

**CONTINUOUS TIME MARKOV CHAIN BASED
APPROXIMATIONS OF STATIONARY AND WEAK
KAM HAMILTON-JACOBI EQUATIONS**YU. AVERBOUKH *Communicated by P.P. PETROV*

Abstract: The main objects of this paper are the stationary and weak KAM Hamilton-Jacobi equations on the finite-dimensional torus. The key idea of the paper is to replace the underlying calculus of variations problems with continuous time Markov decision problems. This leads directly to an approximation of the stationary Hamilton-Jacobi equation by the Bellman equation for a discounted Markov decision problem. Developing elements of the weak KAM theory for the Markov decision problem, we obtain an approximation of the effective Hamiltonian. Additionally, convergences of the functional parts of the discrete weak KAM equations and Mather measures are shown. It turns out that the approximating equations are systems of algebraic equations. Thus, the paper's result can be seen as numerical schemes for stationary and weak KAM Hamilton-Jacobi equations.

Keywords: Hamilton-Jacobi equation, discrete weak KAM theory, Markov chain approximation, discrete Mather measure.

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1 Introduction

Overview of the main results. The paper is concerned with the discrete approximation of solutions to the stationary Hamilton-Jacobi equation

$$\lambda u_\lambda + H(x, -\nabla u_\lambda) = 0 \tag{1}$$

and the weak KAM equation

$$H(x, -\nabla u) = \bar{H}. \tag{2}$$

The Hamiltonian H is assumed to be periodic in x , i.e., we consider them on the d -dimensional torus \mathbb{T}^d . Notice that in the weak KAM equation, the unknowns are a function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ and a constant \bar{H} .

The stationary Hamilton-Jacobi equation describes the value of the discounted infinite horizon problem of calculus of variations:

$$\text{minimize } \int_0^{+\infty} e^{-\lambda s} L(x(s), \dot{x}(s)) ds \tag{3}$$

subject to

$$x(\cdot) \in \text{AC}([0, +\infty); \mathbb{T}^d), \quad x(t) = x_*. \tag{4}$$

In this case (see [19, 4]), the value function for problem (3), (4) is equal to $u_\lambda(x_*)$, where u_λ is a viscosity solution of (1).

Similarly, weak KAM equation (2) provide (see [18, 6]) the fixed points of the backward Lax-Oleinik operator. This means that a pair (u, \bar{H}) is a viscosity solution of (2) if and only if, for each $T > 0$,

$$u(x_*) = \min \left\{ \int_0^T L(x(t), \dot{x}(t)) dt + u(x(T)) : \right. \\ \left. x(\cdot) \in \text{AC}([0, T]; \mathbb{T}^d), x(0) = x_* \right\} + \bar{H}T. \tag{5}$$

The number \bar{H} is called the effective Hamiltonian, also known as the Mañé critical value.

The notion of ‘weak KAM’ derives its name from the celebrated Kolmogorov–Arnold–Moser (KAM) theory. It signifies that the pair $(u(\cdot), \bar{H})$, as given by equation (5), corresponds to a torus in the value space associated with the calculus of variations problem for the Lagrangian L .

Our approach relies on approximating of an absolutely continuous curve $x(\cdot)$ by a continuous-time Markov chain on the regular lattice $\Lambda_N \triangleq \mathbb{Z}^d / (N^{-1}\mathbb{Z}^d)$ with the generator $Q^N(v) = (Q_{x,y}^N(v))_{x,y \in \Lambda_N}$ that depends on a vector $v = (v_1, \dots, v_d)^T$ and is defined by the rule:

$$Q_{x,y}^N(v) \triangleq \begin{cases} N|v_i|, & y = x + h \text{sgn}(v_i)e_i, \\ -N \sum_{j=1}^N |v_j|, & y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Hereinafter, $h = N^{-1}$, $\text{sgn}(a)$ denotes the sign of the number a , whilst e_i stands for the i -th coordinate vector.

For this Markov chain, we consider the criterion

$$\text{minimize } \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t, V_t) dt,$$

where X_t and V_t are a state and a stochastic control respectively. The corresponding Bellman equation turns out to be a system of algebraic equations.

The first main result is the convergence rate for the approximation of the solution to the stationary Hamilton-Jacobi equation (1) by the Bellman equation associated with the discounted Markov decision process. We show that it is of order $N^{-1/2}$ (see Theorem 1).

The study of the limit behavior of the discounted Markov decision problem in the case where $\lambda \rightarrow 0$ makes it possible to develop some elements of the weak KAM theory for continuous time Markov chains. In particular, we derive the weak KAM equation on the lattice, which is also a system of algebraic equations. As for the continuous phase space case, we are seeking for a function and an effective Hamiltonian.

The second main result is the fact that the effective Hamiltonian for the lattice Λ_N approximates the number \bar{H} with an error of the order $N^{-1/2}$ (see Theorem 3). Additionally, the functional parts of the discrete weak KAM equations converge, up to a subsequence, to the functional part of the solution of weak KAM equation (2) (see Theorem 4).

Recall that, in the continuous phase space case, the effective Hamiltonian can be characterized via Mather measures that minimize the action $\nu \mapsto \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) \nu(d(x, v))$, where L stands for the Lagrangian. We also introduce the concept of Mather measures for the weak KAM theory on lattices and show their convergence up to subsequence to a Mather measure for the classical weak KAM theory (see Theorems 5, 6).

Literature overview. Modern theory for first-order Hamilton–Jacobi equations, centered on the concept of viscosity solutions, was solidified by Crandall and Lions [11] and Subbotin [39], following earlier contributions from Kruzhkov [28] and Oleinik [34]. Additionally, we refer to the idempotent analysis [32] proposed by Maslov and Kolokoltsov. This concept, in particular, provides a characterization of the value of optimal control problems through Bellman equations. The exposition of the viscosity equation and its applications to the optimal control theory can be found in [4, 19]. Notice that for the case of smooth and convex Hamiltonian the earlier approaches of [28, 34] also work well.

Passing to the limit in the stationary Hamilton-Jacobi equation as the discounted factor tends to the zero leads to the weak KAM equation [30]. This result is primarily used within the study of homogenization of the evolutionary Hamilton-Jacobi equation [30, 15]. Another source of weak KAM equation is the weak KAM theory itself [17, 37]. Recall that this theory studies the long time behavior of the calculus of variation problems and the Euler-Lagrange flows. In particular, the infinitesimal form of the definition of calibrated curve

immediately yields the weak KAM equation [17, 18, 14, 13, 6]. Notice that the effective Hamiltonian also known as the Mañé critical value provides the averaged optimal outcome for the long time calculus of variations problem. The effective Hamiltonian can be calculated using a Mather measure [32] (see also [6, 37, 16, 17]) that is a measure on the tangent bundle minimizing the action of the Lagrangian over the set of measure invariant w.r.t. the Euler-Lagrange flow (the equivalent form of this condition is obtained in [5, 31]).

There are several extensions and analogs of the weak KAM theory (see, in particular, [2, 12, 22, 20]). We especially mention papers [21, 36, 24, 33] where stochastic analogs of the weak KAM theory are developed. Papers [21, 24, 33] deal with a problem arising from perturbation of the dynamics by the Brownian motion and give an insights into the selection problem for Mather measures, whilst in [36] the weak KAM theory for the discrete time random walk on the regular lattice is constructed. There, in fact, it is shown that the corresponding weak KAM equations converge to a solution of the continuous phase space weak KAM equation if the time and space steps vanish. More precisely, effective Hamiltonians converge to the continuous phase space effective Hamiltonian, while the function parts converge up to the choice of a subsequence. Thus, one can regard the results of [36] as a numerical method for weak KAM equation (2). This concept is close to one of the paper. However, in the contrast to [36], we start with the continuous time Markov chain. This gives half the number of algebraic equations needed to assure the same order of approximation. Moreover, we impose milder conditions and do not require the completeness of the Euler-Lagrange flow. The close scheme based on time discretization of the Lax-Oleinik operator was realized in [7, 9]. Weak KAM theory provides a counterpart to turnpike theorems. Both theories identify a specific point in the product of phase and control spaces that contributes dominantly to an optimal control over long time intervals [26, 40]. Moreover, weak KAM theory can be interpreted as an eigenvalue problem for a certain operator in the max-plus algebra (see [1, 29, 41] and references therein for earlier work in this direction). A weak KAM theory for controlled systems, together with the corresponding limit of values for discounted payoff problems, has recently been developed in [10].

Finally, we refer to [3], where an approximation of an optimal control problem by a continuous time Markov decision problem was developed for the case of compact control space. Additionally, the approximation technique for the Hamilton-Jacobi equation based on discrete-time Markov chains was examined in [35].

Organization of the paper. In Section 2, we give the general notation and assumptions. Additionally, here we remind the definitions of viscosity solutions for equations (1), (2). The next section (Section 3) is concerned with the controlled continuous-time Markov chain defined on a regular lattice that plays a crucial role in the paper. We recall the required concepts from the theory of stochastic processes and compute the Hamiltonian for

corresponding Markov decision problems. Furthermore, we evaluate the L^2 -distance between an absolutely continuous curve and a controlled Markov chains generated by a stochastic strategy. The Bellman equation for the discounted Markov decision problem and the existence of an optimal strategy are discussed in Section 4. Then (see Section 5), we prove the approximation result for the stationary Hamilton-Jacobi equation. Elements of the weak KAM theory for the continuous time Markov decision problem on the lattice are developed in Section 6. Below, we obtain the convergence of the weak KAM equations for the Markov decision problem to the weak KAM equation on the torus (see Section 7). Finally, in Section 8 we introduce the concept of Mather measure for the Markov decision problem on the lattice and derive the convergence of the Mather measures on regular lattices to a continuous phase space Mather measure.

2 Preliminaries

2.1. General notation and assumptions. If (X, ρ_X) is a metric space, $x \in X$, $r > 0$, then $\mathbb{B}_r(x)$ stands for the open ball in X of radius r centered at x , i.e.,

$$\mathbb{B}_r(x) \triangleq \{y \in X : \rho_X(x, y) < r\}.$$

Let $\mathbb{T}^d \triangleq \mathbb{R}^d / \mathbb{Z}^d$ denote the d -dimensional flat torus. Recall that an element of \mathbb{T}^d is a set given by the rule:

$$x = \{\tilde{x} + n : n \in \mathbb{Z}^d\}$$

for some $\tilde{x} \in \mathbb{R}^d$. We denote the standard Euclidean norm on \mathbb{R}^d by $|\cdot|$. Further, with some abuse of notation, for $x, y \in \mathbb{T}^d$, we put

$$|x - y| \triangleq \min\{|x' - y'| : x' \in x, y' \in y\}.$$

The quantity $|x - y|$ is a distance on \mathbb{T}^d .

Elements of \mathbb{R}^d are considered as column vectors, while $\mathbb{R}^{d,*}$ consists of row vectors. Below, we regard \mathbb{R}^d as the tangent space to \mathbb{T}^d , and $\mathbb{R}^{d,*}$ as the cotangent space.

Further, if $x \in \mathbb{T}^d$, ϕ is a function defined in a neighborhood of x that is differentiable at x , then $\nabla\phi(x)$ denotes the derivative of ϕ at x . We assume that $\nabla\phi(x)$ is a row-vector.

Assume that we are given a function $L : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ that is called a *Lagrangian*. We impose the following condition on the function L :

- (L1) L , L_x , and L_v are continuous;
- (L2) L is convex;
- (L3) L satisfies the superlinear growth condition, i.e., for each $a \geq 0$ there exists a constant $g(a)$ such that

$$L(x, v) \geq a|v| + g(a).$$

In the following, we will widely use the notation

$$K(c) \triangleq \sup_{x \in \mathbb{T}^d, |v| \leq c} |L_x(x, v)|. \tag{6}$$

The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^{d,*} \rightarrow \mathbb{R}$ is defined by the Legendre transform:

$$H(x, p) \triangleq \max_{v \in \mathbb{R}^d} [pv - L(x, v)].$$

2.2. Stationary Hamilton-Jacobi and weak KAM equations. The first main object of the paper is the stationary Hamilton-Jacobi equation (1).

We consider the concept of viscosity solutions [4, 11]. A function u_λ is a viscosity solution of (1) if, for every point $x \in \mathbb{T}^d$ and each smooth function ψ defined in some neighborhood of x such that the mapping $y \mapsto u_\lambda(y) - \psi(y)$ attains the minimum (respectively, maximum) at the point x , one has that $\lambda u_\lambda(x) + H(x, -\nabla\psi(x)) \geq 0$ (respectively, $\lambda u_\lambda(x) + H(x, -\nabla\psi(x)) \leq 0$).

The second object of the paper is weak KAM equation (2). As above, its solution is considered in the viscosity sense: a pair (u, \bar{H}) is a viscosity solution of (2) provided that, for every point $x \in \mathbb{T}^d$ and each function $\psi : \mathbb{B}_r(x) \rightarrow \mathbb{R}$ for some $r > 0$ such that ψ is differentiable at x and the mapping $y \mapsto u(y) - \psi(y)$ attains the minimum (respectively, maximum) at the point x , one has that $H(x, -\nabla\psi(x)) \geq \bar{H}$ (respectively, $H(x, -\nabla\psi(x)) \leq \bar{H}$).

In [30], it is shown that, if, for each sufficiently small $\lambda > 0$, u_λ solves (1), then up to a subsequence $(u_\lambda(x) - u_\lambda(z), -\lambda u_\lambda(x))$ converge uniformly on \mathbb{T}^d to a pair $(u(x), \bar{H})$ that is a viscosity solution of weak KAM equation (2) (here z is a fixed point on \mathbb{T}^d).

The constant \bar{H} can be characterized through a so called Mather measure μ . To introduce this concept, we follow [6] and put

$$\omega(v) \triangleq \inf_{x \in \mathbb{T}^d} L(x, v) \vee 1. \tag{7}$$

Further, we denote by \mathcal{M} the set of probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ with finite integral of ω , i.e.,

$$\mathcal{M} \triangleq \left\{ \nu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d) : \int_{\mathbb{T}^d \times \mathbb{R}^d} \omega(v) \nu(d(x, v)) < \infty. \right\}$$

A Mather measure [6] for the torus \mathbb{T}^d is a probability on $\mathbb{T}^d \times \mathbb{R}^d$ that minimizes the functional

$$\nu \mapsto \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) \nu(dx dv)$$

over the set of measures $\nu \in \mathcal{M}$ satisfying the holonomic constraints:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla\phi(x) v \nu(d(x, v)) = 0$$

for each $\phi \in C^1(\mathbb{T}^d)$. It is shown (see, in particular, [6]) that, if μ is a Mather measure for \mathbb{T}^d , then

$$-\bar{H} = \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) \mu(d(x, v)).$$

3 Controlled continuous time Markov chain

3.1. Construction of approximating Markov chain. The approximation results derived in the paper rely on a construction of continuous time Markov chains on infinite or finite time interval. To unify the presentation, we denote a finite or infinite time interval by \mathcal{I} , i.e., either $\mathcal{I} = [0, T]$ or $\mathcal{I} = [0, +\infty)$.

To explain the main idea of the construction of the approximating continuous-time Markov chain, we note that each function $x(\cdot) \in AC(\mathcal{I}, \mathbb{R}^d)$ is entirely determined by $x(0)$ and a velocity $v(\cdot) \in L^1(\mathcal{I}, \mathbb{R}^d)$. We say that a pair $(x(\cdot), v(\cdot))$ is a control process on \mathcal{I} provided that $x(\cdot) \in C(\mathcal{I}, \mathbb{T}^d)$, $v(\cdot) \in L^1_{\text{loc}}(\mathcal{I}, \mathbb{R}^d)$ and

$$\frac{d}{dt}x(t) = v(t).$$

In the Introduction, we already briefly described the main concept of approximating Markov chain. We fix

- (1) a natural number N ;
- (2) a regular lattice $\Lambda_N \triangleq (h\mathbb{Z}^d)/\mathbb{Z}^d \subset \mathbb{T}^d$.

Hereinafter, we denote

$$h = N^{-1}.$$

Additionally, as we mentioned above, e_i stands for the i -th coordinate vector on \mathbb{R}^d .

The Kolmogorov matrix of the approximating Markov chain is constructed as follows. If $v = (v_1, \dots, v_d)^T \in \mathbb{R}^d$, then we define the matrix $Q^N(v) = (Q^N_{x,y}(v))_{x,y \in \Lambda_N}$ by the following rule:

$$Q^N_{x,y}(v) \triangleq \begin{cases} h^{-1}|v_i|, & y = x + \text{sgn}(v_i)he_i, \\ -h^{-1} \sum_{j=1}^d |v_j|, & y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Now let us define stochastic control processes that are determined by this Kolmogorov matrix. We start with open-loop controls.

Definition 1. We say that a 6-tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X, V)$ is a controlled Markov chain process for the lattice Λ_N provided that

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P})$ is a filtered probability space;
- X is a $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -adapted process with values in Λ_N ;
- V is a $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -progressively measurable process with values in \mathbb{R}^d such that $\mathbb{E} \int_0^t |V_s| dt < \infty$ for each $t \in \mathcal{I}$;

- for each $\phi : \Lambda_N \rightarrow \mathbb{R}$ the process

$$\phi(X_t) - \int_0^t Q^N(V_s)\phi(X_s)ds$$

is a $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -martingale.

Hereinafter, \mathbb{E} stands for the expectation corresponding to the probability \mathbb{P} .

Below, we denote the set of all controlled Markov chain process for the lattice Λ_N on the time interval \mathcal{I} satisfying the initial condition $X(0) = x_*$ \mathbb{P} -a.s. by $\text{MCP}_N(\mathcal{I}, x_*)$.

The most convenient tool in the theory of controlled Markov chain is feedback strategies (policies).

Definition 2. A feedback strategy on Λ_N is a measurable mapping $\pi : \mathcal{I} \times \Lambda_N \rightarrow \mathbb{R}^d$. A stationary feedback strategy is a mapping $\pi : \Lambda_N \rightarrow \mathbb{R}^d$.

A feedback strategy π defines a Kolmogorov matrix $Q^N[\pi, t] = (Q_{x,y}^N[\pi, t])_{x,y \in \Lambda_N}$ with entries

$$Q_{x,y}^N[\pi, t] = Q_{x,y}^N[\pi(t, x)].$$

If π is a stationary feedback strategy, we will write simply $Q^N[\pi]$ instead of $Q^N[\pi, t]$.

Definition 3. Given a feedback strategy π , we say that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X)$ is a motion produced by π provided that

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P})$ is a filtered probability space;
- X is $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -adapted process with values in Λ_N ;
- for every $t \in \mathcal{I}$,

$$\mathbb{E} \int_0^t |\pi(s, X_s)| ds < \infty;$$

- for each $\phi : \Lambda_N \rightarrow \mathbb{R}$, the process

$$\phi(X_t) - \int_0^t Q^N[\pi, s]\phi(X_s)ds \tag{8}$$

is a $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -martingale.

If $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X)$ is a motion produced by π , then, letting

$$V_t \triangleq \pi(t, X_t),$$

we obtain a controlled Markov chain process $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X, V)$.

Further, a distribution on Λ_N is a sequence $m = (m_x)_{x \in \Lambda_N}$ with nonnegative entries such that

$$\sum_{x \in \Lambda_N} m_x = 1.$$

We say that a motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X)$ produced by a feedback strategy π has the initial distribution $m_0 = \{m_{0,x}\}_{x \in \Lambda_N}$ if

$$\mathbb{P}(X_0 = x) = m_{0,x}, \quad x \in \Lambda_N.$$

Additionally, we say that a motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X)$ produced by π has the initial state z provided that

$$X(0) = z, \quad \mathbb{P} - \text{a.s.}$$

This corresponds to the initial distribution $\mathbb{1}_z = (\mathbb{1}_{z,x})_{x \in \Lambda_N}$ with entries

$$\mathbb{1}_{z,x} \triangleq \begin{cases} 1, & x = z, \\ 0, & x \neq z. \end{cases}$$

There exists at least one motion produced by the stationary feedback strategy π with the initial distribution equal to m_0 (see [25, Theorem 5.4.1]).

Notice that an evolution of distributions produced by the feedback strategy π and the initial distribution m_0 is described by a mapping $\mathcal{I} \ni t \mapsto m(t) = (m_x(t))_{x \in \Lambda_N}$ such that

$$m_x(t) = \mathbb{P}(X_t = x). \quad (9)$$

Notice that $m(t)$ is a finite-dimensional vector. The function $m(\cdot)$ satisfies the Kolmogorov equation

$$\frac{d}{dt}m(t) = m(t)\mathcal{Q}^N[\pi, t], \quad m(t) = m_0. \quad (10)$$

Here, we regard $m(t)$ as a row-vector.

3.2. Finite differences. Let $\phi : \Lambda_N \rightarrow \mathbb{R}$, $x \in \Lambda_N$ and $i \in \{1, \dots, d\}$. Define, for $x \in \Lambda_N$,

$$\begin{aligned} \Delta_{N,i}^+ \phi(x) &\triangleq \frac{\phi(x + he_i) - \phi(x)}{h}, \\ \Delta_{N,i}^- \phi(x) &\triangleq \frac{\phi(x - he_i) - \phi(x)}{h}. \end{aligned} \quad (11)$$

If there exists a smooth function u such that $\phi(x) = u(x)$ on Λ_N , then the quantity $\Delta_{N,i}^+ \phi(x)$ is the standard right finite-difference approximation of the partial derivative of u at x w.r.t. x_i . Analogously, $\Delta_{N,i}^- \phi(x)$ is a left finite-difference approximation of $-\partial_{x_i} u(x)$.

Notice that

$$\begin{aligned} \Delta_{N,i}^+ \phi(x) &= -\Delta_{N,i}^- \phi(x + he_i), \\ \Delta_{N,i}^- \phi(x) &= -\Delta_{N,i}^+ \phi(x - he_i). \end{aligned} \quad (12)$$

Further, we set

$$\Delta_N \phi(x) = ((\Delta_{N,1}^+ \phi(x), \Delta_{N,1}^- \phi(x)), \dots, (\Delta_{N,d}^+ \phi(x), \Delta_{N,d}^- \phi(x))).$$

Thus, it is convenient to work with sequences of pairs. If $\xi = ((\xi_1^+, \xi_1^-), \dots, (\xi_d^+, \xi_d^-))$ is a d -tuple of pairs, where $\xi_i^+, \xi_i^- \in \mathbb{R}$, $v \in \mathbb{R}^d$, then, with some abuse of notation, we denote

$$\xi \cdot v \triangleq \sum_{i=1}^d [\xi_i^+ v_i^+ + \xi_i^- v_i^-].$$

Notice that, for each $\phi : \Lambda_N \rightarrow \mathbb{R}$ and $x \in \Lambda_N$, one has that

$$\sum_{y \in \Lambda_N} Q_{x,y}^N[v] \phi(y) = \Delta_N \phi(x) \cdot v. \quad (13)$$

Thus, if $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X, V)$ is an controlled Markov chain process for the lattice Λ_N , then, for each function $\phi : \Lambda_N \rightarrow \mathbb{R}$, and $s, r \in \mathcal{I}$, $s < r$,

$$\mathbb{E} \phi(X_r) - \mathbb{E} \phi(X_s) = \mathbb{E} \int_s^r (\Delta_N \phi(X_t) \cdot V_t) dt.$$

In particular, given a feedback strategy π and a corresponding motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X)$, we have that

$$\mathbb{E} \phi(X_r) - \mathbb{E} \phi(X_s) = \mathbb{E} \int_s^r (\Delta_N \phi(X_t) \cdot \pi(t, X_t)) dt.$$

Now, for $x \in \Lambda_N$, $\xi = ((\xi_i^+, \xi_i^-))_{i=1}^d$, where $\xi_i^+, \xi_i^- \in \mathbb{R}$, we set

$$\mathcal{H}(x, \xi) \triangleq \sup_{v \in \mathbb{R}^d} [\xi \cdot v - L(x, v)]. \quad (14)$$

This and (13) give that

$$\begin{aligned} \mathcal{H}(x, (-\Delta_N) \phi(x)) &= \max_{v \in \mathbb{R}^d} [(-\Delta_N) \phi(x) \cdot v - L(x, v)] \\ &= - \min_{v \in \mathbb{R}^d} \left[\sum_{y \in \Lambda_N} Q_{x,y}^N[v] \phi(y) + L(x, v) \right]. \end{aligned} \quad (15)$$

Here,

$$\begin{aligned} (-\Delta_N) \phi(x) &= (((-\Delta_{N,1}^+) \phi(x), (-\Delta_{N,1}^-) \phi(x)) \\ &\quad , \dots, ((-\Delta_{N,d}^+) \phi(x), (-\Delta_{N,d}^-) \phi(x))). \end{aligned}$$

3.3. Distance between the deterministic evolution and the Markov chain. We recall that $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, while $\Lambda_N = (h\mathbb{Z}^d) / \mathbb{Z}^d$. It is convenient to denote by $[\tilde{x}]$ the equivalence class of $\tilde{x} \in \mathbb{R}^d$, i.e.,

$$[\tilde{x}] \triangleq \{\tilde{x} + n : n \in \mathbb{Z}^d\}.$$

Let π be a feedback strategy on Λ_N . We assume that its entries are uniformly bounded. Denote by $\hat{\pi}$ a feedback strategy on $h\mathbb{Z}^d$ such that, for $\tilde{x} \in h\mathbb{Z}^d$, $\hat{\pi}(t, \tilde{x}) = \pi(t, [\tilde{x}])$.

Now we define the infinite matrix $\widehat{\mathcal{Q}}^N[\pi, t] = \{\widehat{\mathcal{Q}}_{\tilde{x}, \tilde{y}}^N[\pi, t]\}_{\tilde{x}, \tilde{y} \in h\mathbb{Z}^d}$ indexed with elements of $h\mathbb{Z}^d$ by the rule:

$$\widehat{\mathcal{Q}}_{\tilde{x}, \tilde{y}}^N[\pi, t] \triangleq \begin{cases} h^{-1}|\hat{\pi}_i(t, \tilde{x})|, & \tilde{y} = \tilde{x} + h \operatorname{sgn}(\hat{\pi}_i(t, \tilde{x}))e_i, \\ -h^{-1} \sum_{j=1}^d |\hat{\pi}_j(t, \tilde{x})|, & \tilde{y} = \tilde{x}, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, if $\tilde{y} = \tilde{x} \pm he_i$, then $\widehat{\mathcal{Q}}_{\tilde{x}, \tilde{y}}^N[\pi, t] = \mathcal{Q}_{x, y}^N[\pi, t]$, where $x = [\tilde{x}]$, $y = [\tilde{y}]$.

Let us introduce the generator \mathcal{L}_t^π on $\mathbb{R}^d \times (h\mathbb{Z}^d)$ by the following rule: for $\phi \in C^1(\mathbb{R}^d \times (h\mathbb{Z}^d))$ with at most quadratic growth,

$$\mathcal{L}_t^\pi \phi(\tilde{x}^1, \tilde{x}^2) \triangleq \nabla_{\tilde{x}^1} \phi(\tilde{x}^1, \tilde{x}^2) \pi(t, \tilde{x}^2) + \sum_{\tilde{y} \in h\mathbb{Z}^d} \widehat{\mathcal{Q}}_{\tilde{x}^2, \tilde{y}}^N[\pi, t] \phi(\tilde{x}^1, \tilde{y}).$$

Hereinafter, we denote by $C^1(\mathbb{R}^d \times (h\mathbb{Z}^d))$ the set of continuous functions from $\mathbb{R}^d \times (h\mathbb{Z}^d)$ to \mathbb{R} those are continuously differentiable w.r.t. the first variable.

It follows from [25, Theorem 5.4.1] that, given $(\tilde{x}_*^1, \tilde{x}_*^2) \in \mathbb{R}^d \times (h\mathbb{Z}^d)$, there exist a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ adapted process $(\tilde{X}^1, \tilde{X}^2)$ such that

$$(\tilde{X}_0^1, \tilde{X}_0^2) = (\tilde{x}_*^1, \tilde{x}_*^2) \quad (16)$$

and, for each $\phi \in C^1(\mathbb{R}^d \times (h\mathbb{Z}^d))$ with at most quadratic growth, one has that the process

$$\phi(\tilde{X}_t^1, \tilde{X}_t^2) - \int_0^t \mathcal{L}_s^\pi \phi(\tilde{X}_s^1, \tilde{X}_s^2) ds \quad (17)$$

is a $\{\mathcal{F}_t\}_{t \in \mathcal{I}}$ -martingale. Notice that, in this case, \tilde{X}^1 satisfies \mathbb{P} -a.s. the system of ODEs

$$\frac{d}{dt} \tilde{X}_t^1 = \hat{\pi}(t, \tilde{X}_t^2),$$

while \tilde{X}^2 is a continuous time Markov chain with Kolmogorov matrix $\widehat{\mathcal{Q}}^N[\pi, \cdot]$. Furthermore, if we let

$$X_t^1 \triangleq [\tilde{X}_t^1], \quad X_t^2 \triangleq [\tilde{X}_t^2], \quad (18)$$

then

$$\frac{d}{dt} X_t^1 = \pi(t, X_t^2),$$

and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P}, X^2)$ is a motion produced by the feedback strategy π and the initial state $x_*^2 = [\tilde{x}_*^2]$.

The key result of this section is the following.

Lemma 1. *Let*

- π be a feedback strategy such that, for some constant $c > 0$, $|\pi(t, x)| \leq c$;
- $x_*^1 \in \mathbb{T}^d$, $x_*^2 \in \Lambda_N$,
- $\tilde{x}_*^1 \in \mathbb{R}^d$, $\tilde{x}_*^2 \in h\mathbb{Z}^d$ be such $x_*^1 = [\tilde{x}_*^1]$, $x_*^2 = [\tilde{x}_*^2]$ and $|x_*^1 - x_*^2| = |\tilde{x}_*^1 - \tilde{x}_*^2|$;

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P})$ be a filtered probability space;
- $(\tilde{X}^1, \tilde{X}^2)$ be a stochastic process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{I}}, \mathbb{P})$ with values in $\mathbb{R}^d \times (h\mathbb{Z}^d)$ satisfying conditions (16) and (17);
- (X^1, X^2) be defined by (18).

Then, for each $t \in \mathcal{I}$,

$$\mathbb{E}|X_t^1 - X_t^2|^2 \leq \mathbb{E}|\tilde{X}_t^1 - \tilde{X}_t^2|^2 \leq |x_*^1 - x_*^2|^2 + \sqrt{dc}N^{-1}t. \quad (19)$$

Proof. The first inequality in (19) directly follows from the construction of (X^1, X^2) .

To prove the second inequality, we first notice that, due to the assumption that π has uniformly bounded entries, $\mathbb{E}|\tilde{X}_t^1 - \tilde{X}_t^2|^2$ is bounded. Further, let $\varphi(\tilde{x}_1, \tilde{x}_2) \triangleq |\tilde{x}_1 - \tilde{x}_2|^2$. Plugging the function φ in (17), we obtain that

$$\mathbb{E}|\tilde{X}_t^1 - \tilde{X}_t^2|^2 \leq \mathbb{E}|\tilde{X}_0^1 - \tilde{X}_0^2|^2 + \mathbb{E} \int_0^t \mathcal{L}_s^\pi \varphi(\tilde{X}_s^1, \tilde{X}_s^2) ds.$$

Direct computation gives that

$$\mathcal{L}_t^\pi \varphi(x_1, x_2) = h \sum_{i=1}^d |\pi_i(t, x_2)| \leq hc\sqrt{d}.$$

Therefore, taking into account that $h = N^{-1}$, we obtain the second inequality in (19). \square

Remark 1. Lemma 1 establishes $N^{-1/2}$ -convergence of the Markov chain to the deterministic evolution. For the strategy π with higher regularity, N^{-1} -convergence can be expected (see [27, Theorem 2.4.1]).

4 Discounted Markov decision problem

In this section, we work with the infinite time interval, i.e., we put $\mathcal{I} = [0, +\infty)$.

If $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X, V)$ is a controlled Markov chain process for the lattice Λ_N , then its quality is evaluated by the quantity

$$\mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t, V_t) dt.$$

The outcome for the feedback strategy π and the initial distribution m_0 is equal to

$$\mathcal{J}_{N, \lambda}[\pi, m_0] \triangleq \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t, \pi(t, X_t)) dt,$$

where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X)$ is a motion produced by π and m_0 . The quantity $\mathcal{J}_{N, \lambda}[\pi, m_0]$ does not depend on the concrete choice of the process $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X)$ produced by π and m_0 . Indeed, (9) implies that

$$\mathcal{J}_{N, \lambda}[\pi, m_0] = \int_0^{+\infty} e^{-\lambda t} \sum_{x \in \Lambda_N} L(x, \pi(t, x)) m_x(t) dt.$$

Here $m(\cdot) = (m_x(\cdot))_{x \in \Lambda_N}$ satisfies (10). If m_0 is equal to 1 at z and zero elsewhere, then we write $\mathcal{J}_{N,\lambda}[\pi, z]$ instead of $\mathcal{J}_{N,\lambda}[\pi, m_0]$.

The Bellman equation for the examined discounted Markov decision problem takes the form:

$$\lambda \varphi_{N,\lambda}(x) + \mathcal{H}(x, (-\Delta_N)\varphi_{N,\lambda}(x)) = 0, \quad x \in \Lambda_N. \quad (20)$$

This fact is formalized in the following statement.

Proposition 1. *Equation (20) has a unique solution. Moreover, the stationary feedback strategy $\pi_{\lambda,N}^*$ defined by the rule:*

$$\pi_{\lambda,N}^*(x) \in \operatorname{Argmax}_{v \in \mathbb{R}^d} \left[(-\Delta_N)\varphi_{N,\lambda}(x) \cdot v - L(x, v) \right] \quad (21)$$

is optimal, i.e.,

$$\varphi_{N,\lambda}(z) = \mathcal{J}_{N,\lambda}[\pi_{\lambda,N}^*, z] \quad (22)$$

and, for each $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X, V) \in \operatorname{MCP}_N([0, +\infty), z)$, one has

$$\mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t, V_t) dt \geq \varphi_{N,\lambda}(z). \quad (23)$$

Proof. The proof relies on truncation arguments. Let $A > 0$. We consider the Markov decision problem

$$\text{minimize } \mathbb{E} \left[\int_0^{+\infty} e^{-\lambda t} L(X_t, V_t) dt \right]$$

over the set of 6-tuples $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X, V) \in \operatorname{MCP}_N([0, +\infty), z)$ satisfying the additional constraint $|V_t| \leq A$. Due to [23, Lemma 4.4, Theorems 4.6, 4.10], this problem has a value denoted by $\varphi_{N,\lambda}^A$. The latter satisfies the following Bellman equation on Λ_N :

$$\lambda \varphi_{N,\lambda}^A(x) + \max_{|v| \leq A} \left[(-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \right] = 0.$$

Now recall that due to condition (L3)

$$L(x, v) \geq g(0).$$

Thus, $\varphi_{N,\lambda}^A(x) \geq \lambda^{-1}g(0)$ for each $x \in \Lambda_N$. On the other hand, letting $V_t \equiv 0$, we obtain that $\varphi_{N,\lambda}^A(x) \leq \lambda^{-1}L(x, 0)$. Therefore, denoting

$$C'_0 \triangleq |g(0)| \vee \left[\sup_{x \in \mathbb{T}^d} L(x, 0) \right],$$

we conclude that

$$|\varphi_{N,\lambda}^A(x)| \leq \lambda^{-1}C'_0, \quad x \in \Lambda_N.$$

Hence, for each $x \in \Lambda_N$ and $i \in \{1, \dots, d\}$,

$$\left| \Delta_{N,i}^+ \varphi_{N,\lambda}^A(x) \right|, \left| \Delta_{N,i}^- \varphi_{N,\lambda}^A(x) \right| \leq 2N\lambda^{-1}C'_0.$$

Choosing $a = 2N\lambda^{-1}\sqrt{d}C'_0 + 1$ in condition (L3), we have that, if $x \in \Lambda_N$, $v \in \mathbb{R}^d$,

$$\begin{aligned} & (-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \\ & \leq 2N\lambda^{-1}\sqrt{d}C'_0|v| - (2N\lambda^{-1}\sqrt{d}C'_0|v| + 1)|v| - g(2N\lambda^{-1}\sqrt{d}C'_0 + 1) \\ & = -|v| - g(2N\lambda^{-1}\sqrt{d}C'_0 + 1). \end{aligned}$$

Hence,

$$(-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \rightarrow -\infty \text{ as } |v| \rightarrow \infty$$

uniformly w.r.t. $|v|$. Thus, there exists a constant $A_{N,\lambda}$ such that, for every $A > A_{N,\lambda}$,

$$\max_{|v| \leq A} \left[(-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \right] = \sup_{v \in \mathbb{R}^d} \left[(-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \right].$$

Therefore, if $A > A_{N,\lambda}$, the function $\varphi_{N,\lambda}^A$ satisfies

$$\lambda\varphi_{N,\lambda}^A(x) + \sup_{v \in \mathbb{R}^d} \left[(-\Delta_N)\varphi_{N,\lambda}^A(x) \cdot v - L(x, v) \right] = 0.$$

This gives the existence of solution to (20). The fact that $\pi_{\lambda,N}^*$ is optimal, i.e., the fact that equality (22) and inequality (23) are valid, can be proved using the standard verification arguments. They also give the uniqueness of the solution of (20). \square

5 Approximation of a solution of the stationary Hamilton-Jacobi equation

The aim of this section is to prove the following approximation result.

Theorem 1. *There exists a constant C_1 depending only on the Lagrangian L such that if u_λ , $\varphi_{N,\lambda}$ are solutions of (1) and (20) respectively, while $x \in \Lambda_N$, one has that*

$$|u_\lambda(x) - \varphi_{N,\lambda}(x)| \leq C_1\lambda^{-3/2}N^{-1/2}.$$

The proof of this statement relies on several auxiliary statements.

Lemma 2. *There exist constants c_0 and c_1 depending only on L such that, if $\lambda > 0$ and u_λ is a viscosity solution of (1), then, for every $x, y \in \mathbb{T}^d$,*

- (1) $\lambda|u_\lambda(x)| \leq c_0$;
- (2) $|u_\lambda(x) - u_\lambda(y)| \leq c_1|x - y|$.

Proof. To prove the first statement, we use condition (L3). It gives that

$$L(x, v) \geq g(0).$$

Since u_λ is a value function for the discounted control problem (3), (4), we have that

$$\lambda u_\lambda(z) \geq \lambda \int_0^{+\infty} e^{-\lambda t} g(0) dt = g(0).$$

On the other hand,

$$\lambda u_\lambda(z) \leq \lambda \int_0^{+\infty} e^{-\lambda t} \sup_{x \in \mathbb{T}^d} L(x, 0) dt = \sup_{x \in \mathbb{T}^d} L(x, 0).$$

Letting

$$c_0 \triangleq g(0) \vee \left[\sup_{x \in \mathbb{T}^d} L(x, 0) \right],$$

we derive the first statement.

To prove the second statement, we argue as in the proof of [4, Proposition 4.1]. We set

$$c_1 \triangleq c_0 + \sup_{x \in \mathbb{T}^d, |v|=1} L(x, v) + 1,$$

fix $x \in \mathbb{T}^d \times \mathbb{R}^d$ and consider a function $\phi(y) \triangleq c_1 |y - x|$. We wish to prove that the maximum of $u_\lambda(y) - \phi(y)$ attains at x . Indeed, in the converse case, let $\hat{y} \neq x$ be a point where the maximum of $u_\lambda(y) - \phi(y)$ is achieved. Notice that the function ϕ is differentiable at \hat{y} . Thus, the function ϕ can serve as a test function for this point. Moreover, $\nabla \phi(\hat{y}) = c_1 \frac{\hat{y} - x}{|\hat{y} - x|}$. Using the fact that u_λ is a solution of (1), we have that

$$\lambda u_\lambda(\hat{y}) + H\left(\hat{y}, -c_1 \frac{\hat{y} - x}{|\hat{y} - x|}\right) \leq 0.$$

Plugging in the right-hand side of the Hamiltonian (see (14)) $v \triangleq -\frac{\hat{y} - x}{|\hat{y} - x|}$, we have that

$$c_1 - \sup_{x \in \mathbb{T}^d, |v|=1} L(x, v) \leq c_0.$$

This contradicts with the choice of c_1 . Therefore, for each $y \in \mathbb{T}^d$,

$$u_\lambda(y) - c_1 |y - x| \leq u_\lambda(x).$$

Interchanging the variable, we conclude that the function u_λ is Lipschitz continuous with the constant c_1 that does not depend on λ . \square

Lemma 3. *Let $\lambda > 0$, $x_* \in \mathbb{T}^d$ and let $(x(\cdot), v(\cdot))$ be such that $v(\cdot) \in L^1([0, +\infty); \mathbb{T}^d)$, $x(0) = x_*$, $\frac{d}{dt}x(t) = v(t)$ and*

$$u_\lambda(x_*) = \int_0^{+\infty} e^{-\lambda t} L(x(t), v(t)) dt.$$

Then,

$$|v(t)| \leq c_2 \text{ for a.e. } t \in [0, +\infty).$$

Here c_2 is a constant determined only by the Lagrangian L .

Proof. Let τ be a Lebesgue point of the function $v(\cdot)$. For $r > 0$, from the dynamic programming arguments we have that

$$e^{-\lambda \tau} u_\lambda(x(\tau)) = \int_0^r e^{-\lambda(\tau+t)} L(x(\tau+t), v(\tau+t)) dt + e^{-\lambda(\tau+r)} u_\lambda(x(\tau+r)).$$

Therefore,

$$\begin{aligned} \int_0^r e^{-\lambda t} L(x(\tau+t), v(\tau+t)) dt \\ = u_\lambda(x(\tau)) - u_\lambda(x(\tau+r)) + u_\lambda(x(\tau+r)) [1 - e^{-\lambda r}]. \end{aligned}$$

Using Lemma 2, we have that

$$\int_0^r e^{-\lambda t} L(x(\tau+t), v(\tau+t)) dt \leq c_1 \int_0^r |v(\tau+t)| dt + c_0 r. \quad (24)$$

Now assume that $r \leq \lambda^{-1} \ln 2$. Using this, inequality (24) and choosing in superlinear growth condition (L3) $a = 2c_1 + 2$, we obtain the following inequality

$$\begin{aligned} (c_1+1) \int_0^r |v(\tau+t)| dt + 2^{-1} g(2c_1+2)r \\ \leq (2c_1+1) \int_0^r e^{-\lambda t} |v(\tau+t)| dt + g(2c_1+2) \int_0^r e^{-\lambda t} dt \\ \leq \int_0^r e^{-\lambda t} L(x(\tau+t), v(\tau+t)) dt \leq c_1 \int_0^r |v(\tau+t)| dt + c_0 r. \end{aligned}$$

This gives,

$$\int_0^r |v(\tau+t)| dt \leq [c_0 - g(2c_1+2)/2] r.$$

Since τ is a Lebesgue point of the function $v(\cdot)$, we conclude that

$$|v(\tau)| \leq c_2 \triangleq [c_0 - g(2c_1+2)/2].$$

This and the fact that almost every point of $[0, +\infty)$ is a Lebesgue point of the function $v(\cdot)$ completes the proof. \square

Lemma 4. *Let $\varphi_{N,\lambda}$ solve stationary Hamilton-Jacobi equation (20). Then, for each $x, y \in \Lambda_N$, $i \in \{1, \dots, d\}$,*

- $\lambda |\varphi_{N,\lambda}(x)| \leq c_0$ for each $x \in \Lambda_N$;
- $|\Delta_{N,i}^+ \varphi_{N,\lambda}(x)|, |\Delta_{N,i}^- \varphi_{N,\lambda}(x)| \leq c_3$;
- $|\varphi_{N,\lambda}(x) - \varphi_{N,\lambda}(y)| \leq c_4 |x - y|$.

Here, c_0 is the same constant as in Lemma 2, c_3 and c_4 are determined only by the Lagrangian L .

Proof. The proof of the first statement is the same as in Lemma 2.

Now let us prove the second statement. From the first statement, we have that, given $x \in \Lambda_N$,

$$\sup_{v \in \mathbb{R}^d} [(-\Delta_N) \varphi_{N,\lambda}(x) - L(x, v)] \leq c_0.$$

If $\Delta_{N,i}^+ \varphi_{N,\lambda}(x) \leq 0$, then, letting $v = e_i$ in the expression of the lattice Hamiltonian (15), we obtain that

$$\Delta_{N,i}^+ \varphi_{N,\lambda}(x) \geq -(c_0 + L(x, e_i)).$$

Analogously, if $\Delta_{N,i}^- \varphi_{N,\lambda}(x) \geq 0$, we have that

$$\Delta_{N,i}^- \varphi_{N,\lambda}(x) \leq c_0 + L(x, -e_i).$$

Combining this, we conclude that,

$$\sup_{x \in \mathbb{T}^d, i \in \{1, \dots, d\}} \left(\Delta_{N,i}^- \varphi_{N,\lambda}(x) \right)^+, \quad \sup_{x \in \mathbb{T}^d, i \in \{1, \dots, d\}} \left(\Delta_{N,i}^+ \varphi_{N,\lambda}(x) \right)^- \leq c_3.$$

Here,

$$c_3 \triangleq c_0 + \sup_{x \in \mathbb{T}^d, |v| \leq 1} L(x, v).$$

From this, using equalities (12), we arrive at the second statement of the lemma.

The third statement obviously follows from this, the definition of right and left differences (see (11)) and the Hölder inequality. \square

Lemma 5. *There exists a constant c_5 determined only by the Lagrangian L such that, if $\pi_{\lambda,N}^*$ satisfies (21), then*

$$|\pi_{\lambda,N}^*(x)| \leq c_5.$$

Proof. From the definition of the feedback strategy $\pi_{\lambda,N}^*$ and the first statement of Lemma 4, we have that

$$\sum_{i=1}^d \left[\Delta_{N,i}^+ \varphi_{N,\lambda}(x) (\pi_{\lambda,N}^*(x))_i^+ + \Delta_{N,i}^- \varphi_{N,\lambda}(x) (\pi_{\lambda,N}^*(x))_i^- \right] + L(x, \pi_{\lambda,N}^*(x)) \leq c_0.$$

Here $(\pi_{\lambda,N}^*(x))_i^+$ (respectively, $(\pi_{\lambda,N}^*(x))_i^-$) stands for the positive (respectively, negative) part of the i -th coordinate of the vector $\pi_{\lambda,N}^*(x)$.

Using condition (L3) with $a = \sqrt{d}c_3 + 1$, the second statement of Lemma 4 and the Hölder inequality, we obtain the following estimate:

$$-\sqrt{d}c_3 |\pi_{\lambda,N}^*(x)| + (\sqrt{d}c_3 + 1) |\pi_{\lambda,N}^*(x)| \leq c_0 - g(\sqrt{d}c_3 + 1).$$

This implies the conclusion of the lemma with $c_5 \triangleq c_0 - g(\sqrt{d}c_3 + 1)$. \square

Proof of Theorem 1. Let $z \in \Lambda_N$. First, let $v(\cdot)$ be such that

$$u_\lambda(z) = \int_0^{+\infty} e^{-\lambda t} L(x(t), v(t)) dt, \quad (25)$$

where

$$x(t) = z + \int_0^t v(s) ds.$$

The existence of such open-loop control follows from the Tonelli theorem [8, Theorem 3.7]. Due to Lemma 3,

$$|v(t)| \leq c_2.$$

Let π be a feedback strategy defined by the rule:

$$\pi(t, x) \triangleq v(t).$$

Now, we construct a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P})$ and a stochastic process (X^1, X^2) defined on it as in Lemma 1 for $x_*^1 = x_*^2 = z$ and $\mathcal{I} = [0, +\infty)$. Notice that $X_t^1 = x(t)$ for $t \in [0, +\infty)$. We have that

$$\varphi_{N,\lambda}(x_*) \leq \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t^2, \pi(t, X_t^2)) dt = \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t^2, v(t)) dt. \quad (26)$$

Lemma 1 gives that

$$\mathbb{E}|X_t^1 - X_t^2| \leq (c_2)^{1/2} d^{1/4} t^{1/2} N^{-1/2}.$$

Therefore,

$$\mathbb{E}|L(X_t^2, v(t)) - L(x(t), v(t))| \leq C'_1 \mathbb{E}|X_t^1 - X_t^2| \leq C'_2 t^{1/2} N^{-1/2},$$

where

$$C'_1 \triangleq K(c_2), \quad C'_2 = C'_1(c_2)^{1/2} d^{1/4},$$

while the function K is defined by (6). This, equality (25) and inequality (26) imply

$$\begin{aligned} \varphi_{N,\lambda}(z) - u_\lambda(z) &\leq C'_2 N^{-1/2} \int_0^{+\infty} t^{1/2} e^{-\lambda t} dt \\ &= C'_2 N^{-1/2} \lambda^{-3/2} \int_0^{+\infty} \alpha^{1/2} e^{-\alpha} d\alpha. \end{aligned} \quad (27)$$

To prove the opposite inequality, recall that by Proposition 1, there exists an optimal feedback strategy $\pi_{\lambda,N}^*$ that is stationary. By Lemma 5,

$$|\pi_{\lambda,N}^*(x)| \leq c_5.$$

Furthermore, we construct a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P})$ and a process (X^1, X^2) as in Lemma 1 for $\pi = \pi_{\lambda,N}^*$, $\mathcal{I} = [0, +\infty)$ and $x_*^1 = x_*^2 = z$. Denoting

$$V_t \triangleq \pi_{\lambda,N}^*(X_t^2),$$

we conclude that

$$\frac{d}{dt} X_t^1 = V_t, \quad \mathbb{P}\text{-a.s.}$$

Therefore,

$$u_\lambda(z) \leq \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t^1, V_t) dt = \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t^1, \pi_{\lambda,N}^*(X_t^2)) dt. \quad (28)$$

Additionally, the choice of $\pi_{\lambda,N}^*$ gives that

$$\varphi_{N,\lambda}(z) = \mathbb{E} \int_0^{+\infty} e^{-\lambda t} L(X_t^2, \pi_{\lambda,N}^*(X_t^2)) dt. \quad (29)$$

Taking into account the boundness of $\pi_{\lambda,N}^*$, Lemma 1 and assumption (L1), we have that

$$\mathbb{E}|L(X_t^1, \pi_{\lambda,N}^*(X_t^2)) - L(X_t^2, \pi_{\lambda,N}^*(X_t^2))| \leq C'_1 \mathbb{E}|X_t^1 - X_t^2| \leq C'_3 t^{1/2} N^{1/2}.$$

Here, C'_1 is as above, while $C'_3 \triangleq C'_1(c_5)^{1/2}d^{1/4}$. This, (28) and (29) give that

$$\begin{aligned} u_\lambda(z) - \varphi_{N,\lambda}(z) &\leq C'_3 N^{-1/2} \int_0^{+\infty} t^{1/2} e^{-\lambda t} dt \\ &= C'_3 N^{-1/2} \lambda^{-3/2} \int_0^{+\infty} \alpha^{1/2} e^{-\alpha} d\alpha. \end{aligned}$$

The conclusion of the theorem with

$$C_1 \triangleq (C'_2 \vee C'_3) \cdot \int_0^{+\infty} \alpha^{1/2} e^{-\alpha} d\alpha$$

follows from this and (27). \square

6 Weak KAM theory on the lattice

In this section, we consider the weak KAM problem for the controlled Markov chain: find a function $\varphi_N : \Lambda_N \rightarrow \mathbb{R}$ and a constant $\bar{\mathcal{H}}_N$ such that, for each $T > 0$ and $z \in \Lambda_N$,

$$\varphi_N(z) = \min \left\{ \mathbb{E} \left[\int_0^T L(X_t, V_t) dt + \varphi_N(X_T) \right] : \right. \\ \left. (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}, X, V) \in \text{MCP}_N([0, T], z) \right\} + \bar{\mathcal{H}}_N T.$$

The dynamic programming arguments give that a pair $(\varphi_N, \bar{\mathcal{H}}_N)$ is a solution of this problem if and only if they satisfies the following weak KAM equation on the lattice Λ_N :

$$\mathcal{H}(x, (-\Delta_N)\varphi_N(x)) = \bar{\mathcal{H}}_N, \quad x \in \Lambda_N. \quad (30)$$

Theorem 2. *For each N , there exists a solution of (30). Moreover, the constant $\bar{\mathcal{H}}_N$ is unique.*

Proof. We use a method borrowed from [30]. Due to Lemma 4, one has that on Λ_N

$$|\lambda\varphi_{N,\lambda}(x)| \leq c_0, \quad (31)$$

while, for each $x, y \in \Lambda_N$,

$$|\varphi_{N,\lambda}(x) - \varphi_{N,\lambda}(y)| \leq c_4|x - y|.$$

Hereinafter, $\varphi_{N,\lambda}$ is the unique solution of (20).

Furthermore, we fix, $z \in \Lambda_N$ and set

$$\psi_{N,\lambda}(x) \triangleq \varphi_{N,\lambda}(x) - \varphi_{N,\lambda}(z).$$

By construction, we have that, for each $x \in \Lambda_N$,

$$|\psi_{N,\lambda}(x)| \leq C'_4 \triangleq c_4\sqrt{d}/2. \quad (32)$$

Since $\varphi_{N,\lambda}$ solves (20), we have that

$$\lambda\varphi_{N,\lambda} + \mathcal{H}(x, (-\Delta_N)\psi_{N,\lambda}(x)) = 0, \quad x \in \Lambda_N. \quad (33)$$

From (31) and (32), it follows that there exist a sequence $\{\lambda_k\}_{k=1}^\infty$, a number $\overline{\mathcal{H}}_N$ and a function $\varphi_N : \Lambda_N \rightarrow \mathbb{R}$ such that

- $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$;
- $\lambda_k \varphi_{N, \lambda_k} \rightarrow -\overline{\mathcal{H}}_N$ as $k \rightarrow \infty$ uniformly on Λ_N ;
- for each $x \in \Lambda_N$, $\psi_{N, \lambda_k}(x) \rightarrow \varphi_N(x)$ as $k \rightarrow \infty$.

Passing to the limit in (33), we obtain the existence of a solution to the weak KAM equation on the lattice Λ_N (30).

Now let us prove the uniqueness of the number $\overline{\mathcal{H}}_N$. Assume that there exist two solutions of (30), namely, $(\varphi'_N(\cdot), \overline{\mathcal{H}}'_N)$, $(\varphi''_N(\cdot), \overline{\mathcal{H}}''_N)$.

We choose a feedback strategy π'_N such that

$$\pi'_N(x) \in \operatorname{Argmax}_{v \in \mathbb{R}^d} \left[(-\Delta_N) \varphi'_N(x) \cdot v - L(x, v) \right]. \quad (34)$$

Let $z \in \Lambda_N$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, +\infty)}, \mathbb{P}, X')$ be a motion produced by the strategy π'_N and the initial state z . For each $T > 0$, we have that

$$\begin{aligned} \mathbb{E}[\varphi'(X'(T)) - \varphi'(z)] &= \mathbb{E} \int_0^T \Delta_N \varphi'_N(X'_t) dt \\ &= \mathbb{E} \int_0^T \left[\Delta_N \varphi'_N(X'_t) + L(X'_t, \pi'_N(X'_t)) \right] dt - \mathbb{E} \int_0^T L(X'_t, \pi'_N(X'_t)) dt. \end{aligned}$$

Due to the choice of the strategy π'_N (see (34)) and the fact that $(\varphi'_N, \overline{\mathcal{H}}'_N)$ is a solution of the weak KAM solution, we have that

$$\mathbb{E}[\varphi'(z) - \varphi'(X'(T))] = T \overline{\mathcal{H}}'_N + \mathbb{E} \int_0^T L(X'_t, \pi'_N(X'_t)) dt. \quad (35)$$

Simultaneously,

$$\begin{aligned} \mathbb{E}[\varphi''(z) - \varphi''(X'(T))] &= -\mathbb{E} \int_0^T \left[\Delta_N \varphi''_N(X'_t) + L(X'_t, \pi'_N(X'_t)) \right] dt \\ &\quad + \mathbb{E} \int_0^T L(X'_t, \pi'_N(X'_t)) dt. \end{aligned}$$

Since, for each $x \in \Lambda_N$,

$$\begin{aligned} \left[(-\Delta_N) \varphi''_N(x) - L(x, \pi'_N(x)) \right] &\leq \max_{v \in \mathbb{R}^d} \left[(-\Delta_N) \varphi''_N(x) - L(x, v) \right] \\ &= \mathcal{H}(x, (-\Delta_N) \varphi''_N(x)) = \overline{\mathcal{H}}''_N, \end{aligned}$$

we have that

$$\mathbb{E}[\varphi''(z) - \varphi''(X'(T))] \leq T \overline{\mathcal{H}}''_N + \mathbb{E} \int_0^T L(X'_t, \pi'_N(X'_t)) dt.$$

This and (35) yield that

$$T \overline{\mathcal{H}}'_N - \mathbb{E}[\varphi''(z) - \varphi''(X'(T))] \geq T \overline{\mathcal{H}}'_N - \mathbb{E}[\varphi'(x_0) - \varphi'(X'(T))].$$

Since the functions φ'_N and φ''_N are bounded on Λ_N , dividing both parts of this inequality by T and passing to the limit when $T \rightarrow \infty$, we conclude that

$$\bar{\mathcal{H}}''_N \geq \bar{\mathcal{H}}'_N.$$

The opposite inequality is proved in the same way. Thus, we conclude that $\bar{\mathcal{H}}'_N = \bar{\mathcal{H}}''_N$. \square

7 Limit weak KAM equations on lattices

7.1. Limit of effective Hamiltonians. In this section, we prove the convergence result for the sequence $\{\bar{\mathcal{H}}_N\}_{N=1}^\infty$.

Theorem 3. *Let (u, \bar{H}) satisfy weak KAM equation for the continuous phase space (2), N be a natural number, and let $(\varphi_N, \bar{\mathcal{H}}_N)$ solve weak KAM equation on the lattice Λ_N (30). Then,*

$$|\bar{H} - \bar{\mathcal{H}}_N| \leq C_2 N^{-1/2},$$

where C_2 is the constant determined only by L .

The proof of this theorem relies on several auxiliary statements.

Lemma 6. *Let (u, \bar{H}) solve (2). Then,*

- (1) $|\bar{H}| \leq c_0$;
- (2) $|u(x) - u(y)| \leq c_1|x - y|$.

Proof. Recall that, in [30], it is shown that

$$\bar{H} = - \lim_{k \rightarrow \infty} \lambda_k u_{\lambda_k}(x)$$

for some sequence $\{\lambda_k\}_{k=1}^\infty$ converging to zero and every $x \in \mathbb{T}^d$. This and the first statement of Lemma 2 give that $|\bar{H}| \leq c_0$.

The estimate $|u(x) - u(y)| \leq c_1|x - y|$ is proved in the same way as the second statement of Lemma 2. \square

Lemma 7. *There exists a constant c_6 determined only by the Lagrangian such that, if*

- $(u(\cdot), \bar{H})$ is a solution of weak KAM equation (2),
- $T > 0$,
- $v(\cdot) \in L^1([0, T]; \mathbb{R}^d)$,
- $x(\cdot) \in C([0, T]; \mathbb{R}^d)$,

satisfy

$$x(t) = x(0) + \int_0^t v(s)ds, \quad u(x_0) - u(x(T)) = \int_0^T L(x(t), v(t))dt + \bar{H}T,$$

then, for a.e. $r \in [0, T]$,

$$|v(t)| \leq c_6.$$

Proof. We argue as in the proof of Lemma 3. Let τ be a Lebesgue point for the function $v(\cdot)$. From the dynamic programming principle, we have that, for $t \in (0, T - \tau]$

$$u(x(\tau)) - u(x(\tau + r)) = \int_{\tau}^{\tau+r} L(x(t), v(t))dt + \bar{H}r.$$

Using the second statement of the Lemma 6 and assumption (L3) for $a = c_1 + 1$, we have that

$$c_1 \int_{\tau}^{\tau+r} |v(t)|dt \geq (c_1 + 1) \int_{\tau}^{\tau+r} |v(t)|dt + (g(c_1 + 1) + \bar{H})r.$$

Therefore,

$$\int_{\tau}^{\tau+r} |v(t)|dt \leq -(g(c_1 + 1) + \bar{H})r.$$

Since τ is a Lebesgue point for $v(\cdot)$, we obtain the conclusion of the lemma with $c_6 = -g(c_1 + 1) + c_0$. \square

Lemma 8. *Let $(\varphi_N, \bar{\mathcal{H}}_N)$ be a solution of (30). Then, for each $x, y \in \Lambda_N$, $i \in \{1, \dots, d\}$,*

- $\lambda|\varphi_{N,\lambda}(x)| \leq c_0$ for each $x \in \Lambda_N$;
- $|\Delta_{N,i}^+ \varphi_{N,\lambda}(x)|, |\Delta_{N,i}^- \varphi_{N,\lambda}(x)| \leq c_3$;
- $|\varphi_{N,\lambda}(x) - \varphi_{N,\lambda}(y)| \leq c_4|x - y|$.

Here, c_0 is the same constant as in Lemma 2, whereas c_3, c_4 are determined in Lemma 4.

The proof of this lemma mimics the proof of Lemma 4.

Finally, let us evaluate the norm of a feedback strategy determined by a solution of the weak KAM equation on the lattice Λ_N .

Lemma 9. *Let $(\varphi_N, \bar{\mathcal{H}}_N)$ be a solution of weak KAM equation on the lattice Λ_N (30) and let π_N^* be such that*

$$\pi_N^*(x) \in \operatorname{Argmax}_{v \in \mathbb{R}^d} \left[(-\Delta_N) \varphi_N(x) \cdot v - L(x, v) \right], \quad (36)$$

then

$$|\pi_N^*(x)| \leq c_5 \text{ for each } x \in \Lambda_N.$$

The proof literally follows the proof of Lemma 5, where we utilize the results of Lemma 8 instead of Lemma 4.

Proof of Theorem 3. We use the trick apparently first proposed in [36].

Let $z_{\natural} \in \Lambda_N$ be such that

$$\varphi_N(z_{\natural}) - u(z_{\natural}) \leq \varphi_N(x) - u(x) \text{ for each } x \in \Lambda_N. \quad (37)$$

Furthermore, let π_N^* be chosen by rule (34). Applying Lemma 1 with $\mathcal{I} = [0, 1]$, $x_*^1 = x_*^2 = z_{\natural}$ and $\pi(t, x) = \pi_N^*(x)$, we obtain a filtered probability

space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ and a stochastic process (X^1, X^2) defined on it that satisfies conditions (16), (18). Therefore,

$$\begin{aligned} & \mathbb{E} \left[\varphi_N(X_1^2) - \varphi_N(z_{\natural}) \right] \\ &= \mathbb{E} \int_0^1 \left[\Delta_N \varphi_N(X_t^2) \pi_N^*(X^2(t)) + L(X_t^2, \pi_N^*(X_t^2)) \right] dt \\ & \quad - \mathbb{E} \int_0^1 L(X_t^2, \pi_N^*(X_t^2)) dt \\ &= -\bar{\mathcal{H}}_N - \mathbb{E} \int_0^1 L(X_t^2, \pi_N^*(X_t^2)) dt. \end{aligned}$$

Analogously, due to the fact that (u, \bar{H}) solves (2) and the characterization of this solution as a fixed point of Lax-Oleinik operator, we have that

$$\mathbb{E} \left[u(z_{\natural}) - u(X_1^1) \right] \leq \bar{H} + \mathbb{E} \int_0^1 L(X_t^1, \pi_N^*(X_t^2)) dt.$$

Therefore,

$$\begin{aligned} \bar{\mathcal{H}}_N - \bar{H} &\leq \mathbb{E} \left[(\varphi_N(z_{\natural}) - u(z_{\natural})) - (\varphi_N(X_1^2) - u(X_1^1)) \right] \\ & \quad - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt \\ &= \mathbb{E} \left[(\varphi_N(z_{\natural}) - u(z_{\natural})) - (\varphi_N(X_1^2) - u(X_1^2)) \right] + \mathbb{E} (u(X_1^1) - u(X_1^2)) \\ & \quad - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt. \end{aligned}$$

Due to the choice of z_{\natural} (see (37)), we have that

$$\begin{aligned} \bar{\mathcal{H}}_N - \bar{H} &\leq \mathbb{E} (u(X_1^1) - u(X_1^2)) \\ & \quad - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt. \end{aligned}$$

Recall that $|\pi_N^*(x)| \leq c_5$ (this is due to Lemma 9). Thus, Lemmas 1, 6 and the Jensen's inequality give that

$$\mathbb{E} (u(X_1^1) - u(X_1^2)) \leq C'_5 N^{-1/2},$$

where $C'_5 \triangleq c_1 d^{1/4} (c_5)^{1/2}$.

Analogously, Lemma 1, assumption (L1) and the Jensen's inequality yield that

$$\begin{aligned} & \mathbb{E} |L(X_t^1, \pi_N^*(X_t^2)) - L(X_t^2, \pi_N^*(X_t^2))| \\ & \leq K(c_5) \cdot \mathbb{E} |X_t^1 - X_t^2| \leq C'_6 N^{-1/2} t^{1/2}, \end{aligned}$$

where the function K is introduced by (6), and

$$C'_6 \triangleq d^{1/4} (c_5)^{1/2} \cdot K(c_5).$$

This, equality (7.1) and inequality (7.1) imply the estimate

$$\overline{\mathcal{H}}_N - \bar{H} \leq C'_7 N^{-1/2}, \quad (38)$$

for $C'_7 \triangleq \frac{3}{2}C'_6 + C'_5$.

To derive the opposite inequality, we first choose z^\natural such that

$$\varphi_N(z^\natural) - u(z^\natural) \geq \varphi_N(x) - u(x) \text{ for each } x \in \Lambda_N. \quad (39)$$

There exist functions $v(\cdot) \in L^1([0, 1]; \mathbb{R}^d)$ and $x(\cdot) \in C([0, T]; \mathbb{R}^d)$ such that

$$x(t) = z^\natural + \int_0^t v(s) ds$$

and

$$u(z^\natural) - u(x(1)) = \int_0^1 L(x(t), v(t)) + \bar{H}. \quad (40)$$

Lemma 7 says that $|v(t)| \leq c_6$.

Letting $\pi(t, x) \triangleq v(t)$, we construct a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ and a stochastic process (X^1, X^2) defined on it that satisfies conditions (16), (18) for the initial points $x_*^1 = x_*^2 = z^\natural$. Notice that X^1 now is deterministic with

$$\frac{d}{dt} X_t^1 = v(t), \quad \mathbb{P}\text{-a.s.}$$

Due to the fact that $(\varphi_N, \overline{\mathcal{H}}_N)$ is a solution of (30), we have that

$$\mathbb{E}[\varphi_N(z^\natural) - \varphi_N(X_1^2)] \leq \overline{\mathcal{H}}_N + \mathbb{E} \int_0^1 L(X_t^2, v(t)) dt.$$

From this and (40), we have that

$$\begin{aligned} & \overline{\mathcal{H}}_N - \bar{H} \\ & \geq \mathbb{E}[(\varphi_N(z^\natural) - u(z^\natural)) - (\varphi_N(X_1^2) - u(X_1^1))] \\ & \quad - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt \\ & = \mathbb{E}[(\varphi_N(z^\natural) - u(z^\natural)) - (\varphi_N(X_1^2) - u(X_1^2))] + \mathbb{E}(u(X_1^1) - u(X_1^2)) \\ & \quad - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt \\ & \geq \mathbb{E}(u(X_1^1) - u(X_1^2)) - \mathbb{E} \int_0^1 [L(X_t^2, \pi_N^*(X_t^2)) - L(X_t^1, \pi_N^*(X_t^2))] dt. \end{aligned} \quad (41)$$

In the latter inequality we used the choice of z^\natural (see (39)). Since $|v(t)| \leq c_6$, Lemmas 1, 8 and the Jensen's inequality yield the estimate

$$\mathbb{E}(u(X_1^1) - u(X_1^2)) \geq -C'_8 N^{-1/2},$$

where $C'_8 \triangleq c_1 d^{1/4} (c_6)^{1/2}$. Using the same arguments and assumption (L1), we conclude that

$$\mathbb{E}|L(X_t^1, v(t)) - L(X_t^2, v(t))| \leq K(c_6) \cdot \mathbb{E}|X_t^1 - X_t^2| \leq C'_9 N^{-1/2} t^{1/2},$$

where

$$C'_9 \triangleq K(c_6)d^{1/4}(c_6)^{1/2}.$$

Evaluating the right-hand side of (41) by means of these inequalities, we conclude that

$$\overline{\mathcal{H}}_N - \overline{H} \geq -C'_{10}N^{-1/2},$$

where $C'_{10} \triangleq C'_8 + \frac{3}{2}C'_9$. This and estimate (38) provide the conclusion of the theorem with $C_2 \triangleq C'_7 \vee C'_{10}$. \square

7.2. Limits of the functions φ_N . Let $(\varphi_N, \overline{\mathcal{H}}_N)$ be a solution of (30). Lemma 8 implies that the function φ_N is Lipschitz continuous on Λ_N with the Lipschitz constant equal to c_4 . Below, we denote by $\hat{\varphi}_N$ an extension of the function φ_N onto the whole torus \mathbb{T}^d that is Lipschitz continuous with the same constant, i.e., $\hat{\varphi}_N : \mathbb{T}^d \rightarrow \mathbb{R}$ is such that

- $\hat{\varphi}_N(x) = \varphi_N(x)$ whenever $x \in \Lambda_N$;
- $\hat{\varphi}_N$ is c_4 -Lipschitz continuous function.

The existence of such function can be shown using, for example, the McShane's methods. In this case, we let

$$\hat{\varphi}_N(x) \triangleq \min \{ \varphi_N(y) + c_4|x - y| : y \in \Lambda_N \}.$$

Notice that the functional part of a solution of the weak KAM equation is defined up to an additive constant. Thus, it is reasonable to fix a value at some point. For definiteness, we choose this point equal to $\mathbf{0}$ that is the equivalent class corresponding to points with integer coordinates. Thus, without loss of generality, we assume that

$$\hat{\varphi}_N(\mathbf{0}) = 0.$$

Theorem 4. *The sequence of functions $\{\hat{\varphi}_N\}_{N=1}^\infty$ is precompact in $C(\mathbb{T}^d)$. If a function $u : \mathbb{T}^d \rightarrow \mathbb{R}$ is its accumulation point, then (u, \overline{H}) is a solution of the weak KAM equation on the torus (2).*

Proof. The fact that the sequence $\{\hat{\varphi}_N\}_{N=1}^\infty$ is precompact directly follows from its definition.

To prove the second part, we assume that there exist a sequence $\{N_l\}_{l=1}^\infty$ and a Lipschitz continuous function u such that

$$\|\hat{\varphi}_{N_l} - u\| \rightarrow 0 \text{ as } l \rightarrow \infty.$$

Further, let

- $x \in \mathbb{T}^d$;
- $\{z_l\}_{l=1}^\infty \subset \mathbb{T}^d$ such that $z_l \in \Lambda_{N_l}$ and $z_l \rightarrow x$ as $l \rightarrow \infty$;
- $v \in \mathbb{R}^d$.

For each natural l , we construct a 6-tuple $(\Omega^l, \mathcal{F}^l, \{\mathcal{F}^l\}_{t \in [0, +\infty)}, \mathbb{P}^l, X^{l,1}, X^{l,2})$ satisfying the conditions of Lemma 1 for $N = N_l$, the strategy $\pi(t, x) \equiv v$ and $x_*^1 = x_*^2 = z_l$. Thus,

$$\mathbb{E}^l |X_t^{l,1} - X_t^{l,2}|^2 \leq \sqrt{d}|v|tN_l^{-1}. \quad (42)$$

Here \mathbb{E}^l is an expectation corresponding to the probability \mathbb{P}^l .

Notice that the process $X^{1,l}$ is deterministic and satisfies

$$X_t^{l,1} = z_l + vt, \quad \mathbb{P}\text{-a.s.}$$

Let T be a positive number. We have that

$$\varphi_{N_l}(z_l) - \mathbb{E}^l \varphi_{N_l}(X_T^{l,2}) \leq \mathbb{E}^l \int_0^T L(X_t^{l,2}, v) dt + \bar{\mathcal{H}}_{N_l} T.$$

From (42), Lemma 8 and the definition of the function K (see (6)), we conclude that

$$\begin{aligned} & \varphi_{N_l}(x) - \varphi_{N_l}(x + vT) - L(z_l, v)T \\ & \leq \bar{\mathcal{H}}_{N_l} T + c_4(d^{1/4}|v|^{1/2}T^{1/2}N_l^{-1/2} + 2|z_l - x|) \\ & \quad + K(|v|)d^{1/4}|v|^{1/2}T^{3/2}N_l^{-1/2} + K(|v|)|v|T^{3/2}. \end{aligned}$$

Passing to the limit when $l \rightarrow \infty$, we obtain that

$$u(x) - u(x + vT) - L(x, v)T \leq \bar{H}T + K(|v|)|v|T^{3/2}.$$

Now let ψ be a smooth function such that, for some $r > 0$, the mapping $\mathbb{B}_r(x) \ni y \mapsto u(y) - \psi(y)$ attains the maximum at x . If $|v|T < r$, we have that

$$\psi(x) - \psi(x + vT) - L(x, v)T \leq \bar{H}T + K(|v|)|v|T^{3/2}.$$

Dividing both parts by T and passing to the limit when $T \rightarrow 0$, we obtain that

$$-\nabla\psi(x)v - L(x, v) \leq \bar{H}.$$

Since the choice of v is arbitrarily, we conclude that

$$H(x, -\nabla\psi(x)) \leq \bar{H}. \quad (43)$$

This provides the first part of the definition of the viscosity solution. Let us prove the second part.

As above, given $x \in \mathbb{T}^d$, we consider a sequence $\{z_l\}_{l=1}^\infty \subset \mathbb{T}^d$ converging to x such that $z_l \in \Lambda_{N_l}$. For each natural l , let $\pi_{N_l}^*$ be such that

$$\pi_{N_l}^* \in \underset{v \in \mathbb{R}^d}{\text{Argmax}} \left[(-\Delta_{N_l})\varphi_{N_l}(x) \cdot v - L(x, v) \right].$$

Lemma 9 says that

$$|\pi_{N_l}^*| \leq c_5.$$

There exists a 6-tuple $(\Omega^l, \mathcal{F}^l, \{\mathcal{F}^l\}_{t \in [0, +\infty)}, \mathbb{P}^l, X^{l,1}, X^{l,2})$ satisfying conditions of Lemma 1 for $x_*^1 = x_*^2 = z_l$ and the stationary strategy $\pi_{N_l}^*$. Arguing as in the proof of the first viscosity inequality, we conclude that

$$\mathbb{E}^l |X_t^1 - X_t^2|^2 \leq \sqrt{dc_5} t N_l^{-1}.$$

Here, as above, \mathbb{E}^l stands for the expectation corresponding to the probability \mathbb{P}^l . Due to the choice of the strategy $\pi_{N_l}^*$, we have that

$$\varphi_{N_l}(z_l) - \mathbb{E}^l \varphi_{N_l}(X_T^{l,2}) = \mathbb{E}^l \int_0^T L(X_t^{l,2}, \pi_{N_l}^*(X_t^{l,2})) dt + \overline{\mathcal{H}}_{N_l} T. \quad (44)$$

Put $V^l(t) \triangleq \pi_{N_l}^*(X_t^{2,l})$. The construction of the processes $X^{1,l}$, $X^{2,l}$ gives that

$$X_t^{1,l} = z_l + \int_0^t V^l(s) ds, \quad \mathbb{P}\text{-a.s.}$$

Thanks to Lemma 6, the Lipschitz continuity of the function $\hat{\varphi}_{N_l}$, the definition of the function K (see (6)) and equality (44), we have that

$$\begin{aligned} u(x) - \mathbb{E}^l u \left(x + \int_0^T V^l(t) dt \right) &\geq \mathbb{E} \int_0^T L(x, V^l(t)) dt + \overline{\mathcal{H}}_{N_l} T \\ &\quad - 2\|u - \varphi_{N_l}\| - c_4(2|z_l - x| + d^{1/4} c_5 T^{1/2} N_l^{-1/2}) \\ &\quad - K(c_5)(|z_l - x| + c_5 d^{1/4} T^{3/2} N_l^{-1/2} + c_5 T^{3/2}). \end{aligned}$$

Now let ψ be a smooth function defined in $\mathbb{B}_r(x)$ for some $r > 0$ such that the mapping $\mathbb{B}_r(x) \ni y \mapsto u(y) - \psi(y)$ attains the minimum at x . For T such that $c_5 T < r$, we have that

$$\psi(x) - \mathbb{E}^l \psi \left(x + \int_0^T V^l(t) dt \right) \geq u(x) - \mathbb{E}^l u \left(x + \int_0^T V^l(t) dt \right).$$

Additionally, since $|V^l(t)| \leq c_5$, \mathbb{P} -a.s., the following inequality holds true:

$$\begin{aligned} H(x, -\nabla \psi(x)) T &\geq \psi(x) - \mathbb{E}^l \psi \left(x + \int_0^T V^l(t) dt \right) \\ &\quad - \mathbb{E}^l \int_0^T L(x, V^l(t)) dt + o(T), \end{aligned}$$

where $o(T)/T \rightarrow 0$ as $T \rightarrow 0$ uniformly w.r.t. l . Here we used the definition of the Hamiltonian H . Thus,

$$\begin{aligned} H(x, -\nabla \psi(x)) T &\geq \overline{\mathcal{H}}_{N_l} T \\ &\quad - 2\|u - \varphi_{N_l}\| - c_1(2|z_l - x| + d^{1/4} c_5 T^{1/2} N_l^{-1/2}) \\ &\quad - K(c_5)(|z_l - x| + c_5 d^{1/4} T^{3/2} N_l^{-1/2} + c_5 T^{3/2}) + o(T). \end{aligned}$$

Passing to the limit when $l \rightarrow \infty$, we have that

$$H(x, -\nabla \psi(x)) T \geq \overline{\mathcal{H}}_{N_l} T + K(c_5) c_5 T^{3/2} + o(T).$$

Dividing this inequality by T and passing to the limit as $T \rightarrow 0$, we have that

$$H(x, -\nabla \psi(x)) \geq \overline{\mathcal{H}}_{N_l}$$

for each smooth function ψ such that the mapping $\mathbb{B}_r(x) \ni y \mapsto u(y) - \psi(y)$ attains the minimum at x . This together with (43) gives the fact that u is a viscosity solution of (2). \square

8 Mather measures on Λ_N and their limit behavior

As in the continuous space case, the constant $\overline{\mathcal{H}}_N$ can be characterized using a Mather measure. To introduce this concept for the lattice system, we first denote by \mathcal{M}_N the set of probabilities on $\Lambda_N \times \mathbb{R}^d$ with finite integral of the function w defined by (7), i.e.,

$$\mathcal{M}_N \triangleq \left\{ \nu \in \mathcal{P}(\Lambda_N \times \mathbb{R}^d) : \int_{\Lambda_N \times \mathbb{R}^d} w(v) \nu(d(x, v)) < \infty \right\}.$$

Here the function w is defined by (7). Obviously, $\mathcal{M}_N \subset \mathcal{M}$.

Definition 4. A measure $\mu \in \mathcal{P}(\Lambda_N \times \mathbb{R}^d)$ is called holonomic for the lattice Λ_N if, for each $\phi : \Lambda_N \rightarrow \mathbb{R}$, the following equality holds true:

$$\int_{\Lambda_N \times \mathbb{R}^d} (\Delta_N \phi(x) \cdot v) \mu(d(x, v)) = 0.$$

Definition 5. A measure $\mu_N \in \mathcal{M}_N$ is called a Mather measure for the lattice Λ_N provided that

- μ_N is holonomic for the lattice Λ_N ;
-

$$\begin{aligned} & \int_{\Lambda_N \times \mathbb{R}^d} L(x, v) \mu_N(d(x, v)) \\ &= \min \left\{ \int_{\Lambda_N \times \mathbb{R}^d} L(x, v) \nu_N(d(x, v)) : \nu \in \mathcal{M}_N, \nu \text{ is holonomic} \right\}. \end{aligned}$$

Below, we denote by $\overline{\mathbb{B}}_c$ the closed ball of the radius c in \mathbb{R}^d centered in the origin.

Theorem 5. There exists a constant C_3 such that, for each N , one can find a Mather measure for the lattice Λ_N concentrated on $\Lambda_N \times \overline{\mathbb{B}}_{C_3}$.

Proof. Let $(\varphi_N, \overline{\mathcal{H}}_N)$ be a solution of the weak KAM equation on the lattice Λ_N . Let π_N^* be defined by (36). By Lemma 9, $|\pi_N^*(x)| \leq c_5$ on Λ_N . The Kolmogorov matrix $\mathcal{Q}^N[\pi_N^*]$ determines a Markov chain on the lattice Λ_N . It is well known (see, for instance, [38, Theorem 5.4.6]) that this Markov chain has at least one stationary distribution, i.e., a sequence $\hat{m}_N = \{\hat{m}_{N,x}\}_{x \in \Lambda_N}$ such that, for each $y \in \Lambda_N$,

$$\sum_{x \in \Lambda_N} \hat{m}_{N,x} \mathcal{Q}_{x,y}^N[\pi_N] = 0. \tag{45}$$

Now let $\phi : \Lambda_N \rightarrow \mathbb{R}$. Taking into account the definition of the matrix \mathcal{Q}^N , from (13) and (45), we obtain that

$$\begin{aligned}
0 &= \sum_{y \in \Lambda_N} \left[\sum_{x \in \Lambda_N} \hat{m}_{N,x} \mathcal{Q}_{x,y}^N[\pi_N^*] \right] \phi(y) \\
&= \sum_{x \in \Lambda_N} \hat{m}_{N,x} \left[\sum_{y \in \Lambda_N} \mathcal{Q}_{x,y}^N[\pi_N^*] \phi(y) \right] \\
&= \sum_{x \in \Lambda_N} \hat{m}_{N,x} \sum_{i=1}^d \left(\Delta_{N,i}^+ \phi(x) (\pi_N^*(x))_i^+ + \Delta_{N,i}^- \phi(x) (\pi_N^*(x))_i^- \right) \\
&= \sum_{x \in \Lambda_N} \hat{m}_{N,x} \left(\Delta_N \phi(x) \cdot (\pi_N^*(x)) \right).
\end{aligned} \tag{46}$$

Set

$$\hat{\mu}_N \triangleq \sum_{x \in \Lambda_N} \hat{m}_{N,x} \delta_{(x, \pi_N^*(x))}. \tag{47}$$

Here, δ_w stands for the Dirac measure concentrated at w . Due to (46), we have that

$$\int_{\Lambda_N \times \mathbb{R}^d} \left(\Delta_N \phi(x) \cdot v \right) \hat{\mu}_N(d(x, v)) = 0,$$

i.e., $\hat{\mu}_N$ is holonomic. Furthermore, $\hat{\mu}_N$ is concentrated on $\Lambda_N \times \overline{\mathbb{B}}_{C_3}$ for the constant $C_3 \triangleq c_5$ that does not depend on a number N .

Now let us prove that

$$\int_{\Lambda_N \times \mathbb{R}^d} L(x, v) \hat{\mu}_N(d(x, v)) = -\overline{\mathcal{H}}_N. \tag{48}$$

Recall that, due to the choice of the strategy π_N^* , we have that, for each $x \in \Lambda_N$,

$$\Delta_N \varphi_N(x) \cdot \pi_N^*(x) + L(x, \pi_N^*(x)) = -\overline{\mathcal{H}}_N.$$

Multiplying each equality on \hat{m}_x and summing up, we obtain that

$$\sum_{x \in \Lambda_N} \left[\hat{m}_x \left(\Delta_N \varphi_N(x) \cdot \pi_N^*(x) + L(x, \pi_N^*(x)) \right) \right] = -\overline{\mathcal{H}}_N.$$

Equality (46) and the definition of the measure $\hat{\mu}_N$ (see (47)) yield that (48) holds true.

To complete the proof it suffices to show that, for each probability $\nu \in \mathcal{M}_N$ that is holonomic, one has that

$$\int_{\Lambda_N \times \mathbb{R}^d} L(x, v) \nu(d(x, v)) \geq -\overline{\mathcal{H}}_N. \tag{49}$$

Indeed, given a holonomic measure ν , let $m_x \triangleq \nu(\{x \times \mathbb{R}^d\})$. Further, denote by ν_x a probability on \mathbb{R}^d such that, for each Borel set $\Upsilon \subset \mathbb{R}^d$

$$\nu_x(\Upsilon) = \nu(\{x\} \times \Upsilon).$$

Additionally, we define a feedback strategy $\bar{\nu}$ by the rule: if $x \in \Lambda_N$,

$$\bar{\nu}(x) \triangleq \int_{\mathbb{R}^d} v \nu_x(dv). \quad (50)$$

It is convenient to set, for each $i \in \{1, \dots, d\}$,

$$\tilde{\nu}_i^+(x) \triangleq \int_{\mathbb{R}^d} v_i^+ \nu_x(dv), \quad \tilde{\nu}_i^-(x) \triangleq \int_{\mathbb{R}^d} v_i^- \nu_x(dv).$$

Obviously,

$$\bar{\nu}(x) \triangleq \sum_{i=1}^d \left(\tilde{\nu}_i^+(x) - \tilde{\nu}_i^-(x) \right) e_i.$$

Notice that the numbers $\tilde{\nu}_i^+(x)$ and $\tilde{\nu}_i^-(x)$ are nonnegative. For each $\phi : \Lambda_N \rightarrow \mathbb{R}^d$ and $x \in \Lambda_N$, the following equalities hold true

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\Delta_N \phi(x) \cdot v \right) \nu_x(dv) &= \int_{\mathbb{R}^d} \sum_{i=1}^d \left(\Delta_{N,i}^+ \phi(x) \cdot v_i^+ + \Delta_{N,i}^- \phi(x) \cdot v_i^- \right) \nu_x(dv) \\ &= \sum_{i=1}^d \left[\Delta_{N,i}^+ \phi(x) \tilde{\nu}_i^+(x) + \Delta_{N,i}^- \phi(x) \tilde{\nu}_i^-(x) \right] \\ &= \Delta_N \phi(x) \cdot \bar{\nu}(x). \end{aligned}$$

This and the definition of the sequence $m = \{m_x\}_{x \in \Lambda_N}$ imply the following property for each function $\phi : \Lambda_N \rightarrow \mathbb{R}^d$:

$$0 = \int_{\Lambda_N \times \mathbb{R}^d} \left(\Delta_N \phi(x) \cdot v \right) \nu(d(x, v)) = \sum_{x \in \Lambda_N} m_x \left(\Delta_N \phi(x) \cdot \bar{\nu}(x) \right). \quad (51)$$

Now recall that $(\varphi_N, \bar{\mathcal{H}}_N)$ is a solution of (30). Therefore, using the definition of the Hamiltonian \mathcal{H} , we have that

$$(-\Delta_N) \varphi_N(x) \cdot \bar{\nu}(x) - L(x, \bar{\nu}(x)) \leq \bar{\mathcal{H}}_N.$$

Taking into account equality (51), equality (50) and the convexity of the function L w.r.t. the second variable, we deduce the following estimates:

$$\begin{aligned} -\bar{\mathcal{H}}_N &\leq \sum_{x \in \Lambda_N} m_x \left[L(x, \bar{\nu}(x)) + \Delta_N \varphi_N(x) \cdot \bar{\nu}(x) \right] \\ &= \sum_{x \in \Lambda_N} m_x L \left(x, \int_{\mathbb{R}^d} v \nu_x(dv) \right) \leq \sum_{x \in \Lambda_N} m_x \int_{\mathbb{R}^d} L(x, v) \nu_x(dv) \\ &= \int_{\Lambda_N \times \mathbb{R}^d} L(x, v) \nu(d(x, v)) \end{aligned}$$

This, in fact, is inequality (49). Thus, $\hat{\mu}_N$ is a Mather measure, that is concentrated on $\Lambda_N \times \bar{\mathbb{B}}_{C_3}$. \square

We complete this section with the statement providing the limit behavior of Mather measures. Following [6, 37], we put

$$C_L^0(\mathbb{T}^d \times \mathbb{R}^d) \triangleq \left\{ \phi \in C(\mathbb{T}^d \times \mathbb{R}^d) : \sup_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \left| \frac{\phi(x,v)}{\omega^+(v)} \right| < \infty, \right. \\ \left. \lim_{|v| \rightarrow \infty} \frac{\phi(x,v)}{\omega(v)} = 0 \right\},$$

where

$$\omega^+(v) \triangleq \sup_{x \in \mathbb{T}^d} L(x,v) \vee 1.$$

We say that a sequence $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}$ converges $(C_L^0)^*$ -weakly to μ iff, for each $\phi \in C_L^0$,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x,v) \mu_n(d(x,v)) \rightarrow \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x,v) \mu(d(x,v)).$$

Notice that if $\{\mu_n\}_{n=1}^\infty$ converges to μ $(C_L^0)^*$ -weakly, then it converges to the same measure in the narrow sense.

The limiting properties of Mather measures are described as follows.

Theorem 6. *Let*

- $\{N_k\}_{k=1}^\infty$ be an increasing sequence of natural numbers;
- for each k , μ_{N_k} be a Mather measure for Λ_{N_k} ;
- $\{\mu_{N_k}\}_{k=1}^\infty$ $(C_L^0)^*$ -weakly converges to a measure μ .

Then, μ is a Mather measure on the torus.

Before the proof of this theorem, we introduce the following result.

Corollary 1. *The sequence of Mather measures $\{\hat{\mu}_N\}_{N=1}^\infty$ defined in Theorem 5 is precompact. Each of its accumulation point is a Mather measure for the continuous phase space. In particular, there exists a Mather measure for the flat torus \mathbb{T}^d supported on the compact set $\Lambda_N \times \overline{\mathbb{B}}_{C_3}$.*

Proof of Theorem 6. Let $\phi \in C^1(\mathbb{T}^d)$, we have that, if $x \in \Lambda_N$,

$$|\Delta_{N,i}^+ \phi(x) - \nabla \phi(x)|, |\Delta_{N,i}^- \phi(x) - \nabla \phi(x)| \leq \varsigma_{N,i},$$

where the number $\varsigma_{N,i}$ is equal to

$$\varsigma_{N,i}[\phi] \triangleq \sup_{\varepsilon \in [-h, h]} |\partial_{x_i} \phi(x + \varepsilon e_i) - \partial_{x_i} \phi(x)|.$$

Denote

$$\varsigma_N[\phi] \triangleq \left[\sum_{i=1}^d (\varsigma_{N,i}[\phi])^2 \right]^{1/2}.$$

Notice that $\varsigma_N[\phi] \rightarrow 0$ as $N \rightarrow \infty$. Therefore, for each $\phi \in C^1(\mathbb{T}^d)$,

$$\left| \Delta_N \phi(x) \cdot v - \nabla \phi(x) v \right| \leq \varsigma_N[\phi] \cdot |v|.$$

We have that

$$\begin{aligned} & \int_{\Lambda_{N_k} \times \mathbb{R}^d} \nabla \phi(x) v \mu_{N_k}(d(x, v)) \\ &= \int_{\Lambda_N \times \mathbb{R}^d} \Delta_{N_k} \phi(x) \cdot v \mu_{N_k}(d(x, v)) \\ & \quad - \int_{\Lambda_{N_k} \times \mathbb{R}^d} \left[\Delta_{N_k} \phi(x) \cdot v - \nabla \phi(x) v \right] \mu_{N_k}(d(x, v)). \end{aligned} \tag{52}$$

The estimate of $\Delta_{N_k} \phi(x) \cdot v - \nabla \phi(x) v$ gives that

$$\begin{aligned} & \int_{\Lambda_{N_k} \times \mathbb{R}^d} \left| \Delta_{N_k} \phi(x) \cdot v - \nabla \phi(x) v \right| \mu_{N_k}(d(x, v)) \\ & \leq \varsigma_{N_k}[\phi] \int_{\Lambda_{N_k} \times \mathbb{R}^d} |v| \mu_{N_k}(d(x, v)). \end{aligned}$$

Since $L(x, v) \geq |v| + g(1)$, we have that the functions $(x, v) \mapsto |v_i|$ and $(x, v) \mapsto |v|$ lies in C_L^0 . Thus, the integrals $\int_{\Lambda_{N_k} \times \mathbb{R}^d} \nabla \phi(x) v \mu_{N_k}(d(x, v))$, $\int_{\Lambda_{N_k} \times \mathbb{R}^d} \Delta_{N_k} \phi(x) v \mu_{N_k}(d(x, v))$ are well-defined. Using $(C_L^0)^*$ -weak convergence of the sequence $\{\mu_{N_k}\}$, we conclude that the quantities $\int_{\Lambda_{N_k} \times \mathbb{R}^d} |v| \mu_{N_k}(d(x, v))$ are uniformly bounded. Using this, the fact that each measure $\{\mu_{N_k}\}$ is holonomic for the lattice Λ_{N_k} , the convergence $\varsigma_{N_k}[\phi] \rightarrow 0$ as $k \rightarrow \infty$, we pass to the limit in (52) and obtain that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla \phi(x) v \mu(d(x, v)) = 0.$$

Thus, μ is holonomic.

Since

$$\int_{\Lambda_{N_k} \times \mathbb{R}^d} L(x, v) \mu_{N_k}(d(x, v)) = -\overline{\mathcal{H}}_{N_k},$$

using the $(C_L^0)^*$ -weak convergence of the sequence $\{\mu_{N_k}\}$ to μ and Theorem 3, we arrive at the equality

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, v) \mu(d(x, v)) = -\overline{H}.$$

This gives that μ is a Mather measure for the flat torus. □

Proof of Corollary 1. Recall that each measure $\hat{\mu}_N$ is concentrated on $\Lambda_N \times \overline{\mathbb{B}}_{C_3} \subset \mathbb{T}^d \times \overline{\mathbb{B}}_{C_3}$. Furthermore, the set $\mathcal{P}(\mathbb{T}^d \times \overline{\mathbb{B}}_{C_3})$ is compact in the topology of narrow convergence. Now, to obtain the conclusion of the corollary, it suffices to notice that on $\mathcal{P}(\mathbb{T}^d \times \overline{\mathbb{B}}_{C_3})$ the narrow convergence coincides with the $(C_L^0)^*$ -weak convergence and use Theorem 6. □

9 Conclusion

This paper presents an approximation scheme based on continuous-time Markov chains for two Hamilton-Jacobi equations arising in the calculus of variations on the d -dimensional torus. We consider two cases: the stationary Hamilton-Jacobi equation, corresponding to calculus of variations problems with discounted payoff, and the weak KAM equation. The latter characterizes a function u and the Mañé critical value \bar{H} , which together provide a fixed point for the calculus of variations problem on any finite time interval (see (5)). Our approach is founded on a calculus of variations framework for continuous-time Markov chains, whose state space is a regular lattice on the d -dimensional torus.

Future progress in this field will involve transferring this technique to Hamilton-Jacobi equations defined on a general compact manifold. In that context, the state space for the Markov chain must be an approximation of the underlying manifold.

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