GENERLAIZED $\star$-$\mathcal{Z}$-SOLITONS ON LPK MANIFOLDS

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Abstract. In this study, we investigate the geometry of Lorentzian para-Kenmotsu manifolds that admit an almost gneralized $\star$-$\mathcal{Z}$-solitons with some special curvature conditions such as the projective and W_1-curvature tensors and obtain important characterizations of the Lorentzian para-Kenmotsu manifold.

1. Introduction

Kenmotsu studied a particular kind of contact Riemannian manifold that satisfies a special requirement, resulting in the description of different geometrical characteristics of class (3) (three-dimensional manifold) manifolds. The resulting structure from this investigation is now known as a Kenmotsu structure. The para-Kenmotsu manifold is recognized as a para-Sasakian manifold analog, an almost product structure, and a specific case of almost paracontact structures [1, 2]. The foundational concept of a paracontact structure inherent to smooth manifolds originated with Sato during 1976. Subsequently, two decades later, Sinha and Sai Prasad developed the category of para-Kenmotsu manifolds. This class represents a specific type within the broader framework of nearly paracontact metric manifolds. Various authors have explored additional structures of this type [3]. Manifolds exhibiting paracontact properties may accommodate even dimensionality, whereas contact structure manifolds exclusively exhibit odd dimensionality.

Hamilton [4] first proposed the idea of Ricci flow in 1982 as a way to find the canonical metric on a smooth manifold. Every closed three-manifold admits a geometric decomposition, according to Thurston's geometric conjecture [5], which was intended to be resolved via this approach. Since then, Ricci flow [4] has emerged as one of the most effective tools for studying Riemannian manifolds [6]. The Ricci flow equation governing the transformation of metrics on a smooth manifold is expressed as:

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij}.$$

In this formulation, g denotes the metric tensor, while S_{ij} represents the Ricci curvature tensor components. When the Ricci flow problem can only be resolved by a one-parameter family of diffeomorphisms and scaling, the solution is termed a Ricci soliton [7]. In the context of contact metric manifolds, Tripathi [8], along with Bejan and Crasmareanu [9], investigated the concept of a Ricci soliton. The Einstein metric naturally generalizes to a Ricci soliton. (g, V, λ) represents a Ricci soliton on a manifold M, g. The condition can be expressed in the form

$$\mathcal{L}_V g + 2S + 2\lambda g = 0,$$

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In this formulation, $\mathcal{L}$ refers to the Lie derivative operator, and λ stands for a constant real parameter.

2. Preliminaries

Assume that M^n is a Lorentzian metric manifold with dimension n and X, Y, Z are the vector fields. Provided that it possesses a structure (φ, ξ, η, g), in which φ represents a $(1,1)$ tensor field, η is a one form on M^n, g is a Lorentzian metric, and ξ is a vector field, then it satisfies

$$\varphi^2 X = X + \eta(X)\xi, \quad g(\varphi X, \varphi U) = g(X, U) + \eta(X)\eta(U), \tag{2.1}$$

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \tag{2.2}$$

for all $X, Y \in \chi(M)$. The relationships below hold in the Lorentzian almost para-contact manifold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0,$$
$$\Phi(X, Y) = \Phi(Y, X),$$

where $\Phi(X, Y) = g(X, \varphi Y)$.

The (para)contact structure is termed K-(para)contact [10, 11, 12] if ξ is a field from the Killing vector. In this particular case, we have

$$\nabla_X \xi = \varphi X.$$

A Lorentzian para-Sasakian manifold Y is a Lorentzian almost paracontact manifold [13, 14] if

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

for all vector fields $X, Y \in \chi(M)$.

The idea of the $\mathcal{Z}$-tensor was initially introduced by Mantica et al. [31] in the framework of manifolds exhibiting weak $\mathcal{Z}$-symmetric. By definition, a symmetric $(0,2)$-tensor $\mathcal{Z}$ is termed a $\mathcal{Z}$-tensor [34] whenever it satisfies the relation

$$\mathcal{Z}(W_1, W_2) = S(W_1, W_2) + \alpha\, g(W_1, W_2),$$

where α denotes a scalar function. Later, Pandey [34] generalized this construction by formulating the notion of a generalized $\mathcal{Z}$-tensor (GZT). In this broader setting, a symmetric $(0,2)$-tensor is said to be a GZT if it fulfills the identity

$$\mathcal{Z}(W_1, W_2) = S(W_1, W_2) + \alpha\, g(W_1, W_2) + \beta\, \eta(W_1)\eta(W_2), \tag{2.3}$$

where η is a 1-form given by $\vartheta(V) = g(V, \xi)$, and α, β are non-vanishing functions. The theme of weak symmetry has also been explored in [26]. From a physical viewpoint, the generalized $\mathcal{Z}$-tensor can be seen as an extension that incorporates the Einstein tensor within the relativistic framework. Various investigations have been carried out concerning manifolds endowed with this tensorial structure: Research conducted by Montica and Suh [32, 33] focused on Riemannian manifolds exhibiting pseudo $\mathcal{Z}$-symmetry, alongside their applications in relativistic spacetime models. Concurrently, investigations led by De and colleagues [28, 35] centered on geometric structures characterized by weak cyclic $\mathcal{Z}$-symmetry. Moreover, De et al.[27] analyzed generalized $\mathcal{Z}$-recurrent spacetimes and their implications in the setting of $f(r, T)$-gravity.

A complete pseudo-Riemannian manifold (M, g) of dimension n admits an almost generalized $\mathcal{Z}$-soliton (AGZS), denoted by $(M^n, g, \alpha, \vartheta, V, \beta, \lambda)$, when a smooth vector field V exists that fulfills the condition

$$\mathcal{Z} + \mathcal{L}_V g + \Lambda g = 0, \tag{2.4}$$

where λ denotes a smooth scalar function and V is as the potential field. Should λ be constant, this structure simplifies to what is known as a generalized $\mathcal{Z}$-soliton (GZS). Based on the sign of λ , the AGZS is classified as shrinking ($\lambda < 0$), steady ($\lambda = 0$), or expanding ($\lambda > 0$). When $V = \nabla\psi$ for a smooth function ψ, equation (2.4) assumes the equivalent form

$$\mathcal{Z} + 2\nabla\nabla\psi + \Lambda g = 0, \tag{2.5}$$

and is then referred to as an almost gradient generalized $\mathcal{Z}$-soliton (gradient AGZS). Equation (2.4) also encompasses several special cases: for $\beta = 0$ one obtains an almost $\mathcal{Z}$-soliton; if $\alpha = \beta = 0$, the equation reduces to an almost Ricci soliton; setting $\alpha = 0$ yields an almost ϑ-Ricci soliton; for $\alpha = r\mu$ with constant μ, one obtains an almost ω-Ricci–Bourguignon soliton; and when $\beta = 0$ with $\alpha = -(p+\frac{1}{n})$, the structure becomes an almost conformal Ricci soliton.

Recently, significant research efforts have focused on geometric solitons within various categories of spacetimes. The research team led by Azami has explored multiple examples within PFSs, such as Riemann solitons [23], gradient Ricci–Bourguignon solitons [30], additional studies of Riemann solitons [24], hyperbolic Ricci solitons [25], and also h-almost conformal η-Ricci–Bourguignon solitons [21]. The same group also analyzed AGZS in the context of magneto-fluid spacetimes under f(r)-gravity [22].

The notion of $*$-Ricci tensor is defined as

$$S^*(X, Y) = g(Q^*(X), Y) = \text{trace}(\varphi \circ R(X, \varphi Y)),$$

where Q^* and S^* are Ricci operator and tensor field of type $(0, 2)$, respectively. We define generalized $*$-$\mathcal{Z}$-tensor as follows

$$\mathcal{Z}^* = S^* + \alpha g + \beta\eta \otimes \eta. \tag{2.6}$$

We define the almost generalized $*$-$\mathcal{Z}$-soliton ($*$-AGZS) as follows

$$\mathcal{L}_V g + \mathcal{Z}^* + \Lambda g = 0, \tag{2.7}$$

here Λ denootes a smooth function.

The LPK manifold [15, 16] M^n is a Lorentzian almost paracontact manifold if and only if

$$(\nabla_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X, \qquad X, Y \in \chi(M^n).$$

On LPK manifolds we get

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.8}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y),$$

where ∇ is the symbol for the operator of covariant differentiation with regard to the Lorentzian metric g.

Moreover, the relations below hold on a LPK manifold M^n [17]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$R(\xi,X)\xi = X + \eta(X)\xi,$$

$$S(X,\xi) = (n-1)\eta(X), \quad S(\xi,\xi) = -(n-1), \tag{2.11}$$

$$Q\xi = (n-1)\xi,$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y).$$

Consider an LPK manifold M^n. It is identified as an η-Einstein manifold if

$$S(X,Y) = \gamma_1\, g(X,Y) + \gamma_2\, \eta(X)\eta(Y),$$

where the functions γ_1 and γ_2 are smooth over M. When the Ricci tensor is extended to include an additional term of the form

$$S(X,U) = \lambda_1\, g(X,U) + \lambda_2\, \eta(X)\eta(U) + \lambda_3\, \Phi(X,UY),$$

with $\Phi(X,U) = g(\varphi X, U)$ and $\lambda_1, \lambda_2, \lambda_3$ smooth functions on M^n is then referred to as a generalized η-Einstein manifold. The classical η-Einstein structure is recovered in the special case $\lambda_3 = 0$. Within this framework, the definition of the $\mathcal{M}$-projective curvature tensor P follows [18].

(2.12)
$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n-2}\left\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right\}.$$

where $R(X,Y)Z$,$S(Y,Z)$, and Q are stands for curvature tensor, Ricci tensor, and the Ricci operator defined by $g(QX,Y) = S(X,Y)$,. In a LPK manifold M^n, the quasic onformal curvature tensor C [19] is given as

$$\begin{aligned} C(X,Y)Z &= aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y) + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad -\frac{r}{n}\left[\left(\frac{a}{n-1} + 2b\right)(g(Y,Z)X + g(X,Z)Y)\right], \end{aligned}$$

where r is the scalar curvature and a and b are constants such that $ab \neq 0$. A manifold M is called an $*$-Einstein manifold if [17]

$$S^*(X,Y) = \lambda g(X,Y), \quad X,Y \in \chi(M)$$

where λ is a constant. A manifold is called an $*$-η-Einstein manifold if it follows as [20]

$$S^*(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y).$$

From [17] we have the following lemma.

Lemma 2.1. The $*$-Ricci tensor of an n-dimensional LPK manifold is given as follows:

$$S^*(X,Y) = S(X,Y) - ng(X,Y) - \eta(X)\eta(Y) + ag(X,\varphi Y), \tag{2.13}$$

for any $X,Y \in \chi(M)$.

3. ∗-AGZSs on LPK manifolds

If an n-dimensional LPK manifold admits an $*$-AGZS, then from equation (2.7), we obtain

$$\mathcal{L}_\xi g(Y,Z) + \mathcal{Z}^*(Y,Z) + \Lambda g(Y,Z) = 0. \tag{3.1}$$

We know that

$$\mathcal{L}_\xi g(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi). \tag{3.2}$$

Therefore, from equations (3.1) and (3.2), we obtain

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) + \mathcal{Z}^*(Y,Z) + \Lambda g(Y,Z) = 0. \tag{3.3}$$

Using equations (2.2) and (2.8) in Equation (3.3), we get

$$\mathcal{Z}^*(Y,Z) + (\Lambda - 2)g(Y,Z) - 2\eta(Y)\eta(Z) = 0. \tag{3.4}$$

Inserting equation (2.6) into (3.4), we acquire

$$S^*(Y,Z) + (\alpha + \Lambda - 2)g(Y,Z) + (\beta - 2)\eta(Y)\eta(Z) = 0. \tag{3.5}$$

From equations (3.5) and (2.13), we have

$$S(Y,Z) = -(\alpha + \Lambda - 2 - n)g(Y,Z) - (\beta - 3)\eta(Y)\eta(Z) - ag(Y, \varphi Z). \tag{3.6}$$

From equation (3.6), replacing $Z = \xi$, we find that

$$S(Y,\xi) = -\left(\alpha + \Lambda - \beta + 1 - n\right)\eta(Y). \tag{3.7}$$

Equations (2.11) and (3.7) give

$$(n-1)\eta(Y) = -\left(\alpha + \Lambda - \beta + 1 - n\right)\eta(Y)$$

or equivalently

$$\left(\alpha + \Lambda - \beta\right)\eta(Y) = 0.$$

Since vector field Y is arbitrary, we get $\Lambda = \beta - \alpha$. Consequently, we have the following outcome.

Theorem 3.1. *When a LPK manifold admits a $*$-AGZS, the manifold transforms into a generalized η-Einstein manifold of the form of equation (3.6) and the scalar functions Λ, α, and β are connected by the relation in $\Lambda = \beta - \alpha$.*

4. LPK manifolds satisfying $R(\xi, X) \cdot S = 0$

Let a LPK manifold satisfy the condition of Ricci semisymmetric, that is, $R(\xi, X) \cdot S(Y,Z) = 0$, hence, we get

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0.$$

Applying equation (2.9) in the above equation, we find

$$S(g(X,Y)\xi - \eta(Y)X, Z) + S(Y, g(X,Z)\xi - \eta(Z)X) = 0.$$

It is clear that

$$g(X,Y)S(\xi,Z) - \eta(Y)S(X,Z) + S(Y,\xi)g(X,Z) - S(Y,X)\eta(Z) = 0. \tag{4.1}$$

Putting $Z = \xi$ in equation (4.1), we obtain

$$g(X,Y)S(\xi,\xi) - \eta(Y)S(X,\xi) + S(Y,\xi)g(X,\xi) - S(Y,X)\eta(\xi) = 0. \tag{4.2}$$

Using equation (2.11) in equation (4.2), we acquire the following:

$$S(Y,X) = \eta(Y)S(X,\xi) + (n-1)g(X,Y) - S(Y,\xi)\eta(X). \tag{4.3}$$

Equations (3.7) and (4.3) give

$$S(X,Y) = (n-1)g(X,Y) - 2(\alpha + \Lambda - \beta + 1 - n)\eta(X)\eta(Y). \tag{4.4}$$

So, it is an η-Einstein manifold.

Theorem 4.1. *If $R(\xi, X) \cdot S = 0$ is satisfied by a LPK manifold allowing $*$-AGZS, then the manifold becomes an η-Einstein manifold.*

Remark 4.2. Note that the condition $R(\xi, X) \cdot S = 0$ is Ricci semisymmetric.

5. $\mathcal{M}$-projectively flat LPK manifolds admitting $\star$-AGZSs

Let an $\mathcal{M}$-projectively flat LPK manifold admits $*$-AGZS. Therefore, equation (2.12) satisfies, that is, $P(X,Y)Z = 0$; thus, we have

$$R(X,Y)Z = \frac{1}{2n-2}(S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY). \tag{5.1}$$

By setting $Z = \xi$ and substituting the expressions from (3.7) and (2.10) into (5.1), one obtains

$$(n-1)(\eta(Y)X - \eta(X)Y) = \eta(Y)QX - \eta(X)QY.$$

Putting $Y = \xi$ and using equation (2.2) in the above equation, we obtain

$$QX = (n-1)X,$$

0r equivalently $S(X,Y) = (n-1)g(X,Y)$. Hence, the following conclusions can be drawn.

Theorem 5.1. *A $\mathcal{M}$-projectively flat LPK manifold admits $*$-AGZS, then the manifold becomes an Einstein manifold.*

Remark 5.2. $\mathcal{M}$-projectively flat means a manifold that is both projectively flat in the sense of the $\mathcal{M}$-projective curvature tensor being zero.

Example 5.3. Consider the five-dimensional manifold $M^5 = \{t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^5\}$, where the standard coordinates in $\mathbb{R}^5$ are t_1, t_2, t_3, t_4, t_5. Let z_1, z_2, z_3, z_4, z_5 be the vector fields on M^5 given by

$$z_1 = t_5\frac{\partial}{\partial t_1}, \quad z_2 = t_5\frac{\partial}{\partial t_2}, \quad z_3 = t_5\frac{\partial}{\partial t_3}, \quad z_4 = t_5\frac{\partial}{\partial t_4}, \quad z_5 = t_5\frac{\partial}{\partial t_5} = \xi,$$

they are linearly independent at every point of M. Define a metric that is Lorentzian on M^5 as

$$g(z_i, z_j) = \begin{cases} 1, & i = j,\ i \in \{1,2,3,4\} \\ -1, & i = j = 5 \\ 0, & i \neq j. \end{cases}$$

Let η be the one form on M^5 defined as

$$\eta(X) = g(X, z_5) = g(X, \xi), \quad \forall X \in \chi(M).$$

And define φ as the $(1,1)$ tensor field on M^5.

$$\varphi z_1 = -z_2, \quad \varphi z_2 = -z_1, \quad \varphi z_3 = -z_4, \quad \varphi z_4 = -z_3, \quad \varphi z_5 = 0.$$

Using φ and g's linearity, we obtain

$$\eta(\xi) = g(\xi,\xi) = -1, \quad \varphi^2 X = X + \eta(X)\xi, \quad \eta(\varphi X) = 0,$$

$$g(X,\xi) = \eta(X), \quad g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M).$$

Assuming that ∇ represents the Levi-Civita connection with regard to the Lorentzian metric g, we have

$$[z_1, z_2] = 0, \quad [z_1, z_3] = 0, \quad [z_2, z_3] = 0,$$
$$[z_1, z_1] = 0, \quad [z_1, z_4] = 0, \quad [z_2, z_4] = 0.$$

Now, the Koszul's formula is defined as

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, Y, Z) + g(Y, Z, X) + g(Z, X, Y).$$

Using Koszul's formula, we can easily calculate the following:

$$\nabla_{z_1} z_1 = -z_5, \quad \nabla_{z_1} z_2 = 0, \quad \nabla_{z_1} z_3 = 0, \quad \nabla_{z_1} z_4 = 0, \quad \nabla_{z_1} z_5 = z_1,$$
$$\nabla_{z_2} z_1 = 0, \quad \nabla_{z_2} z_2 = -z_5, \quad \nabla_{z_2} z_3 = 0, \quad \nabla_{z_2} z_4 = 0, \quad \nabla_{z_2} z_5 = z_2.$$

Also, we can easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

Thus, the manifold is dimension 5 in LPK manifold.

Now, let

$$X = \sum_{i=1}^{5} X_i z_i = X_1 z_1 + X_2 z_2 + X_3 z_3 + X_4 z_4 + X_5 z_5,$$
$$Y = \sum_{i=1}^{5} Y_i z_i = Y_1 z_1 + Y_2 z_2 + Y_3 z_3 + Y_4 z_4 + Y_5 z_5,$$

and

$$Z = \sum_{i=1}^{5} Z_i z_i = Z_1 z_1 + Z_2 z_2 + Z_3 z_3 + Z_4 z_4 + Z_5 z_5.$$

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We get the following:

$$R(z_1, z_2)z_1 = z_2, \quad R(z_1, z_3)z_1 = -z_3, \quad R(z_1, z_4)z_1 = -z_4, \quad R(z_1, z_5)z_1 = -z_5,$$
$$R(z_2, z_3)z_1 = R(z_2, z_4)z_1 = R(z_2, z_5)z_1 = 0,$$
$$R(z_3, z_4)z_1 = R(z_3, z_5)z_1 = R(z_4, z_5)z_1 = 0.$$

$$R(z_1, z_2)z_2 = -z_1, \quad R(z_2, z_3)z_2 = -z_3, \quad R(z_2, z_4)z_2 = -z_4, \quad R(z_2, z_5)z_2 = -z_5,$$
$$R(z_1, z_3)z_2 = R(z_1, z_4)z_2 = R(z_1, z_5)z_2 = 0,$$
$$R(z_3, z_4)z_2 = R(z_3, z_5)z_2 = R(z_4, z_5)z_2 = 0.$$

$$R(z_1, z_3)z_3 = -z_1, \quad R(z_2, z_3)z_3 = -z_2, \quad R(z_3, z_4)z_3 = z_4, \quad R(z_3, z_5)z_3 = z_5,$$
$$R(z_1, z_3)z_2 = R(z_1, z_4)z_2 = R(z_1, z_5)z_2 = 0,$$
$$R(z_3, z_4)z_2 = R(z_3, z_5)z_2 = R(z_4, z_5)z_2 = 0.$$

$$R(z_1, z_4)z_4 = -z_1, \quad R(z_2, z_4)z_4 = -z_2, \quad R(z_3, z_4)z_4 = z_3, \quad R(z_4, z_5)z_4 = -z_5,$$
$$R(z_1, z_2)z_4 = R(z_1, z_3)z_4 = R(z_1, z_5)z_5 = 0,$$
$$R(z_2, z_3)z_4 = R(z_2, z_5)z_4 = R(z_3, z_5)z_4 = 0.$$

$$R(z_1, z_5)z_5 = z_1, \quad R(z_2, z_5)z_5 = z_2, \quad R(z_3, z_5)z_5 = -z_3, \quad R(z_4, z_5)z_5 = -z_5,$$
$$R(z_1, z_2)z_5 = R(z_1, z_3)z_5 = R(z_1, z_4)z_5 = 0,$$
$$R(z_2, z_3)z_5 = R(z_2, z_4)z_5 = R(z_3, z_4)z_5 = 0.$$

Based on the previously described curvature tensor formulas, it can be inferred that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Therefore, the curvature of the manifold is constant. Also, it follows that

$$S(Y, Z) = 4g(Y, Z).$$

On contracting the above equation and putting $Y = Z = z_i$, we get $r = 20$. The Ricci tensor S is given by

$$S(z_1, z_1) = S(z_2, z_2) = S(z_3, z_3) = S(z_4, z_4) = 4 \quad \text{and} \quad S(z_5, z_5) = -4.$$

Then

$$S^*(z_1, z_1) = S^*(z_2, z_2) = S^*(z_3, z_3) = S^*(z_4, z_4) = 0 \quad \text{and} \quad S^*(z_5, z_5) = -1.$$

Therefore

$$\mathcal{Z}^*(z_1, z_1) = \mathcal{Z}^*(z_2, z_2) = \mathcal{Z}^*(z_3, z_3) = \mathcal{Z}^*(z_4, z_4) = \alpha \quad \text{and} \quad \mathcal{Z}^*(z_5, z_5) = -1+\beta-\alpha.$$

If this manifold admits a $*$-AGZS with potential vector field ξ, $\Lambda = 2 - \alpha$, and $\beta = 3$.

Conclusion

In this paper, we investigated Lorentzian para-Kenmotsu (LPK) manifolds admitting $*$-almost generalized η-Ricci solitons ($*$-AGZSs). We showed that LPK manifolds satisfying the Ricci semisymmetric condition $R(\xi, X) \cdot S = 0$ are necessarily η-Einstein, while $\mathcal{M}$-projectively flat LPK manifolds admitting $*$-AGZSs are necessarily Einstein. Furthermore, a five-dimensional example was constructed, which verifies that such manifolds can admit $*$-AGZSs and confirms the theoretical results. These findings provide a clear characterization of the geometric structure of LPK manifolds under $*$-AGZSs and highlight the important role of Ricci semisymmetry and $\mathcal{M}$-projective flatness in determining whether the manifold is η-Einstein or Einstein.

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