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SURJECTIVE ISOMETRY REFLEXIVE STRONGLY FACIALLY SYMMETRIC SPACES

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Abstract: The paper investigates the isometry properties of strongly facially symmetric spaces. It establishes that a linear operator in a reflexive strongly facially symmetric space is a surjective isometry if and only if it maps the set of indecomposable geometric tripotents onto itself and preserves orthogonality relations within this set.

Keywords: Strongly facially symmetric space, symmetric face, geometric tripotent, surjective isometry.

1 Introduction

In the early 1980s, the development of the theory of JB^*-triples began in the works of Kaup; in many respects, these triples are parallel to the function-analytical aspects of the theory of operator algebras [1, 2]. The JB^*-triples that are characterized by holomorphic properties of their unit balls form a wide class of Banach spaces based on ternary algebraic structures, which contains C^*-algebras, Hilbert spaces, and spaces of rectangular matrices. Many axioms used by Alfsen and Schultz show that there are disordered analogs of JB^*-triples [3]. In 1989, Friedman and Russo published papers [4, 5], where they introduced the concept of facially symmetric spaces. The primary purpose of defining these spaces was to provide a geometric characterization of the predual spaces of JB^*-triples that admit an algebraic structure. Many

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of the properties necessary for these characterizations are natural assumptions for state spaces of physical systems. These spaces are considered as a geometric model for representing states in quantum mechanics.

In [6], it was proven that the predual space of a complex von Neumann algebra, as well as more general JB^*-triples, is a neutral strongly facially symmetric space. In [7], a geometric characterization was provided for complex Hilbert spaces and complex spin factors, along with descriptions of JBW^*-triples of ranks 1 and 2 and Cartan factors of types 1 and 4. Later, Friedman and Russo [8] described atomic facially symmetric spaces and demonstrated that a neutral strongly facially symmetric space is isometrically isomorphic to the predual space of one of the Cartan factors of types 1–6. Neil and Russo [9] identified geometric conditions under which a facially symmetric space is isometric to the predual space of a complex JBW^*-triple. In [10], it was proved that the predual space of the real part of a von Neumann algebra is a strongly facially symmetric space if and only if this algebra is the direct sum of an Abelian algebra and an algebra of type I_2. Moreover, it was shown that the predual space of a JBW-algebra is a strongly facially symmetric space if and only if the algebra is the direct sum of an Abelian algebra and an algebra of type I_2 (see [11]). In [12], a classification of finite dimensional real neutral strongly facially symmetric spaces with joint Peirce decomposition property was proposed. Moreover, it was shown in [13] that if there exists a mapping between the sets of minimal geometric tripotents of any two reflexive atomic neutral strongly facially symmetric spaces that preserves both orthogonality and transition pseudo-probabilities, then this mapping admits an extension to an isometric isomorphism between the two spaces.

This paper investigates the isometry properties of strongly facially symmetric spaces. We prove that a linear operator in a reflexive strongly facially symmetric space is a surjective isometry if and only if it maps the set of indecomposable geometric tripotents onto itself while preserving orthogonality relations within this set.

2 Preliminaries

We present necessary information from the theory of facially symmetric spaces, [4, 5]. Let Z be a real or complex normed space, and let Z^* denote its dual space. We say that elements $f, g \in Z$ are orthogonal and write $f \diamond g$ if $\|f+g\| = \|f-g\| = \|f\| + \|g\|$. We say subsets $S, T \subset Z$ are orthogonal and write $S \diamond T$, if $f \diamond g$ for all $(f, g) \in S \times T$. For a subset S of Z, we put $S^\diamond = \{f \in Z : f \diamond g, \forall g \in S\}$; the set $S^\diamond$ is called the orthogonal complement of S. Recall that a face F of a convex set K is a non-empty convex subset of K such that if $g, h \in K$ satisfy $\lambda g + (1-\lambda)h \in F$ for some $\lambda \in (0,1)$, then $g, h \in F$.

A norm exposed face of the unit ball $Z_1 = \{f \in Z : \|f\| \leq 1\}$ of Z is a non-empty set (necessarily $\neq Z_1$) of the form $F_u = \{f \in Z_1 : u(f) = 1\}$, where $u \in Z^*$ with $\|u\| = 1$. While every norm exposed face is a face, the converse does not hold in general. An element $u \in Z^*$ is called a projective unit if $\|u\| = 1$ and $\langle g, u\rangle = 0$ for all $g \in F_u^\diamond$.

Definition 1. *A norm exposed face F_u in Z_1 is called a symmetric face if there exists a linear isometry S_u from Z to Z such that $S_u^2 = I$ whose fixed point set coincides with the topological direct sum of the closure $\overline{\mathrm{sp}}F_u$ of the linear hull of the face F_u and its orthogonal complement $F_u^\diamond$, i.e., with $\overline{\mathrm{sp}}F_u \oplus F_u^\diamond$.*

Definition 2. *A space Z is said weakly facially symmetric (WFS) if each norm exposed face in Z_1 is symmetric.*

A projective unit $u \in Z^*$ is called geometric tripotent if F_u is a symmetric face and $S_u^* u = u$ for the symmetry S_u corresponding to F_u. It should be noted that some properties of geometric tripotents were established in [14, 15]. By $\mathcal{GT}$ and $\mathcal{SF}$ we denote the sets of all geometric tripotents and symmetric faces, respectively; the correspondence $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is one-to-one [5, Proposition 1.6].

Definition 3. *A WFS-space Z is said to be strongly facially symmetric (SFS) if for each norm exposed face F_u of Z_1 and each $y \in Z^*$ satisfying the conditions $\|y\| = 1$ and $F_u \subset F_y$, we have $S_u^* y = y$, where S_u is the symmetry corresponding to F_u.*

For each symmetric face F_u, contractive projections $P_k(u), k = 0, 1, 2$ on Z are defined as follows (see [4]). First, $P_1(u) = (I - S_u)/2$ is the projection on the eigenspace corresponding to the eigenvalue -1 of the symmetry S_u. Next, $P_2(u)$ and $P_0(u)$ are defined as projections of Z onto $\overline{\mathrm{sp}}F_u$ and $F_u^\diamond$, respectively; i.e., $P_2(u) + P_0(u) = (I + S_u)/2$. The projections $P_k(u)$ are called geometric Peirce projections.

We present examples of SFS-spaces.

Example 1. Endowing $l_1(n)$ with the norm $\|x\| = \sum\limits_{i=1}^{n} |x_i|$, where $x = (x_i) \in l_1(n)$, we obtain a strongly facially symmetric space.

Example 2. Every Hilbert space H is a SFS-space. Each element $u \in H$ with $\|u\| = 1$ is a geometric tripotent and $F_u = \{u\}$. Moreover, the symmetry S_u corresponding to a face F_u is defined as follows:

$$S_u(\lambda u + x) = \lambda u - x, \lambda u + x \in \mathrm{sp} u \oplus u^\perp = H,$$

where $u^\perp$ is the orthocomplement of u in the Hilbert space H.

Example 3. The l_1-sum of Hilbert spaces is a SFS-space.

Example 4. The predual space of a spin factor is a strongly facially symmetric space.

3 Main Result

Geometric tripotents u and v are said to be orthogonal if $u \in P_0(v)^* Z^*$ (which implies $v \in P_0(u)^* Z^*$) or, equivalently, $u \pm v \in \mathcal{GT}$ (see [4, Lemma

2.5]). More generally, elements x and y of Z^* are said to be orthogonal, denoted $x \diamond y$, if one of them belongs to $P_2(u)^*Z^*$ and the other belongs to $P_0(u)^*Z^*$ for some geometric tripotent u.

Lemma 1. *Let Z be a SFS-space and $\Phi : Z \to Z$ be a linear isometry. If $u, v \in \mathcal{GT}$ and $u \diamond v$. Then $\Phi^*(u) \diamond \Phi^*(v)$.*

Proof. Let $u, v \in \mathcal{GT}$ and $u \diamond v$. Then by [4, Lemma 2.5] we have $u \pm v \in \mathcal{GT}$. In [4, Lemma 2.4] it was established that any isometry Φ^* maps geometric tripotents to geometric tripotents, i.e. $\Phi^*(u), \Phi^*(v)$ and $\Phi^*(u \pm v)$ are geometric tripotents. Therefore, for all $f \in F_{\Phi^* u}$ we have

$$|1 \pm \langle f, \Phi^*(v)\rangle| = |\langle f, \Phi^*(u)\rangle \pm \langle f, \Phi^*(v)\rangle| =$$

$$= |\langle f, \Phi^*(u \pm v)\rangle| \le \|\Phi^*(u \pm v)\| = 1,$$

i. e. $|1 \pm \langle f, \Phi^*(v)\rangle| \le 1$. But this inequality is valid only for $\langle f, \Phi^*(v)\rangle = 0$. Thus, $\langle f, \Phi^*(u \pm v)\rangle = 1$ and $F_{\Phi^*(u)} \subset F_{\Phi^*(u \pm v)}$. Then by [4, Lemma 2.8] we have

$$\Phi^*(u) + \Phi^*(v) = \Phi^*(u + v) =$$

$$= \Phi^*(u) + P_0(\Phi^*(u))^*\Phi^*(u + v) =$$

$$= \Phi^*(u) + P_0(\Phi^*(u))^*\Phi^*(u) + P_0(\Phi^*(u))^*\Phi^*(v) =$$

$$= \Phi^*(u) + P_0(\Phi^*(u))^*\Phi^*(v).$$

Therefore,

$$\Phi^*(v) = P_0(\Phi^*(u))^*\Phi^*(v),$$

i.e. $\Phi^*(v) \in P_0(\Phi^*(u))Z^*$. Hence, by definition of the orthogonal geometric tripotents it follows that $\Phi^*(u) \diamond \Phi^*(v)$. □

A partial ordering can be defined on the set of geometric tripotents as follows: if $u, v \in \mathcal{GT}$, then $u \le v$ if $F_u \subset F_v$, or equivalently, by [5, Lemma 4.2], $P_2(u)^*v = u$, or $v - u$ is either zero or a geometric tripotent orthogonal to u.

A geometric tripotent u is called maximal if for $v \in \mathcal{GT}$ from $u \le v$, it follows $v = u$.

A geometric tripotent u is called indecomposable if for $v \in \mathcal{GT}$ from $v \le u$, it follows $v = u$. By $\mathcal{I}$ we denote the set of all indecomposable geometric tripotents (see. [7]).

Lemma 2. *Let Z be a SFS-space and $\Phi : Z \to Z$ be a surjective linear isometry.*

i) If u is a maximal geometric tripotent, then $\Phi^*(u)$ is a maximal geometric tripotent.

ii) If u is a indecomposable geometric tripotent, then $\Phi^*(u)$ is a indecomposable geometric tripotent.

Proof. i) Let u be a maximal geometric tripotent. Then by [4, Lemma 2.4] we have that $\Phi^*(u)$ is a geometric tripotent. Suppose that $\Phi^*(u)$ is not a maximal geometric tripotent. Then there exists a geometric tripotent v such that $\Phi^*(u) \leq v$. Therefore, by [5, Lemma 4.2] it follows that $v - \Phi^*(u)$ is geometric tripotent and $\Phi^*(u) \diamond (v - \Phi^*(u))$. Therefore, according to [4, Lemma 2.4] and Lemma 1, we have that

$$\Phi^{*-1}(v), (\Phi^{*-1}(v) - u) \in \mathcal{GT}$$

and

$$u \diamond (\Phi^{*-1}(v) - u).$$

Then again from [5, Lemma 4.2] we get $u \leq \Phi^{*-1}(v)$. This contradicts the fact that u is a maximal geometric tripotent. Hence, $\Phi^*(u)$ is a maximal geometric tripotent.

ii) Let $u \in \mathcal{I}$. By [4, Lemma 2.4] we have that $\Phi^*(u)$ is a geometric tripotent. Suppose that $\Phi^*(u)$ is not an indecomposable geometric tripotent. Then there exists a geometric tripotent v such that $v \leq \Phi^*(u)$. Therefore, by [5, Lemma 4.2] it follows that $\Phi^*(u) - v$ is geometric tripotent and $v \diamond (\Phi^*(u) - v)$. Therefore, according to [4, Lemma 2.4] and Lemma 1, we have that

$$\Phi^{*-1}(v), (u - \Phi^{*-1}(v)) \in \mathcal{GT}$$

and

$$\Phi^{*-1}(v) \diamond (u - \Phi^{*-1}(v)).$$

Then again from [5, Lemma 4.2] we get $\Phi^{*-1}(v) \leq u$. This contradicts the fact that $u \in \mathcal{I}$. Hence, $\Phi^*(u)$ is an indecomposable geometric tripotent. □

Theorem 1. *Let Z be a reflexive strongly facially symmetric space and $\Phi : Z \to Z$ linear operator. Then the following statements are equivalent:*

1) Φ is a surjective isometry;

2) Φ^* maps $\mathcal{I}$ onto itself and preserves orthogonality relations on $\mathcal{I}$.

Proof. 1) ⇒ 2) follows from Lemmas 1 and 2.

2) ⇒ 1). Let Φ^* maps $\mathcal{I}$ onto itself and preserves orthogonality relations on $\mathcal{I}$ and $u \in \mathcal{GT}$ be a decomposable geometric tripotent. Then, by [16, Theorem 4.1], there exist mutually orthogonal indecomposable geometric tripotents u_1, ..., u_n such that $u = u_1 + ... + u_n$. Therefore, according to [4, Lemma 2.1] we have

$$\|\Phi^*(u)\| = \|\Phi^*(u_1) + ... + \Phi^*(u_n)\| =$$

$$= \max\{\|\Phi^*(u_1)\|, ..., \|\Phi^*(u_n)\|\} = 1.$$

Now, let's suppose that $x \in Z^*$ is not geometric tripotent. Then, by the spectral theorem for reflexive SFS-spaces (see [4, Theorem 1]), each element $x \in Z^*$ is uniquely represented in the following form

$$x = \lambda_1 v_1 + ... + \lambda_n v_n,$$

where $\lambda_1 > ...\lambda_n > 0$, and $v_1, ...v_n \in \mathcal{I}$. Then by virtue of [4, Lemma 2.1] it follows that

$$\begin{aligned} \|x\| &= \|\lambda_1 v_1 + ... + \lambda_n v_n\| = \max\{\|\lambda_1 v_1\|, ..., \|\lambda_n v_n\|\} \\ &= \max\{\lambda_1, ..., \lambda_n\} = \lambda_1. \end{aligned} \tag{1}$$

On the other side,

$$\Phi^*(x) = \lambda_1 \Phi^*(v_1) + ... + \lambda_n \Phi^*(v_n). \tag{2}$$

Since Φ^* maps $\mathcal{I}$ onto itself and preserves orthogonality relations on $\mathcal{I}$, then by [4, Lemma 2.1] it follows from equality (2) that

$$\begin{aligned} \|\Phi^*(x)\| &= \|\lambda_1 \Phi^*(v_1) + ... + \lambda_n \Phi^*(v_n)\| \\ &= \max\{\lambda_1 \|\Phi^*(v_1)\|, ..., \lambda_n \|\Phi^*(v_n)\|\} = \\ &= \max\{\lambda_1, ..., \lambda_n\} = \lambda_1. \end{aligned} \tag{3}$$

From equalities (1) and (3) it follows that $\|\Phi^*(x)\| = \|x\|$. Therefore Φ^* is an isometry.

Moreover, Φ is surjective. Indeed, if $y = \mu_1 u_1 + ... + \mu_m u_m$. Since Φ^* maps $\mathcal{I}$ onto itself, then there are indecomposable geometric tripotents v_i, $1 \leq i \leq m$ (orthogonality is not needed now) such that $\Phi^* v_i = u_i$. Therefore, $x = \mu_1 v_1 + ... + \mu_m v_m$ satisfies $\Phi^*(x) = y$. This means that Φ^* is an surjective isometry. Hence, Φ is an surjective isometry. □

Corollary 1. *Any surjective linear isometry Φ^* of the dual space of a reflexive SFS-space Z iz uniquely determined by its values on the set $\mathcal{I}$.*

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