0, +\infty)$.

In order to show that J is a compact mapping, we should prove the following:

"Given a bounded sequence $(f_k) \subset \Lambda_{M,h}$, there exists a subsequence $(f_{m(j)})$ which is $\|\cdot\|_s$ -converging to some $F \in \Lambda_{M,s}$ ".

With this aim, fix a sequence (f_k) as above. Then $f_k = \sum_{n \geq 0} a_{n,k} x^n$ and there is $A \in (0, +\infty)$ such that $|a_{n,k}| \leq Ah^n M_n$ for all $k \geq 1$ and all $n \geq 0$. In particular, taking $n = 0$, the sequence $(a_{0,k})_{k \geq 1}$ is bounded. By the Bolzano-Weierstrass theorem, this sequence possesses a convergent subsequence in $\mathbb{R}$, say $a_{0,k_j^0} \rightarrow b_0$. Now, the sequence $(a_{1,k})_{k \geq 1}$ is also bounded, and so its subsequence $(a_{1,k_j^0})_{j \geq 1}$ is bounded as well. It follows that it possesses a convergent subsequence in $\mathbb{R}$, say $a_{1,k_j^1} \rightarrow b_1$. With this procedure, and having constructed for a given $n \geq 0$ nested subsequences $(k_j^n) \subset (k_j^{n-1}) \subset \dots \subset (k_j^0) \subset \mathbb{N}$ satisfying $a_{l,k_j^l} \rightarrow b_l$ as $j \rightarrow \infty$ ($l = 0, \dots, n$), one obtains by using the Bolzano-Weierstrass theorem a subsequence $(k_j^{n+1}) \subset (k_j^n)$ satisfying $a_{n+1,k_j^{n+1}} \rightarrow$ certain $b_{n+1} \in \mathbb{R}$. Consider the sequence $m(j) := k_j^j$ ($j \geq 1$). This sequence is, eventually, a subsequence of each sequence $(k_j^n)_{j \geq 1}$ ($n = 0, 1, 2, \dots$). Therefore $a_{n,m(j)} \rightarrow b_n$ as $j \rightarrow \infty$ for each $n \geq 0$.

Next, we define $F := \sum_{n \geq 0} b_n x^n \in R[[x]]$:

On the one hand, we have $|a_{n,m(j)}| \leq AM_n h^n$ for all j, n . Letting $j \rightarrow \infty$, we get $|b_n| \leq AM_n h^n$ for all n , from which one derives that $F \in \Lambda_{M,h} \subset \Lambda_{M,s}$, so $F \in \Lambda_{M,s}$.

On the other hand, we are going to prove that $(f_{m(j)})$ converges to F in the norm $\|\cdot\|_s$ (and we would be done).

To this end, we must show that, given $\varepsilon > 0$, there is a positive integer j_0 such that $\sup_{n \geq 0} |a_{n,j} - b_n|/(M_n s^n) < \varepsilon$ for all $j \geq j_0$.

To this end, note that, due to the triangle inequality, we have

$$|a_{n,m(j)} - b_n| \leq 2AM_n h^n \text{ for all } n, j.$$

Now, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $2A(h/s)^n \rightarrow 0$ for all $n > n_0$ (because $h < s$). Then $|a_{n,m(j)} - b_n|/(M_n s^n) \leq 2A(h/s)^n < \varepsilon$ for all j and all $n > n_0$.

Finally, for each fixed $n \in \{0, 1, 2, \dots, n_0\}$, the sequence $(|a_{n,m(j)} - b_n|/(M_n s^n))_{j \geq 1}$ tends to 0. Hence there is j_n such that $|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon$ for all $j \geq j_n$. If we set

$$j_0 := \max\{j_1, \dots, j_{n_0}\},$$

then we get $|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon$ for all $n = 0, 1, \dots, n_0$ and all $j \geq j_0$. Putting all together, we consequently obtain

$$|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon \text{ for all } n \geq 0 \text{ and all } j \geq j_0.$$

But this tells us that $\|f_{m(j)} - F\|_s < \varepsilon$ for all $j \geq j_0$.

In other words, $f_{m(j)} \rightarrow F$ ($j \rightarrow \infty$) in the norm $\|\cdot\|_s$, as required.

REFERENCES

- [1] Atiyah and Macdonald, Introduction to Commutative algebra, university of Oxford, 1969.
- [2] E. Borel, Sur quelques points de la théorie des fonctions, Ann. Sci. Ec. Norm. Sup. XII(1895), 9–55.
- [3] Hassan Sfouli, On a problem concerning quasianalytic local rings. Annales Polocini Mathematici, 111.1 (2014).
- [4] Joseph J. Rotman, An Introduction to Homological Algebra, Second Edition.
- [5] Nestruev, Jet (2002). Smooth manifolds and observables. Berlin: Springer. ISBN 0-387-95543-7.
- [6] V. Thielliez, On quasianalytic local rings, Expo. Math., 26 (1): 1-23, 2008.