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CLASSICAL SOLUTION OF THE CAUCHY PROBLEM FOR A MILDLY QUASILINEAR WAVE EQUATION WITH A VARIABLE COEFFICIENT

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Abstract: We consider the Cauchy problem for a mildly quasilinear wave equation with a variable coefficient in the upper half-plane. We use the method of characteristics to construct the solution in an implicit analytical form as a solution of an integro-differential equation. We study the solvability of this equation, as well as the dependence on the initial data and the smoothness of its solutions. We prove the uniqueness of the solution for the problem in question and establish the conditions under which its classical solution exists.

Keywords: mildly quasilinear wave equation, Cauchy problem, classical solution, apriori estimates, parameter continuation method.

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1 Statement of the problem

In the domain $Q = (0, \infty) \times \mathbb{R}$ of two independent variables $(t, x) \in \overline{Q} \subset \mathbb{R}^2$, for the nonlinear equation

$$\partial_t^2 u(t,x) - a^2(x)\partial_x^2 u(t,x) = f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)), \quad (t,x) \in Q, \tag{1}$$

we consider the following Cauchy problem with the initial conditions

$$u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}, \tag{2}$$

where $0 < a_{\min} \le a(x) \le a_{\max}$ for all $x \in \mathbb{R}$, f is a function given on the set $\overline{Q} \times \mathbb{R}^3$, and φ and ψ are a real-valued functions defined on the set $\mathbb{R}$.

Equations of form (1) are found in mechanics (nonlinear vibrations of strings and rods) [1, 2] and thermodynamics (the Maxwell–Cattaneo model of heat conduction) [3].

Work [4] obtained in an explicit analytical form a classical solution to the problem

$$\begin{aligned}
&(\partial_t - a(t,x)\partial_x - \partial_t(\ln a(t,x)))(\partial_t - a(t,x)\partial_x)u(t,x) = \\
&\quad = a(t,x)\partial_x a(t,x)\partial_x u(t,x) - \partial_t a(t,x)\partial_x u(t,x) + \partial_t(\ln a(t,x))\times \\
&\quad \times (\partial_t u(t,x) + a(t,x)\partial_x u(t,x)) - b(t,x)\partial_t u(t,x) - c(t,x)\partial_x u(t,x) - \\
&\quad - q(t,x)u(t,x), \quad (t,x) \in Q, \\
&u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}.
\end{aligned}$$

Paper [5] considered the following general linear case of problem (1)–(2)

$$\begin{aligned}
&\partial_t^2 u(t,x) - a^2(t,x)\partial_x^2 u(t,x) + a^{(1)}(t,x)\partial_t u(t,x) + a^{(2)}(t,x)\partial_x u(t,x) + \\
&\qquad\qquad + a^{(3)}(t,x)u(t,x) = f(t,x), \\
&u(s(x),x) = \varphi(x), \quad \partial_t u(s(x),x) = \psi(x),
\end{aligned}$$

where the coefficient $a(t,x)$ does not vanish at any point. Communication [6] studied the problem

$$\begin{aligned}
&\partial_t^2 u(t,\mathbf{x}) - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\partial_{x_i}\partial_{x_j} u(t,\mathbf{x}) = 0, \quad (t,\mathbf{x}) \in (0,\infty) \times \mathbb{R}^n, \\
&u(0,\mathbf{x}) = \varphi(\mathbf{x}), \quad \partial_t u(0,\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,
\end{aligned}$$

when the initial data φ and ψ belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Article [7] considered problem (1)–(2) with $f \equiv 0$ in the Triebel–Lizorkin space.

The observability issues for the linear equations of form (1) were considered in works [8, 9]. Strichartz type estimates for linear equations of form (1) were obtained in paper [10]. Weighted L^2-estimates for solutions of the equation

$$\partial_t^2 u(t,\mathbf{x}) - \Delta u(t,\mathbf{x}) + c(\mathbf{x})\partial_t u(t,\mathbf{x}) = 0$$

were derived in [11].

Communication [12] studied problem (1)–(2) in the case $a(x) = \text{const}$. Works [13, 14] investigated problem (1)–(2) with discontinuous initial data in the case $a(x) = \text{const}$.

Local existence and uniqueness of a weak solution of the Cauchy problem for the hyperbolic equation

$$\Big(\sum_{0 \le j,k \le n} g^{(j,k)}(\mathbf{x}) \partial_{x_j} \partial_{x_k} + \sum_{j=0}^{n} b^{(j)}(\mathbf{x}) \partial_{x_j} \partial_{x_j} + c(\mathbf{x}) \Big) u(\mathbf{x}) = \\ = F(u(\mathbf{x}), \partial_{x_0} u(\mathbf{x}), \partial_{x_1} u(\mathbf{x}), \ldots, \partial_{x_n} u(\mathbf{x}))$$

was established in [15]. Energy estimates for solutions to mixed problem of the nonlinear equations of the form

$$\partial_{tt} u(t, \mathbf{x}) + \operatorname{div}(a(\mathbf{x}) \nabla u(t, \mathbf{x})) + \cdots = 0$$

were derived in [16, 17, 18].

Note that Eq. (1) can be reduced to a first-order semilinear hyperbolic system

$$\begin{aligned} &\partial_t u(t, x) = q(t, x), \\ &\partial_t p(t, x) - \partial_x q(t, x) = 0, \\ &\partial_t q(t, x) - a^2(x) \partial_x p(t, x) = f(t, x, u(t, x), q(t, x), p(t, x)), \end{aligned} \tag{3}$$

with respect to the unknown functions u, $q = \partial_t u$, and $p = \partial_x u$. Results on the existence and uniqueness of solutions in the broad sense solutions to the Cauchy problem for system (3) are well known and presented in book [19]. However, generally speaking, this book only establishes local solutions. They will be global classical provided that they are bounded in any characteristic triangle and the given functions f, φ, φ', and ψ are continuously differentiable. The answer to the question “What conditions must be imposed on these functions to obtain global classical solutions?” is not available in [19].

2 Linear problem

We begin by considering the following simplest linear case of problem (1)–(2)

$$f(t, x, u, u_t, u_x) = g(t, x). \tag{4}$$

Following [20, 21], we denote the eikonal

$$T(x) := \int_0^x \frac{ds}{a(s)}. \tag{5}$$

Let X be the inverse of the function T. The function $X \colon \mathbb{R} \mapsto \mathbb{R}$ exists due to the condition $0 < a_{\min} \le a(x) \le a_{\max}$, which implies that the function T

is strictly increasing and the integrals

$$\int_0^{\pm\infty} \frac{ds}{a(s)} \tag{6}$$

diverge, which means that the function T biectivly maps the set $\mathbb{R}$ into itself.

Eikonal (5) defines the characteristics of Eq. (1) as follows: $t \pm T(x) = \text{const}$. The function X has the representation

$$X(x) = \int_0^x a(X(\xi))\, d\xi.$$

Direct differentiation yields the following formulas:

$$T'(x) = \frac{1}{a(x)}, \quad T''(x) = -\frac{a'(x)}{a^2(x)}, \tag{7}$$

$$X'(x) = a(X(x)), \quad X''(x) = a'(X(x))a(X(x)). \tag{8}$$

Thus, if $a \in C^1(\mathbb{R})$, then $T \in C^2(\mathbb{R})$ and $X \in C^2(\mathbb{R})$.

We will look for a solution u of problem (1), (2), and (4) having the form

$$u(t,x) = \sqrt{\frac{a(x)}{a(0)}} v(t, T(x)), \quad (t,x) \in \overline{Q}. \tag{9}$$

Taking into account (7) and (8), we substitute (9) into Eq. (2) and get

$$\begin{aligned} \partial_t^2 v(t,x) - \partial_x^2 v(t,x) &= \sqrt{\frac{a(0)}{a(X(x))}} g(t, X(x)) - \\ -\frac{1}{4} v(t,x) \left(a'(X(x))^2 - 2a(X(x))a''(X(x))\right) &= \sqrt{\frac{a(0)}{a(X(x))}} g(t, X(x)) - \\ -\frac{1}{4}\sqrt{\frac{a(0)}{a(X(x))}} u(t, X(x)) &\left(a'(X(x))^2 - 2a(X(x))a''(X(x))\right), \ (t,x) \in Q. \end{aligned} \tag{10}$$

The initial conditions for the function v are

$$v(0,x) = \sqrt{\frac{a(0)}{a(X(x))}} \varphi(X(x)), \ x \in \mathbb{R}, \tag{11}$$

$$\partial_t v(0,x) = \sqrt{\frac{a(0)}{a(X(x))}} \psi(X(x)), \ x \in \mathbb{R}. \tag{12}$$

We apply [12, Lemma 1] to problem (10)–(12) to obtain

$$u(t,x) = \frac{\sqrt{a(x)}}{2} \left(\frac{\varphi(X(T(x)-t))}{\sqrt{a(X(T(x)-t))}} + \frac{\varphi(X(T(x)+t))}{\sqrt{a(X(T(x)+t))}} \right) +$$

$$+\frac{\sqrt{a(x)}}{2}\int\limits_{T(x)-t}^{T(x)+t}\frac{\psi(X(\xi))}{\sqrt{a(X(\xi))}}\,d\xi+\frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau\int\limits_{T(x)-t+\tau}^{T(x)+t-\tau}\Bigg[\frac{g(\tau,X(\xi))}{\sqrt{a(X(\xi))}}+$$

$$+\frac{u(\tau,X(\xi))\left(2a(X(\xi))a''(X(\xi))-a'(X(\xi))^2\right)}{4\sqrt{a(X(\xi))}}\Bigg]d\xi,\quad (t,x)\in\overline{Q}. \quad (13)$$

Thus, we have proved the following assertion.

Assertion 1. *Let the conditions $a\in C^2(\mathbb{R})$, $g\in C^1(\overline{Q})$, $\varphi\in C^2(\mathbb{R})$, and $\psi\in C^1(\mathbb{R})$ be satisfied. Then a solution $u\in C^2(\overline{Q})$ to problem (1), (2), and (4) satisfies Eq. (5) in $\overline{Q}$.*

3 Integro-differential equation

We consider the integral equation

$$u(t,x)=K[u](t,x)=\frac{\sqrt{a(x)}}{2}\times$$

$$\times\left(\frac{\varphi(X(T(x)-t))}{\sqrt{a(X(T(x)-t))}}+\frac{\varphi(X(T(x)+t))}{\sqrt{a(X(T(x)+t))}}\right)+$$

$$+\frac{\sqrt{a(x)}}{2}\int\limits_{T(x)-t}^{T(x)+t}\frac{\psi(X(\xi))}{\sqrt{a(X(\xi))}}\,d\xi+\frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau\int\limits_{T(x)-t+\tau}^{T(x)+t-\tau}\Psi(\tau,X(\xi),$$

$$u(\tau,X(\xi)),\partial_t u(\tau,X(\xi)),\partial_x u(\tau,X(\xi)))\,d\xi,\quad (t,x)\in\overline{Q}. \quad (14)$$

where

$$\Psi(t,x,u,u_t,u_x)=\frac{f(t,x,u,u_t,u_x)}{\sqrt{a(x)}}+\frac{u\left(2a(x)a''(x)-a'(x)^2\right)}{4\sqrt{a(x)}}.$$

The following theorem on the equivalence of the "differential" and "integral" formulations of problem (1)–(2) holds.

Theorem 1. *Let the conditions $a\in C^2(\mathbb{R})$, $f\in C^2(\overline{Q}\times\mathbb{R})$, $\varphi\in C^2(\mathbb{R})$, and $\psi\in C^1(\mathbb{R})$ be satisfied. The function u belongs to the class $C^2(\overline{Q})$ and satisfies Eq. (1) in $\overline{Q}$ and conditions (2) on the set $\mathbb{R}$ if and only if it is a continuously differentiable solution of Eq. (14) in $\overline{Q}$.*

Proof. 1. Let the function u belong to the class $C^2(\overline{Q})$ and solve Eq. (1) with conditions (2). We seek a representation of the solution u of Eq. (1) problem (1)–(2) in the form $u=v+w$, where v is a solution to the problem

$$\partial_t^2 u(t,x)-a^2(x)\partial_x^2 u(t,x)=0,\quad (t,x)\in Q,$$
$$u(0,x)=\varphi(x),\quad \partial_t u(0,x)=\psi(x),\quad x\in\mathbb{R},$$

and w is a solution to the equation

$$\partial_t^2 w(t,x)-a^2(x)\partial_x^2 w(t,x)=$$

$$= f(t, x, (v+w)(t,x), \partial_t (v+w)(t,x), \partial_x (v+w)(t,x)), \quad (t,x) \in Q,$$

with conditions

$$w(0,x) = \partial_t w(0,x) = 0, \quad x \in \mathbb{R}.$$

Assertion 1 allows us to write Eq. (14).

2. Assume that representation (14) hold for the function $u \in C^1(\overline{Q})$. Then, by virtue of the smoothness conditions $a \in C^2(\mathbb{R})$, $f \in C^2(\overline{Q} \times \mathbb{R})$, $\varphi \in C^2(\mathbb{R})$, and $\psi \in C^1(\mathbb{R})$, similar to [5], it follows that the function u belongs to the class $C^2(\overline{Q})$. We substitute representation (14) into Eq. (1) and conditions (2) and verify that the function u satisfies this equation in $\overline{Q}$ and these conditions in $\mathbb{R}$. □

Before solving Eq. (14), we will study some properties of the operator K. By making the nondegenerate change of variables $X(\xi) = s$, we can rewrite

$$K[u](t,x) = \frac{\sqrt{a(x)}}{2} \times \\ \times \left(\frac{\varphi(X(T(x)-t))}{\sqrt{a(X(T(x)-t))}} + \frac{\varphi(X(T(x)+t))}{\sqrt{a(X(T(x)+t))}} \right) + \\ + \frac{\sqrt{a(x)}}{2} \int\limits_{T(x)-t}^{T(x)+t} \frac{\psi(X(\xi))}{\sqrt{a(X(\xi))}} \, d\xi + \frac{\sqrt{a(x)}}{2} \int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \Phi(\tau, s, \\ u(\tau,s), \partial_t u(\tau,s), \partial_x u(\tau,s)) \, ds, \quad (t,x) \in \overline{Q}. \quad (15)$$

where

$$\Phi(t,x,u,u_t,u_x) = \frac{\Psi(t,x,u,u_t,u_x)}{a(x)}.$$

Now, we see that the operator K maps the space $C^1(\Omega_{(t_0,x_0)})$ into the space $C^1(\Omega_{(t_0,x_0)})$, where

$$\Omega_{(t_0,x_0)} = \{(t,x) : 0 \le t \le t_0 \wedge X(T(x_0) - t_0 + t) \le x \le X(T(x_0) - t_0 + t)\}$$

Note that the set $\Omega_{(t_0,x_0)}$ is the characteristic triangle of Eq. (1).

At the same time $\Omega_{(t_0,x_0)} \subseteq \Omega_{(t_1,x_1)}$ if $(t_0,x_0) \in \Omega_{(t_1,x_1)}$. It follows from the fact that the representation

$$\Omega_{(t_0,x_0)} = \bigcup_{t \in [0,t_0]} \{(t,x) : T(x_0) - t_0 + t \le X(x) \le T(x_0) + t_0 - t\}$$

holds and T is the strictly increasing function. Thus, since $(n,0) \in \Omega_{(n+1,0)}$, $n \in \mathbb{N}$, we have $\Omega_{(n,0)} \subset \Omega_{(n+1,0)}$. It implies $\Omega_{(1,0)} \subset \Omega_{(2,0)} \subset \Omega_{(3,0)} \subset \cdots$. Moreover, we have

$$\bigcup_{i=1}^{\infty} \Omega_{(i,0)} = \overline{Q}. \quad (16)$$

From the definition of the set $\Omega_{(t_0,x_0)}$, we deduce

$$\Omega_{(t_0,x_0)} \subset \mathcal{B}_{(t_0,x_0)} := [0,t_0] \times [X(T(x_0) - t_0), X(T(x_0) + t_0)].$$

Now, we will solve Eq. (14) using the parameter continuation method [22, 23] according to the scheme set forth in [24, 25, 26]. We rewrite Eq. (14) in the operator form

$$u(t,x) = J[u](t,x) + h(t,x), \tag{17}$$

where

$$h(t,x) = \frac{\sqrt{a(x)}}{2}\left(\frac{\varphi(X(T(x)-t))}{\sqrt{a(X(T(x)-t))}} + \frac{\varphi(X(T(x)+t))}{\sqrt{a(X(T(x)+t))}}\right) + \\ + \frac{\sqrt{a(x)}}{2}\int\limits_{T(x)-t}^{T(x)+t} \frac{\psi(X(\xi))}{\sqrt{a(X(\xi))}}\,d\xi,$$

$$J[u](t,x) = \frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \Phi(\tau,\xi, \\ u(\tau,\xi), \partial_t u(\tau,\xi), \partial_x u(\tau,\xi))\,d\xi, \quad (t,x)\in\overline{Q}.$$

For Eq. (17), we consider a family of equations with the parameter $\varepsilon\in[0,1]$,

$$u_\varepsilon(t,x) - \varepsilon(J[u_\varepsilon]-J[0])(t,x) = h(t,x) + J[0](t,x), \quad (t,x)\in\overline{Q}. \tag{18}$$

It is easy to see that any solution u_ε to Eq. (18) with $\varepsilon = 1$ is a solution to Eq. (17) and vice versa. Our goal is to solve Eq. (18) with $\varepsilon = 1$. To do this, we will use the parameter continuation method. First, we must show that the operator on the left-hand side of Eq. (18) satisfies the Lipschitz condition with some constant. Second, we must obtain an a priori estimate of a certain type.

Assume that the function f satisfies the Lipschitz condition

$$|f(t,x,u_1,u_2,u_3) - f(t,x,z_1,z_2,z_3)| \le L(t,x)\sum_{i=1}^{3}|u_i - z_i|, \tag{19}$$

where $L\in C(\overline{Q})$. Then, the function Φ also satisfies the Lipschitz condition

$$|\Phi(t,x,u_1,u_2,u_3) - \Phi(t,x,z_1,z_2,z_3)| \le L_\Phi(t,x)\sum_{i=1}^{3}|u_i - z_i|, \tag{20}$$

where

$$L_\Phi(t,x) = \frac{L(t,x)}{a^{3/2}(x)} + \left|\frac{2a(x)a''(x) - a'(x)^2}{4a^{3/2}(x)}\right|.$$

Now, let's estimate the difference $|J[u_1](t,x) - J[u_1](t,x)|$. We have

$$|J[u_1](t,x) - J[u_1](t,x)| = \left|\frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \Phi(\tau,\xi,u_1(\tau,\xi),\right.$$

$$\partial_t u_1(\tau,\xi), \partial_x u_1(\tau,\xi))\,d\xi - \frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \Phi(\tau,\xi,$$

$$u_2(\tau,\xi), \partial_t u_2(\tau,\xi), \partial_x u_2(\tau,\xi))\,d\xi \Bigg| \le \frac{\sqrt{a(x)}}{2}\times$$

$$\times \int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \big|\Phi(\tau,\xi,u_1(\tau,\xi),\partial_t u_1(\tau,\xi),\partial_x u_1(\tau,\xi)) -$$

$$- \Phi(\tau,\xi,u_2(\tau,\xi),\partial_t u_2(\tau,\xi),\partial_x u_2(\tau,\xi))\big| d\xi \le$$

$$\le \frac{\sqrt{a(x)}}{2}\int\limits_0^t d\tau \int\limits_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \big[L_\Phi(\tau,\xi)|U(\tau,\xi)| + L_\Phi(\tau,\xi)\times$$

$$\times |\partial_t U(\tau,\xi)| + L_\Phi(\tau,\xi)|\partial_x U(\tau,\xi)|\big] d\xi, \quad (21)$$

where $U = u_1 - u_2$. Estimate (21) imply

$$|J[u_1](t,x) - J[u_1](t,x)| \le \frac{\sqrt{\|a\|_{C(\Omega_m)}}\|L_\Phi\|_{C(\Omega_m)}\|u_1-u_2\|_{C^1(\Omega_m)}\mu(\Omega_m)}{2},$$
$$(t,x)\in\Omega_m, \quad m\in\mathbb{N}, \quad (22)$$

where $\Omega_m = \Omega_{(m,0)}$ and μ is the standard Lebesgue measure.

Similarly, we derive the following estimates:

$$\|\partial_t J[u_1] - \partial_t J[u_1]\|_{C(\Omega_m)} \le C^{(1)}\|u_1 - u_2\|_{C^1(\Omega_m)}, \quad (23)$$

$$\|\partial_x J[u_1] - \partial_x J[u_1]\|_{C(\Omega_m)} \le C^{(2)}\|u_1 - u_2\|_{C^1(\Omega_m)}, \quad (24)$$

where $C^{(1)}$ and $C^{(2)}$ are constants depending only on the functions a and L_Φ. In this case, one has the inequality

$$\|J[u_1] - J[u_1]\|_{C^1(\Omega_m)} \le \mathfrak{L}\|u_1 - u_2\|_{C^1(\Omega_m)}$$

where

$$\mathfrak{L} = \frac{\sqrt{\|a\|_{C(\Omega_m)}}\|L_\Phi\|_{C(\Omega_m)}\mu(\Omega_m)}{2} + C^{(1)} + C^{(2)},$$

It shows that the operator $J\colon C^1(\Omega_m) \mapsto C^1(\Omega_m)$ satisfies the Lipschitz condition with the constant $\mathfrak{L}$.

The task now is to obtain an a priori estimate of the following form:

$$\|u_\varepsilon\|_{C^1(\Omega_m)} \le C\|h + J[0]\|_{C^1(\Omega_m)} \quad (25)$$

for all possible solutions u_ε of Eq. (18) for any $\varepsilon\in[0,1]$, where C is some constant independent of h, u_ε, and ε.

For a fixed $m\in\mathbb{N}$, let us introduce a vector function

$$\mathbf{v}^{(m)} : [X(-m), X(m)] \ni t \mapsto \mathbf{v}^{(m)}(t) =$$

$$= (x \mapsto (R_m[\Delta u](t,x), R_m[\partial_t \Delta u](t,x), R_m[\partial_x \Delta u](t,x))) \in$$
$$\in PC\left([X(-m), X(m)]\right) \times PC\left([X(-m), X(m)]\right) \times PC\left([X(-m), X(m)]\right)$$

where $R_m[h]$ is the extension by zero of the restriction $h|_{\Omega_m}$ of the function h, and $PC(\Omega)$ is a space of piecewise continuous functions on Ω.

Thus, we have

$$|u_\varepsilon(t,x)| \leq |h(t,x) + J[0](t,x) - \varepsilon(J[u_\varepsilon] - J[0])(t,x)| \leq \|h + J[0]\|_{C(\Omega_m)} +$$
$$+ |(J[u_\varepsilon] - J[0])(t,x)| \leq \|h + J[0]\|_{C(\Omega_m)} + \frac{\sqrt{a(x)}}{2} \times$$
$$\times \int_0^t d\tau \int_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} \big|\Phi(\tau,\xi,u_\varepsilon(\tau,\xi), \partial_t u_\varepsilon(\tau,\xi), \partial_x u_\varepsilon(\tau,\xi)) -$$
$$- \Phi(\tau,\xi,0,0,0)\big| d\xi \leq \|h + J[0]\|_{C(\Omega_m)} + \frac{\sqrt{a(x)}}{2} \times$$
$$\times \int_0^t d\tau \int_{X(T(x)-t+\tau)}^{X(T(x)+t-\tau)} L_\Phi(\tau,\xi)(u_\varepsilon(\tau,\xi) + \partial_t u_\varepsilon(\tau,\xi) +$$
$$+ \partial_x u_\varepsilon(\tau,\xi)) d\xi \leq \|h + J[0]\|_{C(\Omega_m)} + \sqrt{\|a\|_{C(\Omega_m)}} \|L_\Phi\|_{C(\Omega_m)} \times$$
$$\times \int_0^t \max_{0 \leq \xi \leq \tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \quad (t,x) \in \Omega_m. \quad (26)$$

Estimate (26) yields

$$\|\mathbf{v}_1^{(m)}(t)\| \leq \|h + J[0]\|_{C(\Omega_m)} + C_v^{(1,m)} \int_0^t \max_{0 \leq \xi \leq \tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \quad t \in [0,m]. \quad (27)$$

where $C_v^{(1,m)} = \|L_\Phi\|_{C(\Omega_m)} \sqrt{\|a\|_{C(\Omega_m)}}$. In a similar way, we derive the estimates

$$\|\mathbf{v}_2^{(m)}(t)\| \leq \|\partial_t(h + J[0])\|_{C(\Omega_m)} + C_v^{(2,m)} \int_0^t \max_{0 \leq \xi \leq \tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau,$$
$$t \in [0,m], \quad (28)$$

$$\|\mathbf{v}_3^{(m)}(t)\| \leq \|\partial_x(h + J[0])\|_{C(\Omega_m)} + C_v^{(3,m)} \int_0^t \max_{0 \leq \xi \leq \tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau,$$
$$t \in [0,m], \quad (29)$$

Estimates (27)–(29) imply

$$\|\mathbf{v}^{(m)}(t)\| \leq \|h + J[0]\|_{C^1(\Omega_m)} + C_v^{(m)} \int\limits_0^t \max_{0\leq\xi\leq\tau} \|\mathbf{v}^{(m)}(\xi)\|\, d\tau, \quad t \in [0, m],$$

where $C_v^{(m)} = C_v^{(1,m)} + C_v^{(2,m)} + C_v^{(3,m)}$. Applying [19, p. 39, Lemma 1] to the previous inequality, we obtain

$$\|\mathbf{v}^{(m)}(t)\| \leq \|h + J[0]\|_{C^1(\Omega_m)} \exp\Big(C_v^{(m)} m\Big), \quad t \in [0, m].$$

The last inequality yields estimate (25) with $C = \exp\Big(C_v^{(m)} m\Big)$.

Consider the operator A_ε, which acts according to the formula

$$A_\varepsilon[u] = u(t, x) - \varepsilon(J[u] - J[0]).$$

Note that the operator A_ε is Lipschitz-continuous (this follows from the fact that the operator J is Lipschitz-continuous) and coercive in the sense of the paper [22] (the coercivity follows from the apriori estimate (25)). Furthermore, the function $[0, 1] \ni \varepsilon \mapsto A_\varepsilon$ is continuous in the seminorm of the space of Lipschitz-continuous operators (25). Since the operator A_ε with $\varepsilon = 0$ is continuously invertible in the space $C^1(\Omega_m)$, then by virtue of [22, Theorem 4], it is invertible for all $\varepsilon \in [0, 1]$ in the same space. Thus, Eq. (18) is solvable for $\varepsilon = 1$ on the set Ω_m in the class of continuously differentiable functions. According to the construction, the solution is unique. It means that the equation (17) also has a unique solution in the space $C^1(\Omega_m)$ for any $h \in C^1(\Omega_m)$.

Due to (16), a solution $u^{(\infty)}$ to Eq. (17) in the space $C^1(\overline{Q})$ can be constructed in the form $u^{(\infty)} = \lim\limits_{m\to\infty} u^{(m)}$, where $u^{(m)}$ is its solution on the set Ω_m. As in [27], one can verify that the function $u^{(\infty)} \colon \overline{Q} \mapsto \mathbb{R}$ is indeed the only solution in the class $C^1(\overline{Q})$ to Eq. (17), which continuously depends on the right-hand side.

Thus, we have proven the following theorem.

Theorem 2. *Let the conditions $a \in C^2(\mathbb{R})$, $f \in C^1(\overline{Q} \times \mathbb{R})$, $\varphi \in C^1(\mathbb{R})$, and $\psi \in C(\mathbb{R})$ be satisfied, and let a function f satisfy the Lipschitz condition (19). Then a continuously differentiable solution to Eq. (14) exists, is unique, and depends continuously on the initial data.*

4 Global classical solution

Theorem 3. *Let the conditions $a \in C^2(\mathbb{R})$, $f \in C^2(\overline{Q} \times \mathbb{R})$, $\varphi \in C^2(\mathbb{R})$, and $\psi \in C^1(\mathbb{R})$ be satisfied, and let a function f satisfy the Lipschitz condition (19). Problem (1)–(2) has a unique solution defined by formula (14) in the class $C^2(\overline{Q})$.*

Proof. It follows from Theorems 1 and 2. □

5 Boundedness condition of the variable coefficient

In the present paper, we have considered Eq. (1) provided

$$a(x) \leq a_{\max} \tag{30}$$

for all $a \in \mathbb{R}$. However, this condition can be replaced with the following

$$\int_0^{\pm\infty} \frac{ds}{a(s)} = \pm\infty. \tag{31}$$

Let us prove it. We used condition (30) only to show that integrals (6) diverge when proving the existence of the function X. Condition (31) means that integrals (6) diverge.

6 Conclusions

In this paper, we derive the necessary and sufficient conditions under which there exists a unique classical solution of the Cauchy problem in the upper half-plane for a mildly quasilinear wave equation with a variable coefficient. We also established the dependence of the smoothness of the solution on the smoothness of the initial data. In the future, we plan to use the results obtained in this research to consider mixed problems for Eq. (1).

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