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ON FAMILIES OF SPECIAL OBJECTS OF COMPLETE AND e-COMPLETE NUMBERINGS

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Abstract: The paper presents a number of results concerning Mal'cev's problem of whether an arbitrary complete numbering determines its special object uniquely. It is proved that a special object of the completion of any non-complete numbering is determined uniquely. It follows that a complete numbering ν has a unique special object b if and only if it is equivalent to the completion with respect to b of a numbering for which no element of $\nu(\mathbb{N})$ other than b is a special object. For any at most countable set S with two different elements a and b, its numbering ν is constructed such that $(\nu_a)_b$ is not complete with respect to a and, hence, $(\nu_a)_b \not\equiv ((\nu_a)_b)_a$. The question of the existence of such a numbering ν and elements $a, b \in \nu(\mathbb{N})$ was formulated by Badaev, Goncharov, and Sorbi (2003) in their work, in which the completion operator on arithmetical numberings is deeply studied. It is established that any nonempty subset $S_0 \subseteq S$, $|S| \geqslant 2$, is the set of special objects of some e-complete numbering of S. For finite families of c.e. sets, a description of the special object families of their e-complete computable numberings is obtained, and for finite families of Σ^0_{n+2}-sets, a criterion for the existence of their uniformly complete Σ^0_{n+2}-computable numberings is proved.

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1 Introduction

This paper is concerned with the complete numberings introduced by Mal'cev in [1] and the sets of their special objects (all notation and terminology used in this paper can be found in [2, 3, 4]; in particular, by a *numbering* of a set S we mean an arbitrary surjective mapping of the set of all non-negative integers $\mathbb{N}$ onto S). Despite the fact that complete numberings can enumerate not only families of computably enumerable (c.e.) sets or partial computable (p.c.) functions, but arbitrary non-empty countable sets, they still have many significant properties of their standard Gödel numberings $x \mapsto W_x$ and $x \mapsto \varphi_x$. Namely, complete numberings satisfy effective Kleene's recursion theorem, Rice's theorem, have infinite padding and other properties common in computability theory and termed by Riccardi [5] and Royer [6] *control structures* (see, e.g., [1, 7, 8, 9]).

A numbering ν of a set S is said to be *complete* if there is an element $a \in S$, which is called a *special object of* ν, such that for every p.c. function ψ there exists a computable g satisfying the following condition:

$$\nu(g(x)) = \begin{cases} \nu(\psi(x)), & \text{if } \psi(x)\downarrow, \\ a, & \text{if } \psi(x)\uparrow,\ x \in \mathbb{N}. \end{cases} \tag{1}$$

If an element $a \in S$ is a special object of a complete numbering ν of S, then we will say that ν is *complete with respect to* a.

If a complete numbering of a family of c.e. sets is computable, then its special object is uniquely determined and is the least element of the numerated family under inclusion (see, e.g., [8, 10]). The problem of whether an arbitrary complete numbering uniquely determines its special object was raised by Mal'cev (see, e.g., [11]), and was answered negatively by Ershov in [12]. There he proved that for every $n > 0$ there exists a complete numbering having exactly n special objects. Later Denisov and Lavrov proved [11] that complete numberings can have infinitely many special objects (the question of the existence of such numberings was raised by Ershov in [13]). A joint generalization of the results by Ershov, Denisov, and Lavrov was obtained by Khisamiev in [14] who proved that any subset of a given at most countable non-empty set S is the set of all special objects of some complete numbering of S.

The issues raised in this paper concern refinements of the known solutions to Mal'cev's problem. The first part of the paper proves that if the completion ν_b of a numbering ν is complete with respect to $a \in \nu(\mathbb{N})$ such that $a \neq b$, then ν is complete with respect to a as well. Therefore, a special object of the completion of any non-complete numbering is determined uniquely. This part of the paper also proves that for every at most countable set S with two different elements a and b there exists a numbering ν of S such that $(\nu_a)_b$ is not complete with respect to a. Therefore, $(\nu_a)_b \not\equiv ((\nu_a)_b)_a$ and $(\nu_a)_b \not\equiv (\nu_b)_a$. In addition, some properties of families with uniformly complete numberings introduced by Badaev, Goncharov, and Sorbi in [10] are considered there.

The second part of the paper generalizes the result of Khisamiev's paper [14] which allows us to obtain a solution to Mal'cev's problem for sets of special objects of e-complete numberings introduced by Degtev in [15]. It also provides a characterization of the special object families of e-complete computable numberings of finite families of c.e. sets.

In the notation and terminology of computability theory, we mainly follow Soare's monograph [16]. By φ_d we denote the partial computable function with the Gödel number d. For a partial function ψ, we write $\psi(x)\downarrow$ if the value of ψ on the argument x is defined, and $\psi(x)\uparrow$, otherwise. We denote its domain by $\operatorname{dom}\psi$. For every d, the domain of φ_d will be denoted by W_d. Let $c(x,y)$ denotes the computable pairing function $2^x(2y+1)-1$. Instead of $c(x,c(y,z))$ we will simply write $c(x,y,z)$. By l and r we denote the computable functions such that $l(c(x,y))=x$ and $r(c(x,y))=y$ for all $x,y\in\mathbb{N}$. If η is an equivalence relation, then the notation $[x]_\eta$ is used to denote the η-equivalence class of the element x.

2 Preliminaries on numberings

In the notation and terminology of the theory of numberings, we mainly follow Ershov's monograph [2] and his paper [3]. A pair $\mathfrak{S}=\langle S,\nu\rangle$, where ν is a numbering of a set S, is called a *numbered set*.

Let us give the definition of *completion* ν_a of a numbering ν with respect to $a\in\nu(\mathbb{N})$, introduced by Ershov in [17]:

$$\nu_a(c(d,x)) = \begin{cases} \nu(\varphi_d(x)), & \text{if } \varphi_d(x)\downarrow, \\ a, & \text{if } \varphi_d(x)\uparrow,\ d,x\in\mathbb{N}. \end{cases}$$

The numbering ν_a is complete with respect to a (see, e.g., [10]). An overview of the main properties of the completion operator can be found in [10, 18].

Note that if for the numbering ν of the set S in condition (1), we set

$$\psi(c(d,x))=\varphi_d(x),\ f(d,x)=g(c(d,x))$$

for all d and x, we obtain that

$$\nu(f(d,x)) = \begin{cases} \nu(\varphi_d(x)), & \text{if } \varphi_d(x)\downarrow, \\ a, & \text{if } \varphi_d(x)\uparrow,\ d,x\in\mathbb{N}. \end{cases} \tag{2}$$

Thus, ν is complete with respect to a if and only if there exists a binary computable function f satisfying (2).

Let ν_0 and ν_1 be numberings of sets S_0 and S_1, respectively. The numbering $\nu_0\oplus\nu_1$ of $S_0\cup S_1$ defined by

$$(\nu_0\oplus\nu_1)(2x+k)=\nu_k(x),\ k=0,1,\ x\in\mathbb{N},$$

is called the *direct sum* of ν_0 and ν_1.

A numbering ν of a set S is said to be *reducible* to a numbering μ of S (in this case we use the notation $\nu \leqslant \mu$) if there exists a computable function f such that $\nu(x) = \mu(f(x))$ for each x. We say that ν is *reducible to* ν *via computable* f if $\nu(x) = \mu(f(x))$ for each x. Numberings ν and μ of a set S are said to be *equivalent* ($\nu \equiv \mu$) if $\nu \leqslant \mu$ and $\mu \leqslant \nu$. An arbitrary numbering ν of a set S is complete with respect to $a \in S$ if and only if $\nu \equiv \nu_a$. If ν is complete with respect to $a \in S$ and $\mu \equiv \nu$, then μ is complete with respect to $a \in S$ as well (see, e.g., [10]).

We say that a mapping $\Phi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is an *enumeration operator* or an *e-operator* (see, e.g., [4]) if there exists a c.e. set W (called the *set of axioms* of the operator Φ) such that for every $X \subseteq \mathbb{N}$,

$$\Phi(X) = \{x : \exists\, D \subseteq X\, [(D \text{ is finite})\, \&\, c(x, D) \in W]\}$$

(here and throughout we identify finite sets with their canonical indices). The enumeration operator whose set of axioms is W_p, $p \in \mathbb{N}$, will be denoted by Φ_p. A set A is said to be *enumeration reducible* or *e-reducible* to a set B ($A \leqslant_e B$) if there exists an e-operator Φ such that $A = \Phi(B)$. We say that A is *e-reducible to* B *via an e-operator* Φ if $A = \Phi(B)$.

In Degtev's paper [19], the definition of reducibility of numberings was naturally generalized. A numbering ν of a set S is said to be *enumeration reducible* or *e-reducible* to a numbering μ of S ($\nu \leqslant_e \mu$) if there exists an e-operator Φ such that

$$\nu^{-1}(a) = \Phi(\mu^{-1}(a))$$

for each $a \in S$. We say that the numberings ν and μ are *e-equivalent* ($\nu \equiv_e \mu$) if $\nu \leqslant_e \mu$ and $\mu \leqslant_e \nu$. If $\nu \leqslant \mu$ via a computable function f, then $\nu \leqslant_e \mu$ via the e-operator whose set of axioms is

$$\{c(x, \{f(x)\}) : x \in \mathbb{N}\}.$$

If $\nu \leqslant_e \mu$, then the reducibility $\nu \leqslant \mu$, generally speaking, does not take place (see, e.g., [19]).

In this paper, in addition to numberings of arbitrary countable sets, we consider numberings of families of subsets of $\mathbb{N}$ computable in the arithmetical hierarchy, that are introduced by Goncharov and Sorbi in [20]. A numbering ν of a family of Σ^0_{n+1}-sets, $n \in \mathbb{N}$, is said to be Σ^0_{n+1}-*computable* if

$$G_\nu \coloneqq \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : y \in \nu(x)\} \in \Sigma^0_{n+1}.$$

For $n = 0$, we obtain the classical definition of a *computable* numbering (see, e.g., [2]). Families that have Σ^0_{n+1}-computable numberings are called Σ^0_{n+1}-*computable* as well. It is clear that the completion of any Σ^0_{n+2}-computable numbering is itself Σ^0_{n+2}-computable. An element X of a family $\mathcal{A}$ is said to be *essential* if there exists a set $Y \in \mathcal{A}$ such that $X \subsetneq Y$.

3 Special objects of complete numberings

The following theorem states that for an arbitrary numbering ν, if ν is not complete, then the special objects of the numbering ν_b is determined uniquely, and that the equivalence $(\nu_a)_b \equiv ((\nu_a)_b)_a$, generally speaking, does not hold.

Theorem 1. *Let S be an at most countable set containing two different elements $a, b \in S$.*

1) *Any numbering ν of S such that ν_b is complete with respect to a is also complete with respect to a.*

2) *There exists a numbering ν of S such that $(\nu_a)_b$ is not complete with respect to a.*

Proof. 1) Let ν_b be complete with respect to a. Then there exists a binary computable function f such that

$$\nu_b(f(d,x)) = \begin{cases} \nu_b(\varphi_d(x)), & \text{if } \varphi_d(x)\downarrow, \\ a, & \text{if } \varphi_d(x)\uparrow, \end{cases}$$

for all d and x. Let us choose a computable function g satisfying the condition

$$\varphi_{g(i)}(y) = \begin{cases} c(i,y), & \text{if } \varphi_i(y)\downarrow, \\ \uparrow, & \text{if } \varphi_i(y)\uparrow, \ i, y \in \mathbb{N}, \end{cases}$$

and the p.c. function ψ defined by

$$\psi(i,y) = \varphi_{l(f(g(i),y))}(r(f(g(i),y)))$$

for all integers i and y. Now, we are going to prove that for all i and y, either $\varphi_i(y)\downarrow$ or $\psi(i,y)\downarrow$. In order to do this we fix arbitrary integers i and y with $\varphi_i(y)\uparrow$. Thus, $\varphi_{g(i)}(y)\uparrow$ as well. If $\psi(i,y)\uparrow$, then we have the equalities

$$a = \nu_b(f(g(i),y)) = \nu_b(c(l(f(g(i),y)), r(f(g(i),y)))) = b$$

which contradict the condition $a \neq b$.

Now, by the uniformization theorem (see, e.g., [16]), we choose a binary computable function h such that

$$\varphi_i(y)\downarrow \,\&\, h(i,y) = \varphi_i(y) \ \vee \ \psi(i,y)\downarrow \,\&\, h(i,y) = \psi(i,y)$$

for all i and y. To prove that ν is complete with respect to a, we show that

$$\nu(h(i,y)) = \begin{cases} \nu(\varphi_i(y)), & \text{if } \varphi_i(y)\downarrow, \\ a, & \text{if } \varphi_i(y)\uparrow, \end{cases} \tag{3}$$

for all i and y. If $h(i,y) = \varphi_i(y)$, then (3) is obviously met. Suppose $h(i,y) = \psi(i,y)$. Thus, we have the equalities

$$\nu(h(i,y)) = \nu(\psi(i,y)) = \nu(\varphi_{l(f(g(i),y))}(r(f(g(i),y)))) =$$
$$= \nu_b(c(l(f(g(i),y)), r(f(g(i),y)))) = \nu_b(f(g(i),y)).$$

If $\varphi_i(y)\uparrow$, then $\varphi_{g(i)}(y)\uparrow$. Hence,

$$a=\nu_b(f(g(i),y))=\nu(h(i,y)).$$

If $\varphi_i(y)\downarrow$, then $\varphi_{g(i)}(y)\downarrow$ as well. Hence,

$$\nu(\varphi_i(y))=\nu_b(c(i,y))=\nu_b(\varphi_{g(i)}(y))=\nu_b(f(g(i),y))=\nu(h(i,y)).$$

Therefore, (3) is also true in the case when $h(i,y)=\psi(i,y)$.

2) It is sufficient to prove that there exists a numbering ν of S such that $\nu_a=\nu$ and ν_b is not complete with respect to a. Let η be the c.e. equivalence relation generated by the binary relation

$$\{\langle c(d,x),\varphi_d(x)\rangle : d,x\in\mathbb{N}\,\&\,\varphi_d(x)\downarrow\}.$$

By the recursion theorem, we choose an injective sequence of integers $\{k_t\}_{t\in\mathbb{N}}$ such that

$$\varphi_{k_t}(0)=c(k_t,0),\ t\in\mathbb{N}.$$

It is not hard to verify (for example, using the arguments from the proof of [21, Theorem 1]) that

$$\langle c(k_t,0),c(k_u,0)\rangle\notin\eta$$

for all distinct integers t and u and

$$\langle c(k_t,0),c(d,x)\rangle\notin\eta$$

for each t and all d, x with $\varphi_d(x)\uparrow$.

Let $\{p_t\}_{t\in\mathbb{N}}$ be a sequence of elements of S such that

$$S=\{a,b,p_0,p_1,\dots\}\,\&\,p_t\neq b,\ t\in\mathbb{N}.$$

Now we define the numbering ν by letting

$$\nu(y)=\begin{cases}b, & \text{if } \langle y,c(k_0,0)\rangle\in\eta,\\ p_t, & \text{if } \langle y,c(k_{t+1},0)\rangle\in\eta,\\ a, & \text{in other cases,}\end{cases}$$

for each y. Thus, $\nu(\mathbb{N})=S$ and for all y and z with $\langle y,z\rangle\in\eta$, we have $\nu(y)=\nu(z)$. Let us show that $\nu=\nu_a$. For all d and x, if $\varphi_d(x)\uparrow$, then $\langle c(d,x),c(k_t,0)\rangle\notin\eta$ for each t. Therefore, $\nu(c(d,x))=a$. If $\varphi_d(x)\downarrow$, then $\langle c(d,x),\varphi_d(x)\rangle\in\eta$. Hence, $\nu(c(d,x))=\nu(\varphi_d(x))$. Therefore, $\nu=\nu_a$.

Now, we are going to prove that ν_b is not complete with respect to a. Suppose, for the sake of a contradiction, that there exists a binary computable function f such that

$$\nu_b(f(d,x))=\begin{cases}\nu_b(\varphi_d(x)), & \text{if } \varphi_d(x)\downarrow,\\ a, & \text{if } \varphi_d(x)\uparrow,\end{cases}$$

for all d and x. Fix an integer m with $\nu(m)=a$. By the double recursion theorem there exist indices i and n such that

$$\varphi_i(0)=\begin{cases}m, & \text{if } \langle\varphi_{l(f(n,0))}(r(f(n,0)))\downarrow,c(k_0,0)\rangle\in\eta,\\ \uparrow, & \text{otherwise,}\end{cases}$$

and

$$\varphi_n(0) = \begin{cases} c(i,0), & \text{if } \varphi_{l(f(n,0))}(r(f(n,0)))\downarrow, \\ \uparrow, & \text{otherwise.} \end{cases}$$

Assume that $\varphi_{l(f(n,0))}(r(f(n,0))\uparrow$. Then, by the definition of completion with respect to b, we have

$$\nu_b(f(n,0)) = b.$$

On the other hand, since $\varphi_n(0)\uparrow$, by the choice of the function f, we have

$$\nu_b(f(n,0)) = a.$$

Thus, we obtain a contradiction to the inequality $a \neq b$. Hence, $\varphi_{l(f(n,0))}(r(f(n,0)))\downarrow$ and $\varphi_n(0)\downarrow$. Therefore,

$$\nu_b(f(n,0)) = \nu_b(\varphi_n(0)) = \nu_b(c(i,0)).$$

If

$$\left\langle \varphi_{l(f(n,0))}(r(f(n,0))), c(k_0,0) \right\rangle \notin \eta, \tag{4}$$

then $\varphi_i(0)\uparrow$ and, by the definition of ν_b,

$$\nu_b(c(i,0)) = b.$$

On the other hand, it follows from (4) that

$$b \neq \nu(\varphi_{l(f(n,0))}(r(f(n,0)))) = \nu_b(f(n,0)) = \nu_b(\varphi_n(0)) = \nu_b(c(i,0)).$$

If (4) does not hold, then

$$\nu_b(c(i,0)) = \nu(\varphi_i(0)) = \nu(m) = a,$$

but

$$\nu_b(c(i,0)) = \nu_b(\varphi_n(0)) = \nu_b(f(n,0)) = \nu(\varphi_{l(f(n,0))}(r(f(n,0)))) = b \neq a.$$

These contradictions complete the proof of the theorem. □

Corollary 1. *If a numbering ν of a set S is not complete, then any element $b \in S$ is a unique special object of the numbering ν_b.*

Corollary 2. *For any set S and any two of its distinct elements $a, b \in S$ there exists its numbering ν such that $(\nu_a)_b \not\equiv ((\nu_a)_b)_a$.*

To prove Corollary 2, it suffices to take the numbering ν from the second statement of Theorem 1 and note that $((\nu_a)_b)_a$ is complete with respect to a.

In [10], a natural strengthening of the concept of completeness was considered. A numbering ν of a set S is said to be *uniformly complete* if there exists a ternary computable function h such that for all d, m, x,

$$\nu(h(d,m,x)) = \begin{cases} \nu(\varphi_d(x)), & \text{if } \varphi_d(x)\downarrow, \\ \nu(m), & \text{if } \varphi_d(x)\uparrow. \end{cases} \tag{5}$$

It was also proven there that for every Σ^0_{n+3}-computable numbering μ of a family $\mathcal{A}$ there exists a uniformly complete Σ^0_{n+3}-computable numbering ν of $\mathcal{A}$ such that $\mu \leqslant \nu$. The following theorem shows that for Σ^0_2-computable families, this statement is not true. It also provides a criterion for the existence of uniformly complete Σ^0_2-computable numberings of finite families of Σ^0_2-sets.

Theorem 2. *1) A finite family $\mathcal{A}$ of Σ^0_2-sets has a uniformly complete Σ^0_2-computable numbering if it only if it has the least element under inclusion.*

2) For every Σ^0_2-computable numbering μ of a family $\mathcal{A}$ with the least element under inclusion there exists a uniformly complete Σ^0_2-computable numbering ν of $\mathcal{A}$ such that $\mu \leqslant \nu$.

Proof. 1) It is sufficient to prove that if $\mathcal{A}$ does not contain the least element under inclusion, then it has no uniformly complete Σ^0_2-computable numberings. The converse implication will follow from the second statement of the theorem.

On the contrary, let us assume that $\mathcal{A}$ does not contain the least element under inclusion and has a uniformly complete Σ^0_2-computable numbering ν. Since ν is Σ^0_2-computable, there exists a double computable sequence $\{\nu_s(x)\}_{x,s\in\mathbb{N}}$ such that for all x, y,

$$y \in \nu(x) \Leftrightarrow \exists t \forall s \geqslant t\, [y \in \nu_s(x)].$$

Let $R_0, \dots, R_k$ $(k > 0)$ be all the minimal under inclusion pairwise distinct elements of $\mathcal{A}$. Fix finite sets $F_0, \dots, F_k$ such that for all $j, l \leqslant k$,

$$F_j \subseteq R_l \Leftrightarrow j = l.$$

The existence of such finite sets is guaranteed by the proof of [2, I § 2, Proposition 4].

Let h be a ternary computable functions satisfying (5) for all d, m, and x, and let f be a binary computable function such that

$$\varphi_{f(p,y)}(y) = \varphi_{\varphi_p(y)}(y)$$

for each y. Fix numbers m_0 and m_1 such that $\nu(m_0) = R_0$ and $\nu(m_1) = R_1$. Now we define a partial computable function ψ as follows. By the recursion theorem, we assume we know an index n such that $\psi = \varphi_n$. Choose an arbitrary y such that $\psi(z)\downarrow$ for each $z < y$. Let $u = 0$ if y is even, and let $u = 1$ if y is odd. If there exists $s \geqslant y$ such that

$$F_u \subseteq \nu_s(h(f(n,0), m_0, 0) \text{ and}$$
$$\forall v \leqslant k\, [v \neq u \Rightarrow F_v \not\subseteq \nu_s(h(f(n,0), m_0, 0)], \tag{6}$$

then we define $\psi(y)$ to be equal to an index i such that

$$\varphi_i(y) = h(f(n, y+1), m_{1-u}, y+1). \tag{7}$$

Since we check whether condition (6) holds for some $s \geqslant y$ whenever we define $\psi(y)$, the partial function ψ is not total. Indeed, otherwise (taking into account assignments (7)) we would have that $F_j \not\subseteq \nu_s(h(f(n,0), m_0, 0))$ for each $j \leqslant k$. Let y be the least integer such that $\psi(y)\uparrow$. If $y > 0$, then we have

$$\nu(h(f(n,0), m_0, 0)) = \nu(\varphi_{f(n,0)}(0)) = \nu(h(f(n,1), m_1, 1)) =$$

$$= \nu(\varphi_{f(n,1)}(1)) = \cdots = \nu(h(f(n,y-1), m_w, y-1)) =$$
$$= \nu(\varphi_{f(n,y-1)}(y-1)) = \nu(h(f(n,y), m_{1-w}, y)),$$

where $w = 0$ if y is odd, and $w = 1$, otherwise. Since $\varphi_n(y)\uparrow$, $\varphi_{f(n,y)}(y)\uparrow$ as well. Let $u = 1 - w$ if $y > 0$, and let $u = 0$ if $y = 0$. Now, we have

$$\nu(h(f(n,y), m_u, y)) = \nu(m_u).$$

It follows that

$$F_u \subseteq \nu(m_u) = \nu(h(f(n,y), m_u, y)) = \nu(h(f(n,0), m_0, 0)),$$

$$\forall v \leqslant k\, [v \neq u \Rightarrow F_v \nsubseteq \nu(m_u) = \nu(h(f(n,y), m_u, y)) = \nu(h(f(n,0), m_0, 0))].$$

Therefore, $\psi(y)\downarrow$. This contradiction completes the proof of the theorem.

2) For every y, we define $\nu(2y+1) = \mu(y)$. Thus, μ will be reducible to ν. To define the values of ν on even arguments, we consider the c.e. equivalence η generated by the binary relation

$$\{\langle 2c(d,m,x), \varphi_d(x)\rangle : d, m, x \in \mathbb{N} \,\&\, \varphi_d(x)\downarrow\}.$$

Let A be the least element of $\mathcal{A}$ under inclusion. For every y, we define

$$\nu(2y) = \begin{cases} \mu(z), & \text{if } \langle 2y, 2z+1\rangle \in \eta \,\vee \\ & \vee\, \exists i \exists q\, [\langle 2y, 2c(i, 2z+1, q)\rangle \in \eta \,\&\, \varphi_i(q)\uparrow], \\ A, & \text{if there is no } z \text{ satisfying the previous condition.} \end{cases}$$

By the definition of η, we have that if

$$\langle 2c(d,m,x), 2z_0+1\rangle \in \eta \,\&\, \langle 2c(d,m,x), 2z_1+1\rangle \in \eta,$$

then $z_0 = z_1$. In addition, if

$$\langle 2c(d,m,x), 2c(i_0, 2z_0+1, q_0)\rangle \in \eta \,\&\, \langle 2c(d,m,x), 2c(i_1, 2z_1+1, q_1)\rangle \in \eta \,\&$$
$$\&\, \varphi_{i_0}(q_0)\uparrow \,\&\, \varphi_{i_1}(q_1)\uparrow,$$

then $i_0 = i_1$, $z_0 = z_1$, $q_0 = q_1$, and for no z is it satisfied $\langle 2c(d,m,x), 2z{+}1\rangle \in \eta$. Hence, the definition of the numbering ν is correct. It is not hard to see that ν is Σ^0_2-computable. Finally, defining the ternary function h as follows:

$$h(d,m,c) = 2c(d,m,x),\ \ d, m, x \in \mathbb{N},$$

we obtain that (5) is true for all d, m, and x. This completes the proof of the theorem. □

4 Special objects of e-complete numberings

The complete numberings can be characterized in terms of Ershov's category $\mathfrak{N}$ of numbered sets (see [2, 3, 22]) as follows.

Theorem 3 (Ershov [2, 22])**.** *A numbering ν of a set S is complete if and only if for every e-subobject $\langle \mathfrak{S}_0, \mu_0\rangle$ of an arbitrary numbered set $\mathfrak{S}_1$ and for every morphism $\mu : \mathfrak{S}_0 \to \mathfrak{S} = \langle S, \nu\rangle$ there exists a morphism $\mu_1 : \mathfrak{S}_1 \to \mathfrak{S}$ such that $\mu = \mu_1\mu_0$.*

If in the statement of this theorem we replace the category $\mathfrak{N}$ with Degtev's category of numbered sets $\mathfrak{N}_e$ (see [15]) consistent with the notion of e-reducibility of numberings, then we obtain the definition of an *e-complete* numbering. In what follows, we will use an equivalent definition of e-complete numberings which follows from the following theorem.

Theorem 4 (Degtev [15]). *A numbering ν of a set S, $|S| \geqslant 2$, is e-complete if and only if there exists an element $a \in S$ which is called a special object of ν such that $\overline{\varnothing'} \leqslant_e \nu^{-1}(a)$.*

A numbering ν of a set S is said to be *e-complete with respect to a* if a is a special object of ν (in the sense of Theorem 4). If $\overline{K}_0 = \Phi(\nu^{-1}(a))$, where

$$K_0 = \{c(d, x) : d, x \in \mathbb{N} \,\&\, \varphi_d(x)\downarrow\}$$

and Φ is an e-operator, then we will say that ν is *e-complete with respect to a via* Φ (note that K_0 and $\varnothing'$ are computably isomorphic). By a classical Mal'cev's result [1], the set of numbers of a special object $a \in S$ of any complete numbering ν of a set S, $|S| \geqslant 2$, is productive. Thus, $\overline{\varnothing'} \leqslant_m \nu^{-1}(a)$, and, hence, $\overline{\varnothing'} \leqslant_e \nu^{-1}(a)$. Therefore, every complete numbering is e-complete as well.

The first statement of Theorem 1 remains valid after replacing the complete numberings with e-complete ones.

Proposition 1. *Any numbering ν of a set S such that ν_b is e-complete with respect to a (where $a, b \in S$ and $a \neq b$) is itself e-complete with respect to a.*

Proof. If $\overline{\varnothing'} \leqslant_e \nu_b^{-1}(a)$ via an e-operator whose axiom set is W, then $\overline{\varnothing'} \leqslant_e \nu^{-1}(a)$ via the e-operator whose axiom set is

$$\{c(x, D) : \exists F \subseteq \mathbb{N}\,[c(x, F) \in W \,\&\, \forall c(d, y) \in F\,[\varphi_d(y)\downarrow] \,\&$$
$$\&\, D = \{\varphi_d(y) : c(d, y) \in F\}]\}.$$

This completes the proof of the proposition. □

Corollary 3. *Let μ and ν be numberings of a set S with $|S| \geqslant 2$ and let $b \in S$. If $\mu \equiv_e \nu_b$ and ν is not e-complete with respect to a for each $a \in S$ with $a \neq b$, then b is the unique special object of μ as an e-complete numbering.*

Proof. Suppose μ is e-complete with respect to a. Then

$$\overline{\varnothing'} \leqslant_e \mu^{-1}(a) \leqslant_e \nu_b^{-1}(a).$$

Hence, ν_b is e-complete with respect to a. Therefore, ν is e-complete with respect to a. □

Question 1. *Let μ be a numberings of a set S with $|S| \geqslant 2$ and let $b \in S$. Is it true that if b is a unique special object of μ (as an e-complete numbering), then there exists a numbering ν of S such that $\mu \equiv_e \nu_b$ and ν is not e-complete with respect to a for each $a \in S$ with $a \neq b$?*

It is obvious that if in this question instead of $\equiv_e$-equivalence of numberings we take their $\equiv$-equivalence, then the answer to the question will be positive.

The fact that any nonempty subset of a given at most countable set is the set of all special objects of some complete numbering follows from the following theorem.

Theorem 5 (Khisamiev [14])**.** *Let α and β be Σ^0_{n+2}-computable numberings of families $\mathcal{C}$ and $\mathcal{D}$, respectively. Then there exists a Σ^0_{n+2}-computable numbering ν of the family $\mathcal{C} \cup \mathcal{D} \cup \{\varnothing\}$ satisfying the following conditions:*

1) $\alpha \oplus \beta \leqslant \nu$,

2) ν is complete with respect to any element of $\mathcal{C} \cup \{\varnothing\}$,

3) if $(\mathcal{C} \cup \{\varnothing\}) \setminus \mathcal{D} \neq \varnothing$, then any element of $\mathcal{D} \setminus (\mathcal{C} \cup \{\varnothing\})$ is not a special object of ν.

The following theorem shows that in Theorem 5, instead of $\mathcal{C} \cup \{\varnothing\}$, we can take any Σ^0_{n+2}-computable subfamily $\mathcal{B}$ of an arbitrary Σ^0_{n+2}-computable family $\mathcal{A}$, and instead of $\mathcal{D}$, we can take the difference $\mathcal{A} \setminus \mathcal{B}$. Moreover, none of the elements of this difference will be a special object for ν as for an e-complete numbering.

Theorem 6. *Let $\mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$ be nonempty Σ^0_{n+2}-computable families. Then $\mathcal{A}$ has a complete Σ^0_{n+2}-computable numbering ν whose family of special objects is $\mathcal{B}$. Moreover, ν is not e-complete with respect to A for each $A \in \mathcal{A} \setminus \mathcal{B}$.*

Proof. Let α and β be Σ^0_{n+2}-computable numberings of the families $\mathcal{A}$ and $\mathcal{B}$, respectively. To define the required numbering ν, we construct a sequence $\{\nu_s\}_{s\in\mathbb{N}}$ of partial mappings on $\mathbb{N}$ to $\mathcal{A}$ such that $\nu_s \subseteq \nu_{s+1}$ for each s and $\bigcup_s \nu_s$ is a Σ^0_{n+2}-computable numbering of $\mathcal{A}$, satisfying the statement of the theorem. After that, we let $\nu = \bigcup_s \nu_s$.

Construction

Stage 0. We set ν_0 equal to the partial mapping of $\mathbb{N}$ to $\mathcal{A}$ with empty domain. At each stage $s+1$ of the construction, we will also define some c.e. equivalence η_{s+1} and a strictly increasing computable sequence of integers $\{z^{s+1}_i\}_{i\in\mathbb{N}}$. Let η_0 be the equality relation on $\mathbb{N}$ and let $z^0_i = i$ for each i.

Stage $s+1$. Assume by induction that for $u = s$ the following conditions are met:

1. $\nu_v \subseteq \nu_u$ for each $v < u$,
2. the sequence $\{z^u_i\}_{i\in\mathbb{N}}$ is strictly increasing and computable,
3. $\langle z^u_i, z^u_j \rangle \notin \eta_u$ for all distinct integers i and j,
4. $\operatorname{dom}\nu_u = \mathbb{N} \setminus (\bigcup_i [z^u_i]_{\eta_u})$,
5. for all $x, y \in \operatorname{dom}\nu_u$, if $\langle x, y \rangle \in \eta_u$, then $\nu_u(x) = \nu_u(y)$.

It is not hard to see that when $u = 0$ conditions 1–5 are met. Now let us consider three cases.

i. $s = 3t$ for some t. In this case, we provide the completeness of ν with respect to $\beta(t)$. We define the relation η_{s+1} as the c.e. equivalence relation generated by the binary relation

$$\eta_s \cup \{\langle z^s_{c(d,x)}, \varphi_d(x)\rangle \in \mathbb{N} \times \mathbb{N} : \varphi_d(x)\downarrow\}.$$

Using the recursion theorem and its uniformity, effectively in s, we choose a strictly increasing computable sequence $\{d_i\}_{i\in\mathbb{N}}$ such that

$$\varphi_{d_i}(0) = z^s_{c(d_i,0)} \tag{8}$$

for each i. Then for all i, we let

$$z^{s+1}_i = z^s_{c(d_i,0)}. \tag{9}$$

Thus, condition 2 is met for $u = s+1$. Since condition 3 is met for $u = s$ and condition (8) is met for each i, condition 3 is met for $u = s+1$ as well. Next, we define

$$\nu_{s+1}(x) = \nu_s(x)$$

for each $x \in \operatorname{dom}\nu_s$. Hence, condition 1 is met for $u = s+1$. For every $x \notin \operatorname{dom}\nu_s$, we define

$$\nu_{s+1}(x) = \begin{cases} \nu_s(\varphi_d(y)), & \text{if } \langle x, z^s_{c(d,y)}\rangle \in \eta_{s+1} \,\&\, \varphi_d(y)\downarrow\in \operatorname{dom}\nu_s, \\ \uparrow, & \text{if } \exists i\,[\langle x, z^{s+1}_i\rangle \in \eta_{s+1}], \\ \beta(t), & \text{in other cases.} \end{cases}$$

Let us show that this definition is correct. Let d, j, y, and w be arbitrary integers such that

$$\varphi_d(y)\downarrow\in \operatorname{dom}\nu_s \,\&\, \varphi_j(w)\downarrow\in \operatorname{dom}\nu_s \,\&\, \langle x, z^s_{c(d,y)}\rangle \in \eta_{s+1} \,\&\, \langle x, z^s_{c(j,w)}\rangle \in \eta_{s+1}.$$

Thus, $\langle z^s_{c(d,y)}, z^s_{c(j,w)}\rangle \in \eta_{s+1}$. Assume that $\nu_s(\varphi_d(y)) \neq \nu_s(\varphi_j(w))$. Then

$$\langle \varphi_d(y), \varphi_j(w)\rangle \notin \eta_s.$$

By the definition of equivalence η_{s+1}, we have that $\langle z^s_{c(d,y)}, z^s_{c(j,w)}\rangle \notin \eta_{s+1}$. It follows from this contradiction that the values of ν_{s+1} on elements $x \notin \operatorname{dom}\nu_s$ are defined correctly. Now, it is not hard to see that conditions 4 and 5 are met for $u = s+1$.

By the definition of ν_{s+1}, for all d and y with $z^s_{c(d,y)} \in \operatorname{dom}\nu_{s+1}$, we have

$$\nu_{s+1}\Big(z^s_{c(d,y)}\Big) = \begin{cases} \nu_{s+1}(\varphi_d(y)), & \text{if } \varphi_d(y)\downarrow, \\ \beta(t), & \text{if } \varphi_d(y)\uparrow. \end{cases}$$

In addition, for every i, conditions (8) and (9) are met. Therefore, having met condition 5 for each u, we obtain that

$$\nu\Big(z^s_{c(d,y)}\Big) = \begin{cases} \nu(\varphi_d(y)), & \text{if } \varphi_d(y)\downarrow, \\ \beta(t), & \text{if } \varphi_d(y)\uparrow, \end{cases}$$

for all d and y. Hence, ν will be complete with respect to $\beta(t)$.

ii. $s = 3t + 1$ for some t. In this case we provide the existence of y such that $\nu(y) = \alpha(t)$. Let y be the least element of the set $\bigcup_i [z_i^s]_{\eta_s}$. Fix j such that $y \in [z_j^s]_{\eta_s}$ and define

$$\nu_{s+1}(x) = \alpha(t) \tag{10}$$

for each $x \in \bigcup_{k \leqslant j} [z_k^s]_{\eta_s}$. For every $x \in \operatorname{dom} \nu_s$, we define $\nu_{s+1}(x) = \nu_s(x)$. Let η_{s+1} be the c.e. equivalence relation, generated by the binary relation

$$\eta_s \cup \{\langle x, y \rangle \in \mathbb{N} \times \mathbb{N} : x, y \in \operatorname{dom} \nu_{s+1} \setminus \operatorname{dom} \nu_s\}. \tag{11}$$

Now, for every i, we let

$$z_i^{s+1} = z_{i+j+1}^s.$$

It is not hard to see that in this case conditions 1–5 are met for $u = s + 1$.

iii. $s = 3c(p,t)+2$ for some p and t. In this case we provide that if $\alpha(t) \notin \mathcal{B}$, then ν is not e-complete with respect to $\alpha(t)$ via Φ_p. Let $F_0, \dots, F_k$ be a tuple of all finite sets $F \subseteq \mathbb{N}_{/\eta_s}$ such that for each $x \in \bigcup F$, at least one of the assignments (10), (12) (see below) has been performed at one of the previous stages (the construction will guarantee that the tuple of such sets F is always finite). By the recursion theorem, for every $l \leqslant k$ there exists an index m_l such that

$$W_{m_l} = \begin{cases} \{0\}, & \text{if } \exists D \subseteq \bigcup F_l \, \exists G \subseteq \bigcup_i [z_i^s]_{\eta_s} \, [(D \text{ and } G \text{ are finite}) \, \& \\ & \& \, c(c(m_l, x), D \cup G) \in W_p], \\ \varnothing, & \text{otherwise.} \end{cases}$$

For every $l \leqslant k$ with $W_{m_l} \neq \emptyset$, we choose a finite set $G_l \subseteq \bigcup_i [z_i^s]_{\eta_s}$ such that

$$c(c(m_l, 0), D \cup G_l) \in W_p$$

for some $D \subseteq \bigcup F_l$. For all remaining $l \leqslant k$, we let $G_l = \varnothing$. Fix the least j such that

$$\Big([z_m^s]_{\eta_s}\Big) \cap \Big(\bigcup_{l \leqslant k} G_l\Big) = \varnothing$$

for each $m \geqslant j$ and define

$$\nu_{s+1}(x) = \alpha(t) \tag{12}$$

for each $x \in \bigcup_{i<j} [z_i^s]_{\eta_s}$. Then, for every $x \in \operatorname{dom} \nu_s$, we let $\nu_{s+1}(x) = \nu_s(x)$. Let η_{s+1} be the c.e. equivalence relation, generated by the binary relation (11). Now, for every i, we define

$$z_i^{s+1} = z_{i+j}^s.$$

It follows directly from the construction that conditions 1–5 are met for $u = s + 1$.

End of construction

It remains to prove that for no $A \in \mathcal{A} \setminus \mathcal{B}$ is the numbering ν e-complete with respect to A. Let t be an arbitrary integer with $\alpha(t) \notin \mathcal{B}$. Fix an arbitrary integer p. We are going to show that ν is not e-complete with respect to $\alpha(t)$ via Φ_p. Suppose, for the sake of a contradiction, that $\overline{K}_0 = \Phi_p(\nu^{-1}(\alpha(t)))$. Let $s = 3c(p,t)+2$. Let us choose an integer $l \leqslant k$ such that at the stage $s+1$

of the construction, F_l is the greatest set under inclusion $F \in \{F_0, \dots, F_k\}$ satisfying the condition

$$\forall x \in \bigcup F\,[\nu_s(x) = \alpha(t)].$$

If $0 \notin W_{m_l}$, then $c(m_l, 0) \in \overline{K}_0 = \Phi_p(\nu^{-1}(\alpha(t)))$. Hence, there exist finite sets $D \subseteq \bigcup F_l$ and $G \subseteq \bigcup_i [z_i^s]_{\eta_s}$ such that $c(c(m_l, 0), D \cup G) \in W_p$. It follows directly from the definition of W_{m_l} that $0 \in W_{m_l}$.

If $0 \in W_{m_l}$, then, by the choice of sets $G_0, \dots, G_k$, taking into account assignments (12), there exist finite sets $D \subseteq \bigcup F_l$ and $G \subseteq \bigcup_i [z_i^s]_{\eta_s}$ such that $\nu(G) = \{\alpha(t)\}$ and $c(c(m_l, 0), D \cup G) \in W_p$. Therefore,

$$c(m_l, 0) \in \overline{K}_0.$$

Hence, $0 \notin W_{m_l}$. These contradictions complete the proof of the theorem. □

Corollary 4. *Let S be an arbitrary at most countable set with $|S| \geqslant 2$ and let S_0 be an arbitrary nonempty subset of S. Then S has an e-complete numbering whose set of special objects is S_0.*

Proof. Let us consider the computable family $\mathcal{A}$ defined as follows:

$$\mathcal{A} = \big\{\{y\} : y \in \mathbb{N} \,\&\, y < |S|\big\}.$$

Let $\mathcal{B}$ be a computable subfamily of $\mathcal{A}$ such that $|\mathcal{B}| = |S_0|$ and $|\mathcal{A} \setminus \mathcal{B}| = |S \setminus S_0|$. By Theorem 6, there exists an e-complete Σ_2^0-computable numbering ν of $\mathcal{A}$, whose family of special objects is $\mathcal{B}$. Now, we choose an injective sequence $\{a_y\}_{y<|S|}$ of elements of S such that

$$S = \{a_y : y < |S|\}, \ S_0 = \{a_y : \{y\} \in \mathcal{B}\}.$$

Then the numbering μ defined by

$$\mu(x) = a_y \Leftrightarrow \nu(x) = \{y\}$$

for each x is an e-complete numbering of S whose set of special objects is S_0. □

The following theorem shows that special objects of e-complete computable numberings may not be the least sets under inclusion of numbered families.

Theorem 7. *Let $\mathcal{A}$ be a finite family of c.e. sets. A nonempty family $\mathcal{B} \subseteq \mathcal{A}$ is the family of special objects of an e-complete computable numbering of $\mathcal{A}$ if and only if every element of $\mathcal{B}$ is essential in $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{R_0, \dots, R_n\}$ be a finite family of c.e. sets. Just as in the proof of [2, I § 2, Proposition 4], we fix finite sets $F_0, \dots, F_n$ such that for all $i, j \leqslant n$, the following conditions are met:

- $F_i \subseteq R_i$;
- $F_i \subseteq R_j \Rightarrow R_i \subseteq R_j$;
- $R_i \subseteq R_j \Rightarrow F_i \subseteq F_j$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be the family of special objects of an e-complete computable numbering ν of $\mathcal{A}$. If there exists $k \leqslant n$ such that $R_k \in \mathcal{B}$ and R_k is not essential in $\mathcal{A}$, then

$$\nu^{-1}(R_k) = \{x : F_k \subseteq \nu(x)\}.$$

Thus, $\nu^{-1}(R_k)$ is c.e. Hence, ν is not e-complete with respect to R_k.

Conversely, let every element of $\mathcal{B}$ is essential in $\mathcal{A}$. Without loss of generality, we assume that $\mathcal{B} = \{R_0, \dots, R_m\}$ for some $m \leqslant n$. For each $k \leqslant m$, we will denote by X_k some non-essential element of $\mathcal{A}$ with $R_k \subseteq X_k$. Now, we define a computable numbering ν of the family

$$\mathcal{B} \cup \{X_k : k \leqslant m\}$$

by letting

$$\nu((m+1)x + k) = \begin{cases} R_k, & \text{if } x \notin K_0, \\ X_k, & \text{if } x \in K_0, \end{cases}$$

for all $x \in \mathbb{N}$ and $k \leqslant m$. By the definition of ν, we have that for all $x \in \mathbb{N}$ and $k \leqslant m$, the following equivalence holds:

$$x \notin K_0 \Leftrightarrow \nu((m+1)x + k) = R_k.$$

Thus, $\overline{K}_0 \leqslant_e \nu^{-1}(R_k)$, $k \leqslant m$. Let μ be a computable decidable (i.e., the set $\{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : \mu(x) = \mu(y)\}$ is computable) numbering of the family

$$\{R_{m+1}, \dots, R_n\}.$$

It is not hard to see that then the numbering $\nu \oplus \mu$ is e-complete with respect to each element of $\mathcal{B}$ and, since $(\nu \oplus \mu)^{-1}(R_i)$ is computable for each i with $m < i \leqslant n$, $\mathcal{B}$ contains all its special objects. □

In [23], it was shown that the family of all c.e. sets has a computable numbering that is e-complete with respect to every c.e. set. Therefore, in the statement of Theorem 7, the condition of finiteness of the family $\mathcal{A}$ cannot be omitted. It follows from Theorem 6 that any subfamily of a finite Σ^0_{n+2}-computable family is the family of special objects of some of its e-complete Σ^0_{n+2}-computable numberings. It follows that for finite Σ^0_{n+2}-computable families Theorem 7 also fails.

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