Disjoint cycles with length constraints in the disjoint union of null digraph has at most two vertices and tournament

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Abstract

Let $D = (V, A)$ be a oriented digraph with a partition $V = K \cup I$ such that $D[I]$ is a null graph has at most two vertices and $D[K]$ is a tournament. We show in this paper that every strong D with minimum out-degree 3, except the digraph D_7^3, contains two vertex-disjoint cycles of different lengths.

Key words: strong digraph, tournament, vertex-disjoint cycles, cycles of different lengths, null digraph

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1 Introduction

We consider here only a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loop, and no multiple arc. Unless otherwise indicated, our graph-theoretic terminology will follow [2]. We will also adopt notation and basic definitions that are used in [17].

Thomassen in [19] has proved that every digraph with minimum out-degree at least 3 contains two vertex-disjoint cycles. Recently, in connection with 2-coloring of hypergraphs, Henning and Yeo have begun to study in [7] the existence of vertex-disjoint cycles of different lengths in digraphs and they have posed there several conjectures. One of these conjectures has been solved by Lichiardopol in [11], which asserts that every digraph with minimum out-degree at least 4 contains two vertex-disjoint cycles of different lengths. By the results obtained by Thomassen [19] and Lichiardopol [11], the investigation of structure for digraphs without vertex-disjoint cycles of different lengths can be restricted to digraphs with minimum out-degree 3. Further, by the result obtained by Tan in [18], the investigation of this problem can be reduced to the investigation of such a problem for strong digraphs.

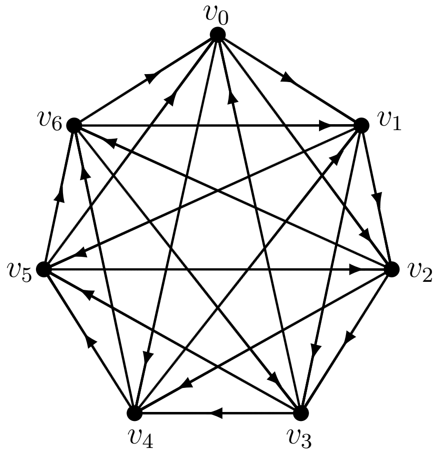

Figure 1: The digraph D_7^3

A digraph $D = (V, A)$ is called *k-regular* if $d_D^+(v) = d_D^-(v) = k$ for every vertex $v \in V$.

A digraph $D = (V, A)$ is called *acyclic* if it has no cycles. An *oriented digraph* is a digraph with no cycle of length 2. A digraph $D = (V, A)$ is called *strong* if for every pair x, y of distinct vertices in D, there exist both a path from x to y and a path from y to x. A digraph with only one vertex is considered to be strong.

A digraph $D = (V, A)$ with $V = \{v_1, v_2, \ldots, v_n\}$ is called a *tournament* if exactly one of the arcs v_iv_j and v_jv_i is in A for every $i \neq j \in \{1, \ldots, n\}$. A digraph $D = (V, A)$ is called a *null digraph* if $A = \emptyset$.

A 3-regular digraph having no vertex-disjoint cycles of different lengths has been given in [1] and is denoted here by D_7^3. This digraph has the vertex set $V(D_7^3) = \{v_0, v_1, \ldots, v_6\}$ and the arc set $A(D_7^3) = \{v_iv_j \mid (j-i)(\text{mod } 7) \in \{1, 2, 4\}\}$. The digraph D_7^3 is illustrated in Figure 1.

Recently, there have been many research results on the existence of disjoint cycles with different lengths in some classes of directed graphs, typically articles [4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16].

We note that the digraph D_7^3 is a strong oriented digraph with minimum out-degree 3, $V(D_7^3) = K \cup I$ such that $D[K]$ is a tournament has six vertices, $D[I]$ is a null digraph has exactly one vertex. So, it is natural to ask whether there are other strong oriented digraphs D with minimum out-degree 3, $V(D) = K \cup I$ such that $D[I]$ is a null graph has at most two vertices and $D[K]$ is a tournament, without vertex-disjoint cycles of different lengths. In this paper, we will give the answer to this question by proving the following main result.

Theorem 1. *Let $D = (V, A)$ be a strong oriented digraph with a partition $V = K \cup I$ such that $D[I]$ is a null graph has at most two vertices, $D[K]$ is a tournament and D has minimum out-degree 3. Then, except the digraph D_7^3, D contains two vertex-disjoint cycles of different lengths.*

2 Preliminary results

From now on, we always assume that $D = (V, A)$ is a strong oriented digraph with a partition $V = K \cup I$ such that $D[I]$ is a null graph has at most two vertices, $D[K]$ is a tournament, D has minimum out-degree 3 and having no vertex-disjoint cycles of different lengths. In further discussions, we will use this assumption implicitly. Then by the result obtained by Thomassen in [19], D has two vertex-disjoint cycles A^0 and A^1. By our assumption, it is clear that A^0 and A^1 must have the same length $t \geq 3$. Let

$$\begin{aligned} A^0 &= (a_0^0, a_1^0, \ldots, a_{t-1}^0, a_0^0), \\ A^1 &= (a_0^1, a_1^1, \ldots, a_{t-1}^1, a_0^1), \text{ and} \\ V' &= V(D) \setminus (V(A^0) \cup V(A^1)). \end{aligned}$$

Further, when we talk about the cycles A^j and A^{j+1} (resp., the vertices $a_i^j, a_{i+1}^j, a_{i+2}^j, \ldots$, of A^j), we implicitly understand that $j \in \{0, 1\}$ and $j + 1$ is taken modulo 2 (resp., $j \in \{0, 1\}$, $i \in \{0, 1, \ldots, t-1\}$ and $i+1, i+2, \ldots$, are taken modulo t).

Then we have the following trivial observations.

Observation 2. *Each of A^0 and A^1 has no chord.*

Observation 3. *Let $V' \neq \emptyset$ and $P = (v_1, \ldots, v_\ell)$ be a path in $D[V']$. Then if v_ℓ has out-neighbors in $V(P)$, then $\ell \geq t$ and $v_{\ell-t+1}$ is the unique out-neighbor of v_ℓ in $V(P)$.*

Observation 4. *Let $V' \neq \emptyset$ and $P = (v_1, \ldots, v_\ell)$ be a path in $D[V']$. Then*

1. if v_1 has an in-neighbor in $V(A^j)$, then v_ℓ has at most one out-neighbor in $V(A^j)$;

2. if v_1 has two in-neighbors in $V(A^j)$, then v_ℓ has no out-neighbor in $V(A^j)$;

3. if v_ℓ has an out-neighbor in $V(A^j)$, then v_1 has at most one in-neighbor in $V(A^j)$;

4. if v_ℓ has two out-neighbors in $V(A^j)$, then v_1 has no in-neighbor in $V(A^j)$.

Observation 5. *Let $V' \neq \emptyset$ and $u \in V'$. Then*

1. *if $a_i^j \to u$ then $u \not\to a_{i+1}^j$;*
2. *if $u \to a_i^j$ then $a_{i+t-1}^j \not\to u$.*

Observation 6. *Let $u \in V'$ and u is adjacent to all vertices in $V(A^j)$. Then*

1. *if u has an in-neighbor in $V(A^j)$ then $A^j \to u$;*
2. *if u has an out-neighbor in $V(A^j)$ then $u \to A^j$.*

The following lemma is simple but it is very useful for many further discussions.

Lemma 7. *Let $u \in V'$ and*

$$X_u = \{v \in V' \mid v \text{ is reachable from } u \text{ in } D[V']\}.$$

Then if u has two in-neighbors in $V(A^j)$, then any vertex in X_u has no out-neighbor in $V(A^j)$ and no in-neighbor in $V(A^{j+1})$.

Proof. Clearly, $u \in X_u$. Let v be a vertex in X_u and P be a path from u to v in $D[V']$. By Observation 4, v has no out-neighbor in $V(A^j)$.

For a contradiction, assume that v has an in-neighbor in $V(A^{j+1})$. Let $Q = (v_1, \ldots, v_\ell)$ be a longest path in $D[V']$ with the initial vertex $v_1 = v$. Then v_ℓ has at most one out-neighbor in X_u by Observation 3, no out-neighbor in $V' \setminus X_u$ by the definition of X_u and no out-neighbor in $V(A^j)$ by the assertion just proved before. Therefore, v_ℓ has at least two out-neighbors in $V(A^{j+1})$. This contradicts Observation 4. Thus, v has no in-neighbor in $V(A^{j+1})$. Lemma 7 is proved. □

Lemma 8. *The length t of A^0 and A^1 is equal to 3. Moreover, each of A^0 and A^1 has at most one vertex in I.*

Proof. For a contradiction, assume that $t \geq 4$. If both a_0^j and a_2^j are in K then a_0^j is adjacent to a_2^j, a contradiction to Observation 2. So, $a_0^j \in I$ or $a_2^j \in I$. Without loss of generality, we may assume that $a_0^j \in I$. Then both a_1^j and a_{t-1}^j are in K. So, a_1^j is adjacent to a_{t-1}^j, a contradiction to Observation 2. Thus, $t = 3$. It is clear that each of A^0 and A^1 has at most one vertex in I. □

Lemma 9. $V' \neq \emptyset$.

Proof. If $V' = \emptyset$, then

$$18 \leq \sum_{v \in V(D)} d^+(v) = |A(D)| \leq \binom{6}{2} = 15,$$

a contradiction. Thus, we must have $V' \neq \emptyset$. □

Lemma 10. *Let $u \in V' \cap K$ and u has two in-neighbors in $V(A^j)$. Then any vertex in $V(A^{j+1})$ has no out-neighbor in $V' \cap K$. Moreover, if $w \in V' \cap I$ and w has an in-neighbor in $V(A^{j+1})$ then w has no out-neighbor in V'.*

Proof. For a contradiction, assume that $v \in V' \cap K$ has an in-neighbor in $V(A^{j+1})$. Since u and v are in K, u adjacent to v. So $u \to v$ or $v \to u$. By Lemma 7, $u \not\to v$. So $v \to u$. Since $u \in K$ and u has no in-neighbor in $V(A^{j+1})$, u has at least two out-neighbors in $V(A^{j+1})$, a contradiction to Observation 4. Thus, any vertex in $V(A^{j+1})$ has no out-neighbor in $V' \cap K$. Now, assume that $w \in V' \cap I$, w has an in-neighbor in $V(A^{j+1})$ and w has an out-neighbor v in V'. It is clear that $v \in K$. By renaming the vertices of $V(A^{j+1})$, we may assume that $a_0^{j+1} \to w$. Because $V(A^{j+1})$ contains at least two vertices belonging to K and v has no in-neighbor in $V(A^{j+1})$, v has two out-neighbors in $V(A^{j+1})$, a contradiction to Observation 4. $\square$

Lemma 11. *Let $u \in V'$ and u is adjacent to all vertices in $V(A^0) \cup V(A^1)$. Then*

1. *if u has an in-neighbor in $V(A^j)$ then $A^j \to u$ and $u \to A^{j+1}$;*
2. *if u has an out-neighbor in $V(A^j)$ then $u \to A^j$ and $A^{j+1} \to u$.*

Proof. 1. If u has an in-neighbor in $V(A^j)$ then by Observation 6, $A^j \to u$. For a contradiction, assume that u has an in-neighbor in $V(A^{j+1})$. Again by Observation 6, $A^{j+1} \to u$. Since D is strong, we can find in D a path $(v_1, \ldots, v_\ell, v_{\ell+1})$ such that $v_1 = u$, $v_{\ell+1} \in V(A^0) \cup V(A^1)$. Then, we get a contradiction to Observation 4. So, u has no in-neighbor in $V(A^{j+1})$. Thus, $A^j \to u$ and $u \to A^{j+1}$.

2. By argument similar to those used in Assertion 1 we can prove Assertion 2. $\square$

Lemma 12. *Let $u \in V' \cap K$, u is not adjacent to exactly one vertex in A^j for every $j = 0, 1$. Then*

1. *u has at most one in-neighbor in $V(A^j)$ for every $j = 0, 1$;*
2. *u has at most one in-neighbor in $V(A^0) \cup V(A^1)$;*
3. *u has at least one in-neighbor in $V(A^0) \cup V(A^1)$;*
4. *u has exactly one in-neighbor and exactly three out-neighbors in $V(A^0) \cup V(A^1)$. If u is not adjacent to a_r^0 and a_s^1, then either $a_{r+2}^0 \to u$ and $u \to \{a_{r+1}^0, a_{s+1}^1, a_{s+2}^1\}$ or $a_{s+2}^1 \to u$ and $u \to \{a_{s+1}^1, a_{r+1}^0, a_{r+2}^0\}$, where $r, s \in \{0, 1, 2\}$ and $r+1, r+2, s+1, s+2$ are taken modulo 3.*

Proof. By renaming the vertices of $V(A^0)$ and $V(A^1)$, we may assume that u is not adjacent to a_0^0 and a_0^1. So, u is adjacent to all vertices in $\{a_1^0, a_2^0, a_1^1, a_2^1\}$. Since $u \in K$, $a_0^0, a_0^1 \in I$. So, $V' \cap I = \emptyset$.

1. By renaming the cycles A^0 and A^1, it suffices to prove that u has at most one in-neighbor in $V(A^0)$. For a contradiction, assume that u has two in-neighbors in $V(A^0)$. Then, we have $\{a_1^0, a_2^0\} \to u$. By Lemma 7, $u \to \{a_1^1, a_2^1\}$. It follows that u has an out-neighbor $v \in V'$ and $v \in K$. By Observation 4 and Lemma 10, $\{a_1^0, a_2^0\} \to v$ and $v \to \{a_1^1, a_2^1\}$. By Lemma 10, each vertex in $V(A^1)$ has no out-neighbor in V'. So, $a_0^1 \to \{a_1^0, a_2^0\}$ and a_1^1 has at least two out-neighbors in $V(A^0)$. It follows that $a_1^1 \to a_1^0$ or $a_1^1 \to a_2^0$. Then, (a_1^1, a_1^0, v, a_1^1) and $(u, a_2^1, a_0^1, a_2^0, u)$, or (a_1^1, a_2^0, v, a_1^1) and $(u, a_2^1, a_0^1, a_1^0, u)$, are two disjoint cycles of different lengths in D, a contradiction.

2. For a contradiction, assume that u has two in-neighbors in $V(A^0) \cup V(A^1)$. By Assertion 1, we may assume that u has one in-neighbor $z^0 \in V(A^0)$ and one in-neighbor $z^1 \in V(A^1)$. By Observation 5, it is easy to see that $u \to \{a_1^0, a_1^1\}$ and $\{a_2^0, a_2^1\} \to u$. So, u has at most two out-neighbors in $V(A^0) \cup V(A^1)$. It follows that u must have an out-neighbor in V'. Let $P = (v_1, \ldots, v_\ell)$ be a longest path in $D[V']$ with the initial vertex $v_1 = u$ and $\ell \geq 2$. If $\ell = 2$ then v_ℓ has at least three out-neighbors in $V(A^0) \cup V(A^1)$. So, v_ℓ has at least two out-neighbors in $V(A^0)$ or $V(A^1)$, we get a contradiction to Observation 4. It follows that $\ell \geq 3$ and v_ℓ has at least two out-neighbors in $V(A^0) \cup V(A^1)$. If $v_\ell \to x \in V(A^0)$ (resp., $v_\ell \to x \in V(A^1)$) then $(a_2^0, P, xA^0a_2^0)$ and A^1 (resp., $(a_2^1, P, xA^1a_2^1)$ and A^0), where $xA^ja_2^j$ are the vertices that go along the cycle A^j from x to a_2^j, are two disjoint cycles of different lengths in D, a contradiction.

3. For a contradiction, assume that u has no in-neighbor in $V(A^0) \cup V(A^1)$. Then $u \to \{a_1^0, a_2^0, a_1^1, a_2^1\}$. Since D is strong, there exists a path $(v_1, \ldots, v_\ell, v_{\ell+1})$ such that $v_1 \in V(A^0) \cup V(A^1)$, $v_{\ell+1} = u$ and $v_2, \ldots, v_\ell \in V'$, from which the contradiction to the Observation 4.

4. By Assertions 2 and 3, u has exactly one in-neighbor, and exactly three out-neighbors in $V(A^0) \cup V(A^1)$.

If u has exactly one in-neighbor in $V(A^0)$ then by Observation 5, it is not difficult to verify that $a_2^0 \to u$ and $u \to \{a_1^0, a_1^1, a_2^1\}$.

If u has exactly one in-neighbor in $V(A^1)$ then by arguments similar to those in the previous paragraph we can prove that $a_2^1 \to u$ and $u \to \{a_1^1, a_1^0, a_2^0\}$. □

Lemma 13. *Let $u \in V' \cap K$, u is not adjacent to exactly one vertex in A^j and is adjacent to all vertices in A^{j+1}. Then*

1. $u \to A^{j+1}$;

2. if u is not adjacent to a_r^j then $\{a_{r+1}^j, a_{r+2}^j\} \to u$, where $r \in \{0, 1, 2\}$, $r+1$ and $r+2$ are taken modulo 3.

Proof. By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that u is not adjacent to a_0^0.

So, $a_0^0 \in I$.

1. Since u is adjacent to all vertices in $V(A^1)$ and by Observation 6, $u \to A^1$ or $A^1 \to u$. For a contradiction, assume that $A^1 \to u$. By Lemma 10, $u \to \{a_1^0, a_2^0\}$. So, u has an out-neighbor in V'. Let $P = (v_1, \ldots, v_\ell)$ be a longest path in $D[V']$ with the initial vertex $v_1 = u$ and $\ell \geq 2$.

First, assume that $v_\ell \in K$. By Lemma 10 and Observation 4, $v_\ell \to \{a_1^0, a_2^0\}$ and $\{a_1^1, a_2^1\} \to v_\ell$. If $a_1^0 \to a_1^1$ (resp., $a_1^0 \to a_2^1$) then (a_1^0, a_1^1, u, a_1^0) and $(v_\ell, a_2^0, a_0^0, a_2^1, v_\ell)$ (resp., (a_1^0, a_2^1, u, a_1^0) and $(v_\ell, a_2^0, a_0^0, a_1^1, v_\ell)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^1, a_2^1\} \to a_1^0$ and a_1^0 has an out-neighbor $v \in V'$. By Lemma 10, $v \in I$ and v has no out-neighbor in V'. By Observation 5, $v \not\to a_2^0$. So, $v \to A^1$. Then (a_1^0, v, a_1^1, a_1^0) and $(v_\ell, a_2^0, a_0^0, a_2^1, v_\ell)$ are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that $v_\ell \in I$. If $\ell = 2$ then v_ℓ has an out-neighbor in $V(A^1)$ because $a_0^0 \in I$, a contradiction to Observation 4. So, $\ell \geq 3$, $v_{\ell-1} \in K$ and $v_{\ell-1} \neq u$. By reasoning similar to the case $v_\ell \in K$, we deduce a contradiction.

2. Suppose that $u \to \{a_1^0, a_2^0\}$. Since D is strong, there exists a path $(v_1, \ldots, v_\ell, v_{\ell+1})$ such that $v_1 \in V(A^0) \cup V(A^1)$, $v_{\ell+1} = u$ and $v_2, \ldots, v_\ell \in V'$, from which the contradiction to the Observation 4. Thus, $a_1^0 \to u$ or $a_2^0 \to u$. If $a_1^0 \to u$ then by Observation 5, $u \not\to a_2^0$. So, $a_2^0 \to u$. Now, assume that $a_2^0 \to u$. For a contradiction, suppose that $u \to a_1^0$.

First, assume that $V(A^1)$ contains one vertex in I. By renaming the vertices of $V(A^1)$, we may assume that $a_0^1 \in I$. First, suppose that a_0^0 has an out-neighbor $v \in V'$. Easy to see $v \in K$. By Observation 6, $A^0 \to v$. By Lemma 10, $v \to \{a_1^1, a_2^1\}$. Again by Lemma 10, each vertex of $V(A^1)$ has no out-neighbor in V'. So, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$ and $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_2^1 \to a_0^0$ (resp., $a_2^1 \to a_1^0$) then (u, a_0^1, a_2^0, u) and $(v, a_1^1, a_2^1, a_0^0, v)$ (resp., $(v, a_1^1, a_2^1, a_1^0, v)$) are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that a_0^0 has no out-neighbor in V'. So, $a_0^0 \to \{a_1^1, a_2^1\}$. If a_0^1 has an out-neighbor $v \in V'$ then by Observation 6, $A^1 \to v$. By Lemma 10, each vertex of $V(A^0)$ has no out-neighbor in V', a contradiction to $a_2^0 \to u$. Thus, a_0^1 has no out-neighbor in V' and $a_0^1 \to \{a_1^0, a_2^0\}$. If a_1^1 has an out-neighbor $v \in V'$ then it is not difficult to verify that $v \in K$, $a_2^1 \to v$. By Lemma 10, each vertex of $V(A^0)$ has no out-neighbor in V', a contradiction to $a_2^0 \to u$. So, a_1^1 has no out-neighbor in V' and $a_1^1 \to \{a_1^0, a_2^0\}$. It follows that a_1^0 has an out-neighbor $v \in V'$. By Observation 5, $a_2^0 \to v$. By Lemma 10, $v \to \{a_1^1, a_2^1\}$ and $a_2^1 \to \{a_1^0, a_2^0\}$. Then, (u, a_0^1, a_2^0, u) and $(v, a_1^1, a_2^1, a_1^0, v)$ are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that all vertices of $V(A^1)$ belong to K. First, suppose that a_0^0 has an out-neighbor $v \in V'$. Easy to see $v \in K$. Lemma 11, $A^0 \to v$ and

$v \to A^1$. If a_i^1 with $i \in \{0,1,2\}$, has an out-neighbor $w \in V'$ then by Lemma 10, $w \in I$ and $w \to \{a_1^0, a_2^0, a_{i+2}^1\}$. It is not difficult to see that a_{i+1}^1 is not adjacent to w, so by Lemma 10, $a_{i+1}^1 \to \{a_1^0, a_2^0\}$. Then, (w, a_{i+2}^1, a_i^1, w) and $(u, a_{i+1}^1, a_1^0, a_2^0, u)$ are two disjoint cycles of different lengths in D, a contradiction. So, each vertex of $V(A^1)$ has no out-neighbor in V'. It follows that each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. If a_0^0 has an in-neighbor a_i^1 with $i \in \{0,1,2\}$, then (v, a_i^1, a_0^0, v) and $(u, a_{i+2}^1, a_1^0, a_2^0, u)$ (or (v, a_i^1, a_0^0, v) and $(u, a_{i+1}^1, a_{i+2}^1, a_2^0, u)$) are two disjoint cycles of different lengths in D, a contradiction. So, a_0^0 has no in-neighbor a_i^1 for every $i \in \{0,1,2\}$. It follows that $A^1 \to \{a_1^0, a_2^0\}$. Then, (v, a_0^1, a_1^0, v) and $(u, a_1^1, a_2^1, a_2^0, u)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that a_0^0 has no out-neighbor in V'. So, a_0^0 has at least two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. Let a_i^1 with $i \in \{0,1,2\}$, has an out-neighbor $v \in V'$. If $v \in K$ and v is not adjacent to a_0^0 then by Assertion 1, $u \to A^1$, a contradiction. If $v \in K$ and v is adjacent to a_0^0 then by Lemma 11, $v \to A^0$ and $A^1 \to v$. So, (a_1^0, a_2^0, u, a_1^0) and $(v, a_0^0, a_0^1 A^1 a_i^1, v)$ (or $(v, a_0^0, a_1^1 A^1 a_i^1, v)$) are two disjoint cycles of different lengths in D, a contradiction. Thus, $v \in I$. If v has an out-neighbor $w \in V'$ then $w \in K$. By Observation 4, $A^1 \to w$. By Assertion 1, w is adjacent to a_0^0. By Lemma 11, $w \to A^0$. So, (a_1^0, a_2^0, u, a_1^0) and $(w, a_0^0, a_0^1, a_1^1, w)$ are two disjoint cycles of different lengths in D, a contradiction. Thus, v has no out-neighbor in V'. It is not difficult to see that $v \to \{a_1^0, a_2^0, a_{i+2}^1\}$ and v is not adjacent to a_{i+1}^1. Since a_{i+1}^1 is not adjacent to exactly one vertex (v, a_{i+2}^1, a_i^1, v) and is adjacent to all vertices in $V(A^0)$, by Assertion 1 we have $a_1^1 \to A^0$, a contradiction to $a_0^0 \to a_1^1$. Thus, a_i^1 has no out-neighbor in V' for every $i \in \{0,1,2\}$. So, $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$. It follows that a_1^0 has an out-neighbor $v \in V'$. It is clear that $v \neq u$. If $v \in K$ then by using Lemma 11, Observation 5 and Assertion 1, we deduce $a_2^0 \to v$ and $v \to A^1$. It follows that (u, a_1^1, a_2^0, u) and $(a_1^0, v, a_2^1, a_0^1, a_1^0)$ are two disjoint cycles of different lengths in D, a contradiction. So, $v \in I$. If $v \to u$ then $(a_0^0, a_1^1, a_2^0, a_0^0)$ and (v, u, a_0^1, a_1^0, v) are two disjoint cycles of different lengths in D, a contradiction. So, $v \not\to u$. Suppose that v has an out-neighbor $w \in V'$. Then $w \in K$ and $w \neq u$. It is clear that $a_2^0 \to w$. If $a_1^0 \to w$ then by Lemma 10, $w \to A^1$. It follows that (u, a_1^1, a_2^0, u) and (v, w, a_0^1, a_1^0, v) are two disjoint cycles of different lengths in D, a contradiction. So, $w \to a_1^0$. Then, (v, w, a_1^0, v) and $(u, a_0^1, a_1^1, a_2^0, u)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that v has no out-neighbor in V'. By Observation 5, $v \to A^1$. Then, (u, a_1^1, a_2^0, u) and $(v, a_2^1, a_0^1, a_1^0, v)$ are two disjoint cycles of different lengths in D, a contradiction.

Thus, we have proved $a_1^0 \to u$. □

Lemma 14. *If $|V'| = 1$ then D is isomorphic to D_7^3.*

Proof. Let there exists $x, y \in V(D)$ such that x is not adjacent to y. Then

$$21 \le \sum_{v \in V(D)} d^+(v) = |A(D)| \le \binom{7}{2} - 1 = 20,$$

a contradiction. So, every pair of vertices $x, y \in V(D)$ are adjacent.

Let $V' = \{u\}$. By renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^1 \to \{a_0^0, a_1^0\}$.

Suppose that $a_1^1 \to a_1^0$, it follows that $a_1^0 \to a_2^1$, $a_2^1 \to \{a_0^0, a_2^0\}$, $a_0^0 \to a_1^1$, $a_1^1 \to a_2^0$, $a_2^0 \to a_0^1$. So, $(a_2^0, a_0^1, a_1^1, a_2^0)$ and $(u, a_2^1, a_0^0, a_1^0, u)$ are two disjoint cycles of different lengths in D. Thus, $a_1^0 \to a_1^1$. It follows that $a_1^1 \to \{a_0^0, a_2^0\}$, $a_0^0 \to a_2^1$, $a_2^1 \to \{a_1^0, a_2^0\}$, $a_2^0 \to a_0^1$. Now let φ be the following mapping from V to $V(D_7^3)$: $u \mapsto v_0$, $a_0^1 \mapsto v_1$, $a_1^1 \mapsto v_2$, $a_0^0 \mapsto v_3$, $a_2^1 \mapsto v_4$, $a_1^0 \mapsto v_5$ and $a_2^0 \mapsto v_6$. Then it is not difficult to verify that φ is an isomorphism between D and D_7^3. □

3 Proof of Theorem 1

Now we continue to prove Theorem 1. If $|V'| = 1$ then by Lemma 14, D is isomorphic to D_7^3. So, we may assume that $|V'| \ge 2$. We consider the following cases separately.

Case 1. V' contains two vertices $u, v \in K$.

Without loss of generality, we may assume that $u \to v$. We again divide Case 1 into several subcases.

Subcase 1.1. $A^j \cap I = \emptyset$ for every $j = 0, 1$.

In this subcase, u and v are adjacent to all vertices in $V(A^0) \cup V(A^1)$. By Lemma 11 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. By Lemma 11 and Observation 4, $A^0 \to v$ and $v \to A^1$.

First, we may assume that there exists vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. Without loss of generality, we may assume that $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (resp., $a_2^1 \to a_2^0$) then (a_0^1, a_0^0, v, a_0^1) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (resp., (a_0^1, a_0^0, v, a_0^1) and $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \to a_2^1$. It follows that a_2^1 has an out-neighbor $w \in V'$. By Lemma 10, $w \in I$ and w has no out-neighbor in V'. So, w has at least two out-neighbors in $V(A^0)$. It follows that $w \to a_1^0$ or $w \to a_2^0$. Then, (a_2^1, w, a_1^0, a_2^1) and $(a_0^1, a_0^0, u, v, a_0^1)$, or (a_2^1, w, a_2^0, a_2^1) and $(a_0^1, a_0^0, u, v, a_0^1)$, are two disjoint cycles of different lengths in D, a contradiction.

Now, we may assume that any vertex of $V(A^1)$ has no out-neighbor in $V(A^0)$. So, $A^0 \rightarrow A^1$. It follows that there exists $z_1, z_2 \in V'$ with $z_1 \neq z_2$ such that $a_0^1 \rightarrow z_1$ and $a_1^1 \rightarrow z_2$. By Lemma 10, $z_1, z_2 \in I$, z_1 and z_2 has no out-neighbor in V'. So, z_i has at least two out-neighbors in $V(A^0)$ for every $i = 1, 2$. It follows that there exists $r, s \in \{0, 1, 2\}$, $r \neq s$ such that $z_1 \rightarrow a_r^0$ and $z_2 \rightarrow a_s^0$. Then, $(a_r^0, a_0^1, z_1, a_r^0)$ and $(a_s^0, u, a_1^1, z_2, a_s^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Subcase 1.2. There exists $j \in \{0, 1\}$ such that $A^j \cap I \neq \emptyset$ and $A^{j+1} \cap I = \emptyset$.

By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^0 \in I$.

First, assume that a_0^0 is not adjacent to u or a_0^0 is adjacent to u but $a_0^0 \rightarrow u$. By Lemma 13 and Lemma 11, $\{a_1^0, a_2^0\} \rightarrow u$ and $u \rightarrow A^1$. By Lemma 10 and Observation 4, $v \rightarrow A^1$ and $\{a_1^0, a_2^0\} \rightarrow v$. Suppose that there exists $i \in \{0, 1, 2\}$ such that a_i^1 has an out-neighbor $w \in V'$. By Lemma 10, $w \in I$ and w has no out-neighbor in V'. So, $w \rightarrow \{a_1^0, a_2^0, a_{i+2}^1\}$ and a_{i+1}^1 is not adjacent to w. If $a_{i+1}^1 \rightarrow a_1^0$ (resp., $a_{i+1}^1 \rightarrow a_2^0$) then $(a_i^1, w, a_{i+2}^1, a_i^1)$ and $(a_{i+1}^1, a_1^0, u, v, a_{i+1}^1)$ (resp., $(a_i^1, w, a_{i+2}^1, a_i^1)$ and $(a_{i+1}^1, a_2^0, u, v, a_{i+1}^1)$ are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \rightarrow a_{i+1}^1$ and a_{i+1}^1 has an out-neighbor $z \in V'$. It is easy to see that $z \in I$, $z \neq w$ and $z \rightarrow \{a_1^0, a_2^0, a_i^1\}$. Then, $(a_i^1, w, a_{i+2}^1, a_i^1)$ and $(z, a_2^0, u, a_{i+1}^1, z)$ are two disjoint cycles of different lengths in D, a contradiction. Thus, a_i^1 has no out-neighbor in V' for every $i = 0, 1, 2$ and a_i^1 has two out-neighbors in $V(A^0)$ for every $i = 0, 1, 2$. Let $a_1^1 \rightarrow a_2^0$. If $a_0^1 \rightarrow a_0^0$ (resp., $a_0^1 \rightarrow a_1^0$) then (a_1^1, a_2^0, u, a_1^1) and $(a_0^1, a_0^0, a_1^0, v, a_0^1)$ (resp., $(a_0^1, a_1^0, v, a_2^1, v, a_0^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \rightarrow a_1^1$. It follows that $a_1^1 \rightarrow \{a_0^0, a_1^0\}$. Similarly, we can prove that $a_2^0 \rightarrow a_0^1$ and $a_0^1 \rightarrow \{a_0^0, a_1^0\}$. It follows that a_0^0 has an out-neighbor $x \in V'$. It is not difficult to verify that $x \in K$, $A^0 \rightarrow x$ and $x \rightarrow A^1$. Then, $(a_1^1, a_1^0, a_2^0, a_1^1)$ and $(a_0^0, x, a_2^1, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that a_0^0 is adjacent to u but $u \rightarrow a_0^0$. By Lemma 11, $u \rightarrow A^0$ and $A^1 \rightarrow u$. If a_0^0 is not adjacent to v then by Lemma 13, $v \rightarrow A^1$, a contradiction to Observation 4. So, a_0^0 is adjacent to v. By Lemma 11 and Observation 4, $v \rightarrow A^0$ and $A^1 \rightarrow v$. By Lemma 10, a_0^0 has no out-neighbor in V'. So, a_0^0 has at least two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \rightarrow \{a_0^1, a_1^1\}$. If $a_2^0 \rightarrow a_1^1$ (resp., $a_2^0 \rightarrow a_2^1$) then (a_0^0, a_0^1, u, a_0^0) and $(a_2^0, a_1^1, v, a_1^0, a_2^0)$ (resp., $(a_2^0, a_2^1, v, a_1^0, a_2^0)$ are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^1, a_2^1\} \rightarrow a_2^0$ and a_2^0 has an out-neighbor $w \in V'$. By Lemma 10, $w \in I$ and w has no out-neighbor in V'. So, w has at least two out-neighbors in $V(A^1)$, it follows that $w \rightarrow a_1^1$ or $w \rightarrow a_2^1$. Then, (a_0^0, a_0^1, u, a_0^0), and $(a_2^0, w, a_1^1, v, a_2^0)$ or $(a_2^0, w, a_2^1, v, a_2^0)$, are two disjoint cycles of different lengths in D, a contradiction.

Subcase 1.3. $A^j \cap I \neq \emptyset$ for every $j = 0, 1$.

By renaming the vertices of $V(A^0)$ and $V(A^1)$, we may assume that $a_0^0, a_0^1 \in I$.

First, assume that u is adjacent to a_0^0 and a_0^1. By Lemma 11 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$ and $u \to A^1$. Suppose that v is adjacent to a_0^0 and a_0^1. Then by Lemma 11 and Observation 4, $A^0 \to v$ and $v \to A^1$. By Lemma 10, a_0^1 has no out-neighbor in V', so $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_1^0$ (resp., $a_1^1 \to a_2^0$) then (a_1^1, a_1^0, u, a_1^1) and $(a_0^1, a_2^0, v, a_2^1, a_0^1)$ (resp., (a_1^1, a_2^0, u, a_1^1) and $(a_0^1, a_1^0, v, a_2^1, a_0^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \to a_1^1$ and a_1^1 has an out-neighbor $w \in V'$. Because I contains at most two vertices, it is easy to see that $w \in K$, a contradiction to Lemma 10. By similar reasoning, we can deduce the contradiction in cases v is adjacent to a_0^0 and is not adjacent to a_0^1, v is adjacent to a_0^1 and is not adjacent to a_0^0, or v is not adjacent to a_0^0 and a_0^1.

Next, assume that u is adjacent to exactly one of the two vertices a_0^0 and a_0^1. By renaming the cycles A^0 and A^1, we may assume that u is adjacent to a_0^0 and is not adjacent to a_0^1. By Lemma 13, $\{a_1^1, a_2^1\} \to u$ and $u \to A^0$. Suppose that v is adjacent to a_0^0 and a_0^1. Then by Lemma 11 and Observation 4, $v \to A^0$ and $A^1 \to v$. By Lemma 10 and I contains at most two vertices, each vertex of $V(A^0)$ has no out-neighbor in V'. So, each vertex of $V(A^0)$ has two out-neighbors in $V(A^1)$. It follows that $a_0^0 \to \{a_1^1, a_2^1\}$. If $a_1^0 \to a_1^1$ (resp., $a_1^0 \to a_2^1$) then (a_1^0, a_1^1, v, a_1^0) and $(a_2^1, u, a_2^0, a_0^0, a_2^1)$ (resp., (a_1^0, a_2^1, v, a_1^0) and $(a_1^1, u, a_2^0, a_0^0, a_1^1)$) are two disjoint cycles of different lengths in D, a contradiction. Using a similar method, we can demonstrate the contradiction in cases v is adjacent to a_0^0 and is not adjacent to a_0^1, v is adjacent to a_0^1 and is not adjacent to a_0^0, or v is not adjacent to a_0^0 and a_0^1.

Finally, assume that u is not adjacent to a_0^0 and a_0^1. By Lemma 12 and by renaming the cycles A^0 and A^1, we may assume that $a_2^0 \to u$ and $u \to \{a_1^0, a_1^1, a_2^1\}$. Suppose that v is adjacent to a_0^0 and a_0^1. Then by Lemma 11 and Observation 4, $v \to A^1$ and $A^0 \to v$. By Lemma 10 and I contains at most two vertices, each vertex of $V(A^1)$ has no out-neighbor in V'. So, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. It follows that $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_1^0$ (resp., $a_1^1 \to a_2^0$) then (a_1^1, a_1^0, v, a_1^1) and $(a_0^1, a_2^0, u, a_2^1, a_0^1)$ (resp., (a_1^1, a_2^0, u, a_1^1) and $(a_0^1, a_1^0, v, a_2^1, a_0^1)$) are two disjoint cycles of different lengths in D, a contradiction. Using a similar method, we can demonstrate the contradiction in cases v is adjacent to a_0^0 and is not adjacent to a_0^1, v is adjacent to a_0^1 and is not adjacent to a_0^0, or v is not adjacent to a_0^0 and a_0^1.

Case 2. V' contains exactly one vertex $u \in K$ and there exists $v \in V'$

such that $u \to v$.

In this case, all vertices in $V' \setminus \{u\}$ also belong to I. We again divide Case 2 into several subcases.

Subcase 2.1. $A^j \cap I = \emptyset$ for every $j = 0, 1$.

By Lemma 11 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. By Observation 4, $v \to A^1$.

First, assume that a_0^1 has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^1 \to \{a_0^0, a_1^0\}$. Suppose that $a_2^1 \to a_1^0$ and $a_1^0 \to a_1^1$. Then, $(a_2^1, a_1^0, a_1^1, a_2^1)$ and $(a_0^0, u, v, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $a_2^1 \to a_1^0$ and $a_1^1 \to a_1^0$. Then, a_1^0 has an out-neighbor $w \in V'$ and $w \neq v$, because if $w = v$ then (a_1^0, w, a_1^1, a_1^0) and $(a_0^0, u, a_2^1, a_0^1, a_0^0)$ are two disjoint cycles of different lengths in D. By Observation 5, $w \not\to a_2^0$. If $w \to a_0^0$ then we set $(A^0)' = (w, a_0^0, a_1^0, w)$, A^1 and two vertices $u, a_2^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 1.2 we deduce the contradiction. So, $w \not\to a_0^0$ and w has two out-neighbors in $V(A^1)$. It follows that $w \to a_1^1$ or $w \to a_2^1$. Then, (w, a_1^1, a_1^0, w) or (w, a_2^1, a_1^0, w), and $(a_0^0, u, v, a_0^1, a_0^0)$, are two disjoint cycles of different lengths in D, a contradiction. Thus, $a_1^0 \to a_2^1$. If $a_1^1 \to a_0^0$ (resp., $a_1^1 \to a_2^0$) then $(a_0^1, a_1^0, a_2^1, a_0^1)$ and $(a_1^1, a_0^0, u, v, a_1^1)$ (resp., $(a_1^1, a_2^0, u, v, a_1^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_0^0, a_2^0\} \to a_1^1$ and a_1^1 has an out-neighbor $w \in V'$. By Lemma 10 and Observation 5, $w \to a_0^0$ or $w \to a_2^0$. Then, (w, a_0^0, a_1^1, w) or (w, a_2^0, a_1^1, w), and $(a_1^0, u, a_2^1, a_0^1, a_1^0)$, are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that a_0^1 has at most one out-neighbor in $V(A^0)$. By renaming the vertices of $V(A^0)$, we may assume that $\{a_0^0, a_1^0\} \to a_0^1$. Then, a_0^1 has an out-neighbor $w \in V'$. By Lemma 10 and Observation 5, w has at least two out-neighbors in $V(A^0)$. Suppose that $w \to a_2^1$. Set A^0, $(A^1)' = (w, a_2^1, a_0^1, w)$ and two vertices $u, a_1^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.2 we deduce the contradiction. Thus, $w \not\to a_2^1$ and $w \to A^0$. If $a_1^1 \to a_i^0$ with $i \in \{1, 2\}$ then (w, a_0^0, a_0^1, w) and $(a_1^1, a_i^0, u, v, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction. If $a_1^1 \to a_0^0$ then (w, a_1^0, a_0^1, w) and $(a_1^1, a_0^0, u, v, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction. So, a_1^1 has no out-neighbor in $V(A^0)$. It follows that a_1^1 has an out-neighbor $z \in V'$, $z \neq w$, contradicts the assumption that I contains at most two vertices.

Subcase 2.2. There exists $j \in \{0, 1\}$ such that $A^j \cap I \neq \emptyset$ and $A^{j+1} \cap I = \emptyset$.

By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^0 \in I$. It is easy to see that V' contains exactly two vertices u and v.

First, assume that $a_0^0 \to u$. By Lemma 11, $A^0 \to u$ and $u \to A^1$. By Observation 4, $v \to A^1$. It is easy to see that a_0^0 has an out-neighbor in $V(A^1)$ and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to a_0^1$. So, $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_1^0$ then $(a_0^0, a_0^1, a_2^0, a_0^0)$ and $(a_1^1, a_1^0, u, v, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \to a_1^1$. It follows that $a_1^1 \to \{a_0^0, a_2^0\}$. Then, $(a_1^1, a_0^0, a_1^0, a_1^1)$ and $(a_0^1, a_2^0, u, v, a_0^1)$ are two disjoint cycles of different lengths in D, a contradiction.

Next, assume that $u \to a_0^0$. By Lemma 11, $u \to A^0$ and $A^1 \to u$. It is easy to see that v has an out-neighbor in $V(A^1)$, a contradiction to Observation 4.

Finally, assume that u is not adjacent to a_0^0. By Lemma 13, $\{a_1^0, a_2^0\} \to u$ and $u \to A^1$. By Observation 4, $v \to A^1$. It is easy to see that a_0^0 has two out-neighbors in $V(A^1)$ and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. So, $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$. Then, $(a_0^0, a_0^1, a_2^0, a_0^0)$ and $(a_1^1, a_1^0, u, v, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction.

Case 3. V' contains exactly one vertex $u \in K$ and there exists $v \in V'$ such that $v \to u$.

In this case, all vertices in $V' \setminus \{u\}$ also belong to I. By Case 2, we may assume that if $w \in V' \setminus \{u\}$ then $u \not\to w$. We again divide Case 3 into several subcases.

Subcase 3.1. $A^j \cap I = \emptyset$ for every $j = 0, 1$.

By Lemma 11 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ sao cho $x \to v$. By Observation 4, $x \in V(A^0)$. By renaming the vertices of $V(A^0)$, we may assume that $x = a_0^0$. By Observation 5, $v \not\to a_1^0$

First, assume that $v \to a_2^0$. Set $(A^0)' = (v, a_2^0, a_0^0, v)$, A^1 and two vertices $u, a_1^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 1.2 we deduce the contradiction.

Now, assume that $v \not\to a_2^0$. Then, v has two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $v \to \{a_0^1, a_1^1\}$. Suppose that $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (resp., $a_2^1 \to a_2^0$) then (a_0^0, v, a_0^1, a_0^0) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (resp., $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \to a_2^1$ and a_2^1 has an out-neighbor $w \in V'$. By Lemma 10 and Observation 5, $w \to a_1^0$ or $w \to a_2^0$. Then, (a_0^0, v, a_0^1, a_0^0), and $(a_2^1, w, a_1^0, u, a_2^1)$ or $(a_2^1, w, a_2^0, u, a_2^1)$, are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $a_0^0 \to a_0^1$. Since

a_0^1 has at most one out-neighbor in V', $a_0^1 \to a_1^0$ or $a_0^1 \to a_2^0$. First, we consider case $a_0^1 \to a_1^0$. If $a_2^1 \to a_0^0$ (resp., $a_2^1 \to a_2^0$) then (a_1^0, u, a_0^1, a_1^0) and $(a_2^1, a_0^0, v, a_1^1, a_2^1)$ (resp., $(a_2^1, a_2^0, a_0^0, v, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_0^0, a_2^0\} \to a_2^1$ and a_2^1 has an out-neighbor $w \in V'$. By Lemma 10 and Observation 5, $w \to a_0^0$ or $w \to a_2^0$. Then, (a_1^0, u, a_0^1, a_1^0), and $(a_2^1, w, a_0^0, v, a_1^1, a_2^1)$ or $(a_2^1, w, a_2^0, a_0^0, a_2^1)$, are two disjoint cycles of different lengths in D, a contradiction. Similarly, we can point out the contradiction when considering case $a_0^1 \to a_2^0$.

Subcase 3.2. There exists $j \in \{0, 1\}$ such that $A^j \cap I \neq \emptyset$ and $A^{j+1} \cap I = \emptyset$.

By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^0 \in I$. It is easy to see that V' contains exactly two vertices u and v.

First, assume that $a_0^0 \to u$. By Lemma 11, $A^0 \to u$ and $u \to A^1$. If a_i^1 with $i \in \{0, 1, 2\}$ has an out-neighbor $w \in V'$ then by Lemma 10 and Observation 5, $w \to a_{i+2}^1$. Set A^0, $(A^1)' = (w, a_{i+2}^1, a_i^1, w)$ and two vertices $u, a_{i+1}^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.3 we deduce the contradiction. So, each vertex of $V(A^1)$ has no out-neighbor in V' and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to a_0^1$. It follows that $a_0^1 \to \{a_1^0, a_2^0\}$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ such that $x \to v$. By Observation 4, $x \in V(A^0)$. First, suppose that $x = a_1^0$. If $a_1^1 \to a_0^0$ (resp., $a_2^1 \to a_1^0$) then $(a_0^0, a_0^1, a_2^0, a_0^0)$ and $(a_1^1, a_1^0, v, u, a_1^1)$ (resp., $(a_2^1, a_1^0, v, u, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \to \{a_1^1, a_2^1\}$ and $\{a_1^1, a_2^1\} \to \{a_0^0, a_2^0\}$. Then, (a_2^0, u, a_1^1, a_2^0) and $(a_0^0, a_0^1, a_1^0, a_2^1, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction. By considering the same thing, in case $x = a_2^0$ and $a_1^0 \not\to v$ we can also point out the contradiction.

Next, assume that $u \to a_0^0$. Then, a_0^0 has two out-neighbors in $V(A^1)$. By Lemma 11, $A^1 \to u$ and $u \to A^0$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ such that $x \to v$. By Observation 4, $x \in V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $x = a_0^1$. So, $v \not\to a_1^1$. If $v \to a_2^1$ then we set A^0, $(A^1)' = (v, a_2^1, a_0^1, v)$ and two vertices $u, a_1^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.3 we deduce the contradiction. So, $v \not\to a_2^1$ and $v \to \{a_1^0, a_2^0\}$. Suppose that $a_1^0 \to a_0^1$. Since a_0^0 has two out-neighbors in $V(A^1)$, $a_0^0 \to a_1^1$ or $a_0^0 \to a_2^1$. Then, (v, a_1^0, a_0^1, v), and $(a_0^0, a_1^1, u, a_2^0, a_0^0)$ or $(a_0^0, a_2^1, u, a_2^0, a_0^0)$, are two disjoint cycles of different lengths in D, a contradiction. Thus, $a_0^1 \to a_1^0$. Since a_0^0 has two out-neighbors in $V(A^1)$, $a_0^0 \to a_0^1$ or $a_0^0 \to a_1^1$. If $a_1^0 \to a_2^1$ then (u, a_1^0, a_2^1, u) and $(a_0^0, a_0^1, v, a_2^0, a_0^0)$, or (u, a_0^0, a_1^1, u) and $(a_1^0, a_2^1, a_0^1, v, a_1^0)$) are two disjoint cycles of different lengths in D, a contradiction. Thus, $a_2^1 \to a_1^0$. It follows that a_1^0 has an out-neighbor $w \in V'$ and $w \to A^1$. Then, (a_0^0, a_1^1, u, a_0^0) and $(a_1^0, w, a_0^1, v, a_1^0)$, or (a_0^0, a_2^1, u, a_0^0) and $(a_1^0, w, a_0^1, v, a_1^0)$, are two disjoint cycles of different lengths in D, a contradiction.

Finally, assume that u is not adjacent to a_0^0. Then, a_0^0 has two out-neighbors in $V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. By Lemma 13, $u \to A^1$ and $\{a_1^0, a_2^0\} \to u$. If a_i^1 with $i \in \{0, 1, 2\}$ has an out-neighbor $w \in V'$ then by Lemma 10 and Observation 5, $w \to a_{i+2}^1$. Set A^0, $(A^1)' = (w, a_{i+2}^1, a_i^1, w)$ and two vertices $u, a_{i+1}^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.3 we deduce the contradiction. So, each vertex of $V(A^1)$ has no out-neighbor in V' and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. So, $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$. Then, $(a_0^0, a_1^1, a_2^0, a_0^0)$ and $(a_0^1, a_1^0, u, a_2^1, a_0^1)$ are two disjoint cycles of different lengths in D, a contradiction.

Case 4. V' contains exactly one vertex $u \in K$ and u is not adjacent to any vertex in $V' \setminus \{u\}$.

Let $v \in V' \setminus \{u\}$. We again divide Case 4 into several subcases.

Subcase 4.1. $A^j \cap I = \emptyset$ for every $j = 0, 1$.

By Lemma 11 and by renaming the cycles A^0 and A^1, we may assume that $A^0 \to u$, $u \to A^1$. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ such that $x \to v$.

First, assume that $x \in V(A^0)$. By renaming the vertices of $V(A^0)$, we may assume that $x = a_0^0$. Easy to see $v \not\to a_1^0$. If $v \to a_2^0$ then we set $(A^0)' = (v, a_2^0, a_0^0, v)$, A^1 and two vertices $u, a_1^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 1.2 we deduce the contradiction. So, $v \not\to a_2^0$ and $v \to A^1$. Suppose that a_i^1 with $i \in \{0, 1, 2\}$ has an out-neighbor $w \in V'$. By Lemma 10 and Observation 5, $w \to a_{i+2}^1$ or $w \to A^0$. If $w \to a_{i+2}^1$ then we set A^0, $(A^1)' = (w, a_{i+2}^1, a_i^1, w)$ and two vertices $u, a_{i+1}^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.2 we deduce the contradiction. So, $w \to A^0$. It is easy to see that each vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. If $a_{i+1}^1 \to a_1^0$ (resp., $a_{i+1}^1 \to a_2^0$) then $(a_{i+1}^1, a_1^0, u, a_{i+1}^1)$ (resp., $(a_{i+1}^1, a_2^0, u, a_{i+1}^1)$) and $(a_i^1, w, a_0^0, v, a_i^1)$ are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_1^0, a_2^0\} \to a_{i+1}^1$ and $a_{i+1}^1 \to a_0^0$. Then, $(a_0^0, a_1^0, a_{i+1}^1, a_0^0)$ and $(a_2^0, u, a_i^1, w, a_2^0)$ are two disjoint cycles of different lengths in D, a contradiction. Thus, each vertex of $V(A^1)$ has no out-neighbor in V' and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. Suppose that $a_0^1 \to a_0^0$. If $a_2^1 \to a_1^0$ (resp., $a_2^1 \to a_2^0$) then (a_0^1, a_0^0, v, a_0^1) and $(a_2^1, a_1^0, u, a_1^1, a_2^1)$ (resp., $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_0^0 \to a_0^1$ and $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_2^1 \to a_0^0$ (resp., $a_2^1 \to a_1^0$) then (a_2^1, a_0^0, v, a_2^1) and $(u, a_0^1, a_1^0, a_2^0, u)$ (resp., (a_2^1, a_1^0, u, a_2^1)

and $(v, a_0^1, a_2^0, a_0^0, v)$) are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that $x \in V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $x = a_0^1$. Easy to see $v \not\to a_1^1$. If $v \to a_2^1$ then easy to see v is not adjacent to a_1^1, it follows that a_1^1 is not adjacent to exactly one vertex in $(A^1)' = (v, a_2^1, a_0^1, v)$ and is adjacent to all vertices in A^0, from which we deduce the contradiction with the Lemma 13. So, $v \not\to a_2^1$ and $v \to A^0$. Suppose that a_1^1 has an out-neighbor in $V(A^0)$. Without loss of generality, we may assume that $a_1^1 \to a_0^0$. If $a_2^0 \to a_0^1$ (resp., $a_2^0 \to a_2^1$) then (a_1^1, a_0^0, u, a_1^1) and $(a_2^0, a_0^1, v, a_1^0, a_2^0)$ (resp., $(a_2^0, a_2^1, a_0^1, v, a_2^0)$) are two disjoint cycles of different lengths in D, a contradiction. So, $\{a_0^1, a_2^1\} \to a_2^0$. If $a_2^0 \to a_1^1$ then $(a_2^0, a_1^1, a_2^1, a_2^0)$ and $(a_0^0, u, a_0^1, v, a_0^0)$ are two disjoint cycles of different lengths in D, a contradiction. So, $A^1 \to a_2^0$. It follows that a_2^0 has an out-neighbor $w \in V'$. It is not difficult to see that $w \to a_1^1$ or $w \to a_2^1$. Then (a_2^0, w, a_1^1, a_2^0) or (a_2^0, w, a_2^1, a_2^0), and $(a_0^0, u, a_0^1, v, a_0^0)$, are two disjoint cycles of different lengths in D, a contradiction. Thus, a_1^1 has no out-neighbor in $V(A^0)$, that is $A^0 \to a_1^1$ and a_1^1 has an out-neighbor $w \in V'$. It is not difficult to see that $w \to A^0$. Then, (a_1^1, w, a_0^0, a_1^1) and $(a_1^0, u, a_0^1, v, a_1^0)$ are two disjoint cycles of different lengths in D, a contradiction.

Subcase 4.2. There exists $j \in \{0, 1\}$ such that $A^j \cap I \neq \emptyset$ and $A^{j+1} \cap I = \emptyset$.

By renaming the cycles A^0 and A^1, we may assume that $j = 0$. By renaming the vertices of $V(A^0)$, we may assume that $a_0^0 \in I$. It is easy to see that V' contains exactly two vertices u and v. Since D is strong, there exists $x \in V(A^0) \cup V(A^1)$ such that $x \to v$. Suppose that $x \in V(A^1)$. By renaming the vertices of $V(A^1)$, we may assume that $x = a_0^1$. By Observation 5, $v \not\to a_1^1$. So, $v \to a_2^1$. Set A^0, $(A^1)' = (v, a_2^1, a_0^1, v)$ and two vertices $u, a_1^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 1.3 we deduce the contradiction. So, $x = a_1^0$ or $x = a_2^0$.

First, assume that $a_0^0 \to u$. By Lemma 11, $A^0 \to u$ and $u \to A^1$. Suppose that $x = a_1^0$. Since $v \not\to \{a_0^0, a_2^0\}$, $v \to A^1$. It is easy to see that a_0^0 has an out-neighbor in $V(A^1)$ and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to a_0^1$. So, $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_2^1 \to a_0^0$ (resp., $a_2^1 \to a_2^0$) then (a_0^1, a_1^0, v, a_0^1) and $(a_2^1, a_0^0, u, a_1^1, a_2^1)$ (resp., $(a_2^1, a_2^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $x = a_2^0$ and $a_1^0 \not\to v$. If $v \to a_1^0$ then we set $(A^0)' = (v, a_1^0, a_2^0, v)$, A^1 and two vertices $u, a_0^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 3.2 we deduce the contradiction. So, $v \not\to a_1^0$ and $v \to A^1$. It is easy to see that a_0^0 has an out-neighbor in $V(A^1)$ and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to a_0^1$.

So, $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_2^1 \to a_0^0$ (resp., $a_2^1 \to a_1^0$) then (a_0^1, a_2^0, v, a_0^1) and $(a_2^1, a_0^0, u, a_1^1, a_2^1)$ (resp., $(a_2^1, a_1^0, u, a_1^1, a_2^1)$) are two disjoint cycles of different lengths in D, a contradiction.

Next, assume that $u \to a_0^0$. By Lemma 11, $A^1 \to u$ and $u \to A^0$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. First, suppose that $x = a_1^0$. It is clear that $v \to A^1$. It is easy to see that each vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. If $a_0^1 \to a_2^0$ (resp., $a_1^1 \to a_2^0$) then $(a_0^1, a_2^0, a_0^0, a_0^1)$ (resp., $(a_1^1, a_2^0, a_0^0, a_0^1)$) and (v, a_2^1, u, a_1^0, v) are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \to \{a_0^1, a_1^1\}$ and $\{a_0^1, a_1^1\} \to a_1^0$. Then, (u, a_2^0, a_1^1, u) and $(v, a_2^1, a_0^1, a_1^0, v)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $x = a_2^0$ and $a_1^0 \not\to v$. If $v \to a_1^0$ then we set $(A^0)' = (v, a_1^0, a_2^0, v)$, A^1 and two vertices $u, a_0^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 2.2 we deduce the contradiction. So, $v \not\to a_1^0$ and $v \to A^1$. It is easy to see that each vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. If $a_2^1 \to a_1^0$ (resp., $a_1^1 \to a_1^0$) then (u, a_0^0, a_0^1, u) and $(v, a_2^1, a_1^0, a_2^0, v)$ (resp., $(v, a_1^1, a_1^0, a_2^0, v)$) are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \to \{a_1^1, a_2^1\}$ and $a_1^1 \to a_2^0$. Then, (u, a_1^0, a_2^1, u) and $(v, a_0^1, a_1^1, a_2^0, v)$ are two disjoint cycles of different lengths in D, a contradiction.

Finally, assume that u is not adjacent to a_0^0. By Lemma 13, $u \to A^1$ and $\{a_1^0, a_2^0\} \to u$. By renaming the vertices of $V(A^1)$, we may assume that $a_0^0 \to \{a_0^1, a_1^1\}$. First, suppose that $x = a_1^0$. It is clear that $v \to A^1$ and each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. So, $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$. Then, (u, a_1^1, a_2^0, u) and $(v, a_2^1, a_0^1, a_1^0, v)$ are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $x = a_2^0$ and $a_1^0 \not\to v$. It is clear that each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$. So, $\{a_0^1, a_1^1\} \to \{a_1^0, a_2^0\}$. Since v has at least two out-neighbor in $V(A^1)$, $v \to a_1^1$ or $v \to a_2^1$. Then, (v, a_1^1, a_2^0, v) and $(u, a_2^1, a_0^1, a_1^0, u)$, or (u, a_1^1, a_1^0, u) and $(v, a_2^1, a_0^1, a_2^0, v)$ are two disjoint cycles of different lengths in D, a contradiction.

Case 5. $V' = I$.

In this case, all vertices in $V(A^0) \cup V(A^1)$ are also in K. Let $V' = \{u, v\}$. Since D is strong, there exists $x, y \in V(A^0) \cup V(A^1)$ such that $x \to u$ and $y \to v$. By renaming the cycles A^0 and A^1, we may assume that $x \in V(A^0)$. By renaming the vertices of $V(A^1)$, we may assume that $x = a_0^0$. By Observation 5, $u \not\to a_1^0$. If $u \to a_2^0$ then we set $(A^0)' = (u, a_2^0, a_0^0, u)$, A^1 and two vertices $v, a_1^0 \in V(D) \setminus (V((A^0)') \cup V(A^1))$. By considering similarly to Subcase 2.2, or Subcase 3.2, or Subcase 4.2, we deduce the contradiction. So, $u \not\to a_2^0$ and $u \to A^1$. We again divide Case 5 into several subcases.

Subcase 5.1. $y \in V(A^0)$.

By similar reasoning as above we see that $v \to A^1$. So, each vertex of $V(A^1)$ has two out-neighbors in $V(A^0)$.

First, assume that $y = a_0^0$. Since a_2^1 has two out-neighbors in $V(A^0)$, $a_2^1 \to a_1^0$ or $a_2^1 \to a_2^0$. Suppose that $a_0^1 \to \{a_0^0, a_1^0\}$. If $a_1^0 \to v$ then (u, a_0^1, a_0^0, u) and $(v, a_1^1, a_2^1, a_1^0, v)$, or (v, a_0^1, a_1^0, v) and $(u, a_2^1, a_2^0, a_0^0, u)$, are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \not\to v$. If $a_1^0 \to u$ then (v, a_0^1, a_0^0, v) and $(u, a_1^1, a_2^1, a_1^0, u)$, or (u, a_0^1, a_1^0, u) and $(v, a_2^1, a_2^0, a_0^0, v)$, are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \not\to u$. It follows that $a_1^0 \to \{a_1^1, a_2^1\}$ and $\{a_1^1, a_2^1\} \to \{a_0^0, a_2^0\}$. Then, $(a_0^1, a_1^0, a_2^1, a_0^1)$ and $(v, a_1^1, a_2^0, a_0^0, v)$, are two disjoint cycles of different lengths in D, a contradiction. Using a similar method of proof, we can also find the contradiction in cases $a_0^1 \to \{a_0^0, a_2^0\}$ or $a_0^1 \to \{a_1^0, a_2^0\}$.

Next, assume that $y = a_1^0$. Suppose that $a_0^1 \to \{a_0^0, a_2^0\}$. If $a_2^0 \to u$ then (u, a_0^1, a_0^0, u) and $(v, a_1^1, a_2^1, a_1^0, v)$, or (u, a_2^1, a_2^0, u) and $(v, a_0^1, a_0^0, a_1^0, v)$, are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \not\to u$. If $a_2^0 \to v$ then (u, a_0^1, a_0^0, u) and $(v, a_1^1, a_2^1, a_1^0, v)$, or (u, a_0^1, a_0^0, u) and $(v, a_1^1, a_2^1, a_2^0, v)$, are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \not\to v$. It follows that $a_2^0 \to \{a_1^1, a_2^1\}$ and $\{a_1^1, a_2^1\} \to \{a_0^0, a_1^0\}$. Then, (u, a_0^1, a_0^0, u) and $(a_1^1, a_2^1, a_1^0, a_2^0, a_1^1)$ are two disjoint cycles of different lengths in D, a contradiction. We can easily deduce the contradiction in either case $a_0^1 \to \{a_0^0, a_1^0\}$ or $a_0^1 \to \{a_1^0, a_2^0\}$.

Finally, assume that $y = a_2^0$. Suppose that $a_0^1 \to \{a_0^0, a_1^0\}$ or $a_0^1 \to \{a_0^0, a_2^0\}$. Then, (u, a_0^1, a_0^0, u) and $(v, a_2^1, a_1^0, a_2^0, v)$, or (u, a_0^1, a_0^0, u) and $(v, a_1^1, a_2^1, a_2^0, v)$, are two disjoint cycles of different lengths in D, a contradiction. Now, suppose that $a_0^1 \to \{a_1^0, a_2^0\}$. If $a_1^1 \to a_0^0$ then (u, a_1^1, a_0^0, u) and $(v, a_0^1, a_1^0, a_2^0, v)$ are two disjoint cycles of different lengths in D, a contradiction. So, $a_0^0 \to a_1^1$ and $a_1^1 \to \{a_1^0, a_2^0\}$. It follows that $a_1^0 \to u$ or $a_1^0 \to v$. Then, (v, a_1^1, a_2^0, v) and $(a_1^0, u, a_2^1, a_0^1, a_1^0)$, or (a_1^0, v, a_0^1, a_1^0) and $(u, a_1^1, a_2^0, a_0^0, u)$ are two disjoint cycles of different lengths in D, a contradiction.

Subcase 5.2. $y \in V(A^1)$.

By renaming the vertices of $V(A^1)$, we may assume that $y = a_0^1$. By Observation 5, $v \not\to a_1^1$. If $v \to a_2^1$ then we set A^0, $(A^1)' = (v, a_2^1, a_0^1, v)$ and two vertices $u, a_1^1 \in V(D) \setminus (V(A^0) \cup V((A^1)'))$. By considering similarly to Subcase 3.2 we deduce the contradiction. So, $v \not\to a_2^1$ and $v \to A^0$. It is clear that each vertex of $V(A^0)$ has an out-neighbor in $V(A^1)$ and each vertex of $V(A^1)$ has an out-neighbor in $V(A^0)$. If $a_2^1 \to a_1^0$ and $a_1^0 \to a_1^1$ then $(a_1^1, a_2^1, a_1^0, a_1^1)$ and (u, a_0^1, v, a_0^0, u) are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^0 \to a_2^1$ or $a_1^1 \to a_1^0$.

First, assume that $a_1^0 \to a_2^1$. If $a_1^1 \to a_0^0$ then (a_1^1, a_0^0, u, a_1^1) and $(v, a_1^0, a_2^1, a_0^1, v)$ are two disjoint cycles of different lengths in D, a contradiction. So, $a_0^0 \to a_1^1$.

If $a_1^1 \to a_2^0$ then $(a_1^1, a_2^0, a_0^0, a_1^1)$ and $(v, a_1^0, a_2^1, a_0^1, v)$ are two disjoint cycles of different lengths in D, a contradiction. So, $a_2^0 \to a_1^1$ and $a_1^1 \to a_1^0$. Then, $(a_1^1, a_1^0, a_2^0, a_1^1)$ and (u, a_0^1, v, a_0^0, u) are two disjoint cycles of different lengths in D, a contradiction.

Now, assume that $a_2^1 \to a_1^0$ and $a_1^1 \to a_1^0$. So, $a_1^0 \to a_0^1$. If $a_2^0 \to a_1^1$ then $(a_2^0, a_1^1, a_1^0, a_2^0)$ and (u, a_0^1, v, a_0^0, u) are two disjoint cycles of different lengths in D, a contradiction. So, $a_1^1 \to a_2^0$. Then, (v, a_1^0, a_0^1, v) and $(u, a_1^1, a_2^0, a_0^0, u)$ are two disjoint cycles of different lengths in D, a contradiction.

The proof of Theorem 1 is complete.

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References

[1] M. Abreu, E. L. Aldred, M. Funk, B. Jackson, D. Labbate and J. Sheehan, Graphs and digraphs with all 2-factors isomorphic, *J. Combin. Theory Ser. B*, 92 (2004), 395 – 404.

[2] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*, Springer, New York, 2001.

[3] J. Bang-Jensen and F. Havet, Tournaments and semicomplete digraphs, In: *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds), pp. 35 – 124, Springer, New York, 2018.

[4] J. Bensmail, A. Harutyunyan, N. K. Le, B. Li and N. Lichiardopol, Disjoint cycles of different lengths in graphs and digraphs, *Electron. J. Combin.*, 24 (2017), Article P4.37.

[5] Y. Gao and D. Ma, Disjoint cycles with different length in 4-arc-dominated digraphs, *Oper. Res. Lett.*, 41 (2013), 650 – 653.

[6] Z. He, P. Cheng and Y. Gao, Disjoint cycles of different lengths in 3-regular digraphs, *Graphs Combin.*, 38 (3) (2022), Article 74.

[7] M. A. Henning and A. Yeo, Vertex disjoint cycles of different length in digraphs, *SIAM J. Discrete Math.*, 26 (2012), 687 – 694.

[8] L. X. Hung, D. D. Hieu and N. D. Tan, Vertex-disjoint cycles of different lengths in multipartite tournaments, *Discrete Math.*, 345 (2022), Article 112819.

[9] L. X. Hung and N. D. Tan, Vertex-Disjoint Cycles of Different Lengths in Local Tournaments. *Graphs and Combinatorics*, Volume 39, Issue 5, Article 92 (2023).

[10] L. X. Hung, Vertex-disjoint cycles of different lengths in tournaments of cycles, *Siberian Electronic Mathematical Reports*, Vol. 22, No. 1, pp. 46 – 53 (2025).

[11] N. Lichiardopol, Proof of a conjecture of Henning and Yeo on vertex-disjoint directed cycles, *SIAM J. Discrete Math.*, 28 (2014), 1618 – 1627.

[12] R. M. Steiner, Disjoint cycles with length constraints in digraphs of large connectivity or large minimum degree, *SIAM J. Discrete Math.*, 36 (2) (2022), 1343 – 1362.

[13] N. D. Tan, Vertex disjoint cycles of different lengths in d-arc-dominated digraphs, *Oper. Res. Lett.*, 42 (2014), 351 – 354.

[14] N. D. Tan, On vertex disjoint cycles of different lengths in 3-regular digraphs, *Discrete Math.*, 338 (2015), 2485 – 2491.

[15] N. D. Tan, On 3-regular digraphs without vertex disjoint cycles of different lengths, *Discrete Math.*, 340 (2017), 1933 – 1943.

[16] N. D. Tan, On 3-regular digraphs of girth 4, *Discrete Math.*, 342 (2020), Article 111632.

[17] N. D. Tan, Tournaments and bipartite tournaments without vertex disjoint cycles of different lengths, *SIAM J. Discrete Math.*, 35 (1) (2021), 485 – 494.

[18] N. D. Tan, A decomposition for digraphs with minimum outdegree 3 having no vertex disjoint cycles of different lengths, *Discussiones Math. Graph Theory*, 43 (2023), 573 – 581.

[19] C. Thomassen, Disjoint cycles in digraphs, *Combinatorica*, 3 (1983), 393 – 396.

[20] A. Yeo, Semicomplete multipartite digraphs, In: *Classes of Directed Graphs* (J. Bang-Jensen and G. Gutin, eds), pp. 297 – 340, Springer, New York, 2018.