

Authors' response letter

First of all, I would like to thank both reviewers for the valuable and useful comments.

Reviewer A

- *The sentence in the introduction 'The modern theory of first order Hamilton-Jacobi equations relies on the notion of viscosity solutions proposed by Crandall and Lions [9]' is simply wrong. Viscosity solutions were initially developed much earlier by Krushkov and Oleinik (and in some versions by Maslov and Subbotin) in their famous papers. They are of course for convex Hamiltonians only. But only these Hamiltonians are used in the present paper, so reference to Lions theory really concerns further extensions that are not used here.* I rewrote this phrase and added the references to the works of Kruzhkov, Maslov, Oleinik and Subbotin.
- *This wrong attitude is reflected in the definition of viscous solutions given in Section 2.2. The definition is given in the general form of Lions for general Hamiltonians, which is not needed here, as only convex case is considered and only one half of the definition is needed in that case (which is in fact tacitly and therefore misleadingly assumed in the proof of viscosity some place below, where only this half is used).* It seems that in this case one will need some extra condition e.g. the condition that the Hamilton-Jacobi equation is satisfied almost everywhere. Indeed, if one consider the equation

$$\varphi + \frac{(D\varphi)^2}{2} = 0,$$

then, if one let φ to be a positive (resp., negative) constant, then it is supersolution (resp., subsolution). However, only the function $\varepsilon \equiv 0$ is a viscosity solution.

- *Proof of Lemma 2. Strictly speaking one cannot use the suggested function ψ as a test, since it is not smooth (as required in the definition).* The definition of the viscosity solution requires the smoothness of the test function in some neighborhood of the analyzing point. Thus, ϕ given by $\phi(y) \triangleq c_1|y - x|$ is

a good test function for each point $\hat{y} \neq x$. The corresponding explanation is added.

- The number \bar{H} in (2) and more relevantly in f-la without number after (4) and the corresponding stationary solutions were intensively studied from more than century ago in the connection with turnpike theory (stationary solutions of economic optimization often form turnpikes) and/or nonlinear Frobenius theory, and a bunch of existence and uniqueness results were obtained including even non-convex game-theoretic Hamiltonians. Some references to start looking at these results are e.g.

- (1) Vassili Kolokoltsov and Wei Yang. *The turnpike theorems for Markov games. Dynamic Games and Applications* 2: 3 (2012), 294-312. (Eq (3.3) there for discrete time, including controlled Markov chains and games),
- (2) Marianne Akian, Stéphane Gaubert, Antoine Hochart. *A game theory approach to the existence and uniqueness of nonlinear Perron-Frobenius eigenvectors. Discrete and Continuous Dynamical Systems, 2020, 40(1): 207-231 (rather general continuous state space case),*
- (3) Yakovenko and Kontorer. *Nonlinear Semigroups and Infinite Horizon Optimization. Advances in Soviet Mathematics* 13, 1992, p. 167-210 (including continuous time discussion).

Moreover, a vanishing discount limit is also extensively studied, see e.g. (Exists on Arxiv) P. Cannarsa, S. Gaubert, C. Mendico, M. Quincampoix. *Analysis of the vanishing discount limit for optimal control problems in continuous and discrete time. (with relation to weak KAM discussed).* This comment is very profound and useful. It gave me insight into the link between the weak KAM theory and some other areas of mathematics. I have added the aforementioned papers and some other to the literature survey.

- Further on, a general framework for proving convergence estimates for lattice Markov chain approximations to ordinary ODE (with possible control) is developed in (in the spirit of theorem 3 of the paper): V. N. Kolokoltsov and O. A. Malafeyev. *Many Agent Games in Socio-economic Systems: Corruption, Inspection, Coalition Building, Network Growth, Security. Springer Series in Operations Research and Financial Engineering, Springer Nature, 2019.*, where in particular, various regularity assumptions are discussed that can be used to improve $N^{-1/2}$ order of convergence. I added the corresponding remark in §3.
- p.155, l.13. I would suggest writing $m_x(t)$, rather than $m(t)$, as it is not seen what variable the integration is over. Revised. Now the integration is replaced by the summation.
- p.149 , l. -8: 'interval by' should be 'by interval'. Revised.

Reviewer B

1. *The paper focuses on the KAM Hamilton-Jacobi equation. While the KAM abbreviation is well-known to specialists in Hamilton-Jacobi theory, it would be helpful to decipher it upon its first use.* I added the paragraph (see page 2, lines 24-27) explaining the notion ‘weak KAM’. It derives the name from the celebrated Kolmogorov–Arnold–Moser (KAM) theory.
2. *p.21, Lemma 7. It seems like there is a misprint: the constant c_7 is likely meant to be c_6 .* Corrected.
3. *The paper needs a conclusion.* Corrected. The conclusion is added.
4. *A minor revision for grammar and style is recommended.* The grammar and style have been revised.

Sincerely,
Yurii Averboukh,