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A SHARP UPPER BOUND FOR THE LENGTH OF INCIDENCE ALGEBRAS

N.A. KOLEGOV

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Abstract: It is shown that the length of an incidence algebra is bounded above by a value depending on a partition of the poset. A series of examples is constructed when this upper bound is achieved. As a consequence, an inequality is proved that connects three invariants of an arbitrary finite poset: its height, width, and the length of its incidence algebra.

Keywords: length function, incidence algebras, structural matrix rings, generators of algebras, finite posets, graded posets.

1 Introduction

The length is a numerical invariant of finite-dimensional algebras [9, 17, 18]. In a sense, it evaluates complexity of the multiplication operation in the algebra. There are interesting applications of the length function to the theory of matrix computation [1, 15]. Given a concrete algebra, usually it is difficult to calculate its length since all generating subsets must be estimated. For example, the length of the algebra $M_n(\mathbb{F})$ of all $n\times n$ matrices over a field $\mathbb{F}$ is still unknown, see [5, 10, 20] for some key results on this problem. Nevertheless, the collection of algebras with the known length estimations is constantly growing and includes various matrix algebras [14], group algebras [3], non-associative algebras [4].

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This paper considers the task of length computation for incidence algebras. This is the important family of algebras closely connected to partially ordered sets, see the classical book [21]. Two finite posets are isomorphic iff their incidence algebras over a field are isomorphic. It would be interesting to connect the length of an incidence algebra with the properties of its underlying poset. It follows from the known results on the length of triangle matrix algebras [14] that for "large enough" fields the length of an incidence algebra always equals $n-1$, where n is the cardinality of the poset, or the size of matrices. However, for "small" fields, the task of length computation is more difficult. In this case, the linear upper bound and the logarithmic lower bound are known [14].

The main result of the paper is Theorem 1 which provides a new upper bound for the length of incidence algebras. Moreover, a series of examples is constructed which ensures sharpness of the bound in rather general settings. Corollary 1 connects the length with two well-known invariants of the poset: the width and the height.

For natural n, we set $\mathbf{n} = \{1,\ldots,n\}$ and consider a partial order $\preceq$ on $\mathbf{n}$. Denote by $D_n(\mathbb{F})$ the algebra of all diagonal $n\times n$ matrices over a field $\mathbb{F}$. We will deal with the incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq,\mathbb{F}) \subseteq M_n(\mathbb{F})$, see Subsection 2.4. The length function ℓ will be introduced in Subsection 2.2. It is necessary to note that the length of $D_n(\mathbb{F})$ was calculated precisely in [13, Theorem 5.4]:

$$\ell(D_n(\mathbb{F})) = \begin{cases} n-1 \text{ if } |\mathbb{F}| \geq n; \\ (q-1)[\log_q n] + [q^{\{\log_q n\}}] - 1 \text{ if } |\mathbb{F}| = q < n, \end{cases} \tag{1}$$

where $[\cdot]$ is the floor function and $\{\cdot\}$ is the fractional part.

Given two posets $(\mathcal{X}_1,\preceq_1)$, $(\mathcal{X}_2,\preceq_2)$, then a function $f:\mathcal{X}_1\to\mathcal{X}_2$ is called *strictly order-preserving* if for any $a \prec_1 b$, we have $f(a) \prec_2 f(b)$. We are ready now to state the main result of the paper.

Theorem 1. *Let $\mathcal{A} = \mathcal{A}_n(\preceq,\mathbb{F})$ be an incidence algebra over a field $\mathbb{F}$ and a partial order $\preceq$ on $\mathbf{n} = \{1,\ldots,n\}$. We fix any natural number $m \leq n$ and assume that the set $\mathbf{m} = \{1,\ldots,m\}$ is equipped with the standard order $\leq$.*

(1) *Given any strictly order-preserving surjective function $\vartheta : \mathbf{n}\to\mathbf{m}$, then the length of the incidence algebra can be bounded as*
$$\ell(\mathcal{A}) \leq \ell(D_{x_1}(\mathbb{F})) + \ldots + \ell(D_{x_m}(\mathbb{F})) + m - 1,$$
where each x_i equals cardinality of the preimage $\vartheta^{-1}(i)$.

(2) *Moreover, for any $x'_1,\ldots,x'_m \in \mathbb{N}$ satisfying $x'_1+\ldots+x'_m = n$, there exist such partial order $\preceq'$ on $\mathbf{n}$ and such strictly order-preserving surjective function $\vartheta' : \mathbf{n}\to\mathbf{m}$ that $x'_i = |\vartheta'^{-1}(i)|$ and*
$$\ell(\mathcal{A}_n(\preceq',\mathbb{F})) = \ell(D_{x'_1}(\mathbb{F})) + \ldots + \ell(D_{x'_m}(\mathbb{F})) + m - 1.$$

Note that for at least one m, the indicated function ϑ does exist, see Proposition 1 below. Example 1 demonstrates how the bound from the theorem can be calculated for a concrete incidence algebra.

Corollary 1. *Let $\mathcal{A} = \mathcal{A}_n(\preceq, \mathbb{F})$ be an incidence algebra over a field $\mathbb{F}$ and a partial order $\preceq$ on $\mathbf{n} = \{1, \ldots, n\}$. Let h and w denote the height and the width of $(\mathbf{n}, \preceq)$, respectively. Then*

$$\ell(\mathcal{A}) \leq (\ell(D_w(\mathbb{F})) + 1) \cdot h - 1.$$

Moreover, for any $h', w' \in \mathbb{N}$ there exists such $n' \in \mathbb{N}$ and such partial order $\preceq'$ on $\mathbf{n}'$ that the height and the width of $(\mathbf{n}', \preceq')$ equal h' and w', respectively, and $\ell(\mathcal{A}_{n'}(\preceq', \mathbb{F})) = (\ell(D_{w'}(\mathbb{F})) + 1) \cdot h' - 1$.

The paper is organised as follows. In Section 2, the main definitions and necessary known results are given. In Section 3, we prove Item 1 of Theorem 1. Sections 4, 5, 6 are primarily devoted to the proof of Item 2 of Theorem 1. We utilise complete multipartite posets (see Definition 3). In Section 4, an auxiliary notion of *suitable pairs* is introduced. Using this notion we obtain a lower bound for the length of an incidence algebra over a complete multipartite poset in Proposition 9 when some restrictions are imposed. In Section 5, sufficient conditions for a pair of numbers to be suitable are established in Proposition 4. In Section 6, we complete the proof of Theorem 1 and also prove Corollary 1.

2 Preliminaries

2.1. Posets. We denote by $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, $\leq$ is the standard total order on integers.

Let $(\mathcal{X}, \preceq)$ be a finite poset. The maximum cardinality of chains (antichains) is called *height* (respectively, *width*) of the poset. Say that $z \in \mathcal{X}$ *covers* $x \in \mathcal{X}$ and denote it by $x \prec: z$ if $x \prec z$ and there is no such $y \in \mathcal{X}$ that $x \prec y \prec z$.

Definition 1. For $m \in \mathbb{N}$, we introduce an *m-partite*, or *multipartite*, poset as a triple $(\mathcal{X}, \preceq, \vartheta)$ with a strictly order-preserving surjective function $\vartheta : \mathcal{X} \to \mathbf{m}$, where $\mathbf{m}$ is equipped with the standard order $\leq$.

Multipartite posets were studied by [2].

Remark 1 ([2])**.** *Let $\mathcal{X}_i$ denote the preimage $\vartheta^{-1}(i)$ for $i \in \mathbf{m}$. It follows directly from the above definition that*
(a) $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_m$,
(b) $\mathcal{X}_i$ is a nonempty antichain for each i,
(c) $\mathcal{X}_i \cap \mathcal{X}_j = \varnothing$ for any $i \neq j$,
(d) for any $x \prec y$, there exist $1 \leq i < j \leq m$ such that $x \in \mathcal{X}_i$, $y \in \mathcal{X}_j$.
Conversely, any family of subsets $\{\mathcal{X}_i\}_{i=1}^m$ which satisfies (a)–(d) uniquely determines a strictly order-preserving surjective function ϑ by $\vartheta(\mathcal{X}_i) = i$.

A slight modification of Mirsky's theorem [12] shows for which $m \in \mathbb{N}$ a poset can be made m-partite.

Proposition 1. *Consider a finite poset $(\mathcal{X}, \preceq)$ of the height h and the linearly ordered set $(\mathbf{m}, \leq)$. Then there exists a strictly order-preserving surjective function $\vartheta : \mathcal{X} \to \mathbf{m}$ if and only if $h \leq m \leq |\mathcal{X}|$.*

Proof. The case $m > |\mathcal{X}|$ is impossible since ϑ is surjective. If $m < h$, apply the fact that the intersection of a chain and an antichain contains at most one point. For $h \leq m \leq |\mathcal{X}|$, we construct the family $\{\mathcal{X}_i\}_{i=1}^m$ using induction on m. When $m = h$, one may define each $\mathcal{X}_i$ to be the set of minimums of $\mathcal{X} \backslash (\mathcal{X}_1 \cup \ldots \cup \mathcal{X}_{i-1})$. Assume that the required family $\{\mathcal{X}_i\}_{i=1}^m$ was constructed for some m satisfying $h \leq m < |\mathcal{X}|$. Since $m < |\mathcal{X}|$, there exists such k that $|\mathcal{X}_k| \geq 2$. Then $\mathcal{X}_k$ can be replaced by a union of two disjoint nonempty sets $\mathcal{X}_k = \mathcal{Y} \cup \mathcal{Z}$ and we obtain a family with $m + 1$ members. □

Definition 2. An m-partite poset $(\mathcal{X}, \preceq, \vartheta)$ is called *graded* [19, Sec. 13.1] if for any pair $x \prec\colon y$, we have $\vartheta(y) - \vartheta(x) = 1$. In this case, the map ϑ is called the *rank function*, the preimage $\mathcal{X}_i = \vartheta^{-1}(i)$ is the *i-th level*, and the cardinality $\mathcal{W}_i = |\mathcal{X}_i|$ is the *i-th Whitney number (of the second kind).*

Definition 3. An m-partite poset is called *complete m-partite* [22], or a *cobweb poset* [8], in the case when $x \prec\colon y$ if and only if $\vartheta(y) - \vartheta(x) = 1$.

Incidence algebras over such posets were investigated in [7].

Remark 2. *Note that if $(\mathcal{X}, \preceq, \vartheta)$ is complete m-partite, the partial order $\preceq$ and its covering relation $\prec\colon$ can be expressed in terms of direct products of levels:*

$$\prec\colon = \bigcup_{i=1}^{m-1} \mathcal{X}_i \times \mathcal{X}_{i+1}, \qquad \prec = \bigcup_{1 \leq i < j \leq m} \mathcal{X}_i \times \mathcal{X}_j. \tag{2}$$

In particular, the height of $(\mathcal{X}, \preceq)$ coincides with m. Moreover, the tuple of Witney numbers $(\mathcal{W}_1, \ldots, \mathcal{W}_m)$ uniquely determines the poset $(\mathcal{X}, \preceq)$ up to an isomorphism.

2.2. The length function. We will denote by $\mathbb{F}$ an arbitrary field and by $\mathbb{F}_q$ a finite field of cardinality q. If V is a linear space over $\mathbb{F}$ and $M \subseteq V$, then we denote by $\langle M \rangle$ the linear span of M.

Let $\mathcal{A}$ be an associative finite-dimensional algebra with identity $1_\mathcal{A}$ over a field $\mathbb{F}$. If M, N are two nonempty subsets of $\mathcal{A}$, then put $M \cdot N = \{mn \mid m \in M, n \in N\}$. For a subset $S = \{A_1, \ldots, A_m\} \subseteq \mathcal{A}$, let $S^0 = \{1_\mathcal{A}\}$ and $S^k = \underbrace{S \cdot \ldots \cdot S}_{k \text{ times}}$ with $k \in \mathbb{N}$. In other words, $S^k = \{A_{i_i} \cdot \ldots \cdot A_{i_k} \mid A_{i_j} \in S\}$ is the set of all possible products of k elements from S, repetitions are allowed.

Next we put $\mathcal{L}_k(S) = \langle \cup_{i=0}^k S^i \rangle$. Since $\mathcal{L}_k(S) \subseteq \mathcal{L}_{k+1}(S)$, we have an increasing chain of inclusions. Introduce $\mathcal{L}(S) = \cup_{k=0}^\infty \mathcal{L}_k(S)$. Then $\langle \mathcal{L}(S) \rangle = \mathcal{L}(S)$, $\mathcal{L}(S) \cdot \mathcal{L}(S) = \mathcal{L}(S)$, and $1_\mathcal{A} \in \mathcal{L}(S)$. So $\mathcal{L}(S)$ is a unital subalgebra of $\mathcal{A}$ and, even more, it is the minimal subalgebra containing the set $S \cup \{1_\mathcal{A}\}$.

Definition 4. We will call $\mathcal{L}(S)$ the algebra *generated* by the set S.

Since $\dim \mathcal{A} < \infty$, the following quantity is well-defined.

Definition 5. The *length* $\ell(S)$ of a set S is the minimal such $\ell \in \mathbb{N}_0$ that $\mathcal{L}_\ell(S) = \mathcal{L}_{\ell+1}(S)$.

We note that if ℓ is the length of S, then

$$\mathcal{L}_{\ell+2}(S) = \langle \mathcal{L}_1(S) \cdot \mathcal{L}_{\ell+1}(S) \rangle = \langle \mathcal{L}_1(S) \cdot \mathcal{L}_{\ell}(S) \rangle = \mathcal{L}_{\ell+1}(S).$$

Hence, arguing by induction we obtain that $\mathcal{L}_\ell(S) = \mathcal{L}_{\ell+k}(S)$ for all $k \in \mathbb{N}_0$. So $\mathcal{L}_\ell(S) = \cup_{k=0}^{\infty} \mathcal{L}_k(S) = \mathcal{L}(S)$.

Now consider the special case where the set S generates the entire algebra, that is, $\mathcal{L}(S) = \mathcal{A}$. Then we have

$$\langle 1_{\mathcal{A}} \rangle = \mathcal{L}_0(S) \subsetneq \mathcal{L}_1(S) \subsetneq \ldots \subsetneq \mathcal{L}_{\ell-1}(S) \subsetneq \mathcal{L}_\ell(S) = \mathcal{A}.$$

Since all inclusions are strict, we deduce that the sequence of dimensions of the spaces $\mathcal{L}_k(S)$ is strictly increasing. From this, it follows that $\ell(S) \leq \dim \mathcal{A} - 1$ and so we may give the following definition.

Definition 6. The *length* $\ell(\mathcal{A})$ of an algebra $\mathcal{A}$ is the maximal length of all generating subsets.

2.3. Matrix algebras. For $a, b \in \mathbb{N}$, let $M_{a\times b}(\mathbb{F})$ be the linear space of all rectangular $a \times b$ matrices over a field $\mathbb{F}$. Given a matrix $B \in M_{a\times b}(\mathbb{F})$, its i, j-entry is $(B)_{ij}$, or b_{ij}. We denote by $M_n(\mathbb{F})$ the algebra of all $n\times n$ matrices over $\mathbb{F}$ and by $T_n(\mathbb{F})$, $D_n(\mathbb{F})$ its subalgebras of upper-triangular and diagonal matrices, respectively. The diagonal matrix with the elements $d_1, \ldots, d_n \in \mathbb{F}$ on the diagonal is denoted by $\operatorname{diag}(d_1, \ldots, d_n)$. The matrix unit is denoted by $E_{ij}^{(n)} \in M_n(\mathbb{F})$, or simply E_{ij}. Also, I_n, or I, is the identity matrix in $M_n(\mathbb{F})$. For two matrices $A \in M_{a\times b}(\mathbb{F})$, $B \in M_{a\times c}(\mathbb{F})$, we introduce the matrix $(A|B) \in M_{a\times(b+c)}(\mathbb{F})$ obtained by joining of A and B.

Given $P \in M_{a\times b}(\mathbb{F})$, $Q \in M_{c\times d}(\mathbb{F})$, introduce the notations for their direct sum $P \oplus Q$ and Kronecker product $P \otimes Q$.

Let $\alpha_1, \ldots, \alpha_n$ be some elements of $\mathbb{F}$, then the *Vandermonde matrix* is

$$\mathbf{V}(\alpha_1, \ldots, \alpha_n) = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \ldots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \ldots & \alpha_n^{n-1} \end{pmatrix} \in M_n(\mathbb{F}).$$

Lemma 1. *If the Vandermonde matrix $V = \mathbf{V}(\alpha_1, \ldots, \alpha_n)$ is nonsingular, then the n-th row of the inverse matrix V^{-1} contains no zero.*

Proof. It is known that $\det V \neq 0$ if and only if $\alpha_i \neq \alpha_j$ for all $i \neq j$. Then the formula for inverse matrix implies that $(V^{-1})_{ni} = (-1)^{i+n} \cdot \frac{1}{\det V} \cdot \det \mathbf{V}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n) \neq 0$ for all $i = 1, \ldots, n$. □

Definition 7. We introduce the map $\operatorname{col} : D_n(\mathbb{F}) \to M_{n\times 1}(\mathbb{F})$ given by $\operatorname{col}(\operatorname{diag}(d_1, \ldots, d_n)) = (d_1, \ldots, d_n)^t$. In addition, consider another map $\mathcal{F}$ from $\underbrace{D_n(\mathbb{F}) \times \ldots \times D_n(\mathbb{F})}_{n \text{ times}}$ to $M_n(\mathbb{F})$ given by

$$\mathcal{F}(D_1, \ldots, D_n) = (\operatorname{col}(D_1)|\operatorname{col}(D_2)|\ldots|\operatorname{col}(D_n)).$$

Lemma 2. *For natural numbers $m_1, \ldots, m_s$, denote their product by $n = m_1 \cdot \ldots \cdot m_s$. Consider an arbitrary subset of matrices $\{P_1^{(j)}, \ldots, P_{m_j}^{(j)}\} \subset D_{m_j}(\mathbb{F}_q)$ for each $j = 1, \ldots, s$. We introduce the family*

$$\{P_{i_1}^{(1)} \otimes \ldots \otimes P_{i_s}^{(s)} \mid i_j = 1, \ldots, m_j,\ j = 1, \ldots s\} \subset D_n(\mathbb{F}_q), \tag{3}$$

which can be equipped with the lexicographical order with respect to multi-indices $(i_1, \ldots, i_s)$. Using only one index $i = 1, \ldots, n$ we can enumerate all elements of the family from the least to the greatest one according to the lexicographical order. Denote the i-th matrix by D_i. Then

$$\mathcal{F}(D_1, \ldots, D_n) = \mathcal{F}(P_1^{(1)}, \ldots, P_{m_1}^{(1)}) \otimes \ldots \otimes \mathcal{F}(P_1^{(s)}, \ldots, P_{m_s}^{(s)}).$$

Proof. We proceed by induction on s. The base $s = 1$ is obvious. Assume that the lemma holds for $s - 1$. Denote $n' = \frac{n}{m_s} = m_1 \ldots m_{s-1}$. Let $D'_1, \ldots, D'_{n'}$ be lexicographically ordered matrices that were constructed for the sets $\{P_{i_j}^{(j)}\}_{i_j=1}^{m_j}$, $j = 1, \ldots, s-1$. Then $\mathcal{F}(D_1, \ldots, D_n)$ equals

$$\mathcal{F}(D'_1 \otimes P_1^{(s)}, \ldots, D'_1 \otimes P_{m_s}^{(s)}, \ldots\ldots, D'_{n'} \otimes P_1^{(s)}, \ldots, D'_{n'} \otimes P_{m_s}^{(s)}) = \mathcal{F}(D'_1, \ldots, D'_{n'}) \otimes \mathcal{F}(P_1^{(s)}, \ldots, P_{m_s}^{(s)}).$$

It remains to apply the induction assumption to $\mathcal{F}(D'_1, \ldots, D'_{n'})$. □

2.4. Matrix incidence algebras. Next, we turn to incidence algebras. We will deal only with the finite-dimensional case and so an incidence algebra may be naturally considered as a subalgebra of $M_n(\mathbb{F})$.

Definition 8. Let $n \in \mathbb{N}$, consider a partial order $\preceq$ on the set $\mathbf{n} = \{1, \ldots, n\}$. Then we define an *incidence algebra* over a field $\mathbb{F}$ as

$$\mathcal{A}_n(\preceq, \mathbb{F}) = \langle \{E_{ij} \mid i \preceq j\} \rangle \subseteq M_n(\mathbb{F}).$$

Two simple examples of incidence algebras are $\mathbb{F}^n \cong D_n(\mathbb{F}) = \mathcal{A}_n(=, \mathbb{F})$ and $T_n(\mathbb{F}) = \mathcal{A}_n(\leq, \mathbb{F})$. Since a partial order relation is always reflexive, each incidence algebra contains $D_n(\mathbb{F})$ as a subalgebra. At the same time, any incidence algebra can be embedded in $T_n(\mathbb{F})$, see [21, Proposition 1.2.7].

The Jacobson radical [21, Theorem 4.2.5] of an incidence algebra $\mathcal{A}$ equals

$$J(\mathcal{A}) = \{A \in \mathcal{A} \mid (A)_{ii} = 0,\ i \in \mathbf{n}\}.$$

The length of $D_n(\mathbb{F})$ is given by Equation (1). It is known that $\ell(T_n(\mathbb{F})) = n - 1$, see [14, Theorem 4.1].

Theorem 2 ([14, Lemma 4.2, Corollary 4.6])**.** *Consider an incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathbb{F})$. If the field $\mathbb{F}$ is infinite, then $\ell(\mathcal{A}) = n - 1$. If $\mathbb{F} = \mathbb{F}_q$ is a finite field of cardinality q, then*

$$\ell(D_n(\mathbb{F}_q)) \leq \ell(\mathcal{A}) \leq n - 1.$$

In particular, we again have $\ell(\mathcal{A}) = n - 1$ for $q \geq n$.

3 The upper bound for the lengths of incidence algebras

The main result of this section is a proof of Item 1 of the main theorem. Then we compare our new estimation with the known bound $n-1$ from Theorem 2 and provide an example of how the estimation can be calculated for a concrete algebra.

Proof of Item 1 of Theorem 1. Consider the family of sets $\{\mathcal{X}_i\}_{i=1}^m$ from Remark 1. So $x_i = |\mathcal{X}_i|$ for $i \in \mathbf{m}$. For convenience, set $x_0 = 0$.

Let $\mathcal{X}_i = \{k_{i,1}, \ldots, k_{i,x_i}\}$. By Items (a)–(c) of Remark 1, it is possible to define a permutation $\sigma : \mathbf{n} \to \mathbf{n}$ by $\sigma(k_{i,j}) = j + \sum_{t=0}^{i-1} x_t$. Let $\mathcal{X}'_i$ denote the image $\sigma(\mathcal{X}_i)$, that is, $\mathcal{X}'_i = \mathbb{N} \cap [\ \sum_{t=0}^{i-1} x_t + 1,\ \sum_{t=0}^{i} x_t\ ]$.

Then we introduce a new partial order $\preceq'$ on $\mathbf{n}$ given by $i \preceq' j$ if and only if $\sigma^{-1}(i) \preceq \sigma^{-1}(j)$. Note that the map $\vartheta' = \vartheta \circ \sigma^{-1}$ is surjective and strictly order-preserving with respect to $(\mathbf{n}, \preceq')$ and $(\mathbf{m}, \leq)$. Thus, $(\mathbf{n}, \preceq', \vartheta')$ is an m-partite poset. Moreover,

$$\vartheta'^{-1}(i) = \sigma(\vartheta^{-1}(i)) = \sigma(\mathcal{X}_i) = \mathcal{X}'_i.$$

This means that the family $\{\mathcal{X}'_i\}_{i=1}^m$ satisfies Items (a)-(d) of Remark 1 with respect to $\preceq'$.

Consider the incidence algebra $\mathcal{A}' = \mathcal{A}_n(\preceq', \mathbb{F})$. Since the posets $(\mathbf{n}, \preceq)$ and $(\mathbf{n}, \preceq')$ are isomorphic, the algebras $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic as well. Hence, $\ell(\mathcal{A}) = \ell(\mathcal{A}')$.

We will show that the algebra $\mathcal{A}'$ can be viewed as block upper-triangular

$$\mathcal{A}' = \left(\begin{array}{c|c|c|c} D_{x_1}(\mathbb{F}) & * & \ldots & * \\ \hline 0 & D_{x_2}(\mathbb{F}) & \ldots & * \\ \hline \vdots & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \ldots & D_{x_m}(\mathbb{F}) \end{array}\right).$$

Let $(\alpha, \beta) \in \mathbf{n} \times \mathbf{n}$, $\alpha \neq \beta$. Consider two cases:

[*Case* 1.] Assume that $\alpha, \beta \in \mathcal{X}'_i$ for some $i \in \mathbf{m}$. Since $\mathcal{X}'_i$ is an antichain, we have $\alpha \npreceq' \beta$ and so $(A)_{\alpha\beta} = 0$ for each $A \in \mathcal{A}'$.

[*Case* 2.] Now let $\alpha \in \mathcal{X}'_i$, $\beta \in \mathcal{X}'_j$ and $i > j$. According to Item (d) of Remark 1 we again have $\alpha \npreceq' \beta$. Therefore, $(A)_{\alpha\beta} = 0$ for every $A \in \mathcal{A}'$.

Thus, $\mathcal{A}'$ is indeed a block upper-triangular algebra. So it remains to apply [16, Corollary 3]. □

Once we have obtained a new estimation, it is necessary to compare it with the known bound $n-1$ form Theorem 2.

Proposition 2. *Consider the upper bound $\phi(x_1, \ldots, x_m) = \ell(D_{x_1}(\mathbb{F})) + \ldots + \ell(D_{x_m}(\mathbb{F})) + m - 1$ from Item 1 of Theorem 1.*

(1) *Let $\mathbb{F} = \mathbb{F}_q$ and $x_k > q$ for at least one $k \in \mathbf{m}$. Then we have $\phi(x_1, \ldots, x_m) < n - 1$.*
(2) *Let $\mathbb{F} = \mathbb{F}_q$ and $x_k \leq q$ for each $k \in \mathbf{m}$. Then $\phi(x_1, \ldots, x_m) = n - 1$.*
(3) *Let $\mathbb{F}$ be infinite. Then $\phi(x_1, \ldots, x_m) = n - 1 = \ell(\mathcal{A})$.*

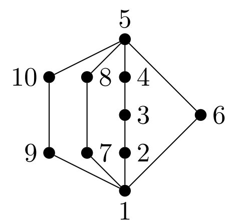

FIG. 1. Hasse diagram of a 5-partite poset.

Proof. Let us prove Item 1. Choose such index $k_0 \in \mathbf{m}$ that $x_{k_0} > q$. The minimal polynomial of each matrix from $D_{x_{k_0}}(\mathbb{F})$ has degree strictly less than x_{k_0}. From this, we deduce that $\ell(D_{x_{k_0}}(\mathbb{F})) < x_{k_0} - 1$ according to [13, Lemma 5.3]. Theorem 2 implies the inequality $\ell(D_{x_k}(\mathbb{F})) \leq x_k - 1$ for any k. Also, $\sum_{k=1}^{m} x_k = n$ by virtue of Item (a) of Remark 1. Thus,

$$\sum_{i=1}^{m} \ell(D_{x_i}(\mathbb{F})) + m - 1 < \sum_{i=1}^{m}(x_i - 1) + m - 1 = n - m + m - 1 = n - 1.$$

To prove Items 2, 3 we note that $\ell(D_{x_k}(\mathbb{F})) = x_k - 1$ for each $k \in \mathbf{m}$ by Equation (1). Hence, the above inequality turns to equality. The fact that $\ell(\mathcal{A}) = n - 1$ for an infinite field was taken from Theorem 2. □

However, often there is more than one way to introduce the function ϑ on a poset. Item 1 of Theorem 1 may provide several distinct estimations depending on the choice of ϑ.

Example 1. *Consider the partial order $\preceq$ on $\mathbf{10} = \{1, 2, \ldots, 10\}$ which Hasse diagram is depicted in Figure 1. This poset can be considered as 5-partite at least in two different ways. The first partition is given by*

$$\mathcal{X}_1 = \{1\},\ \mathcal{X}_2 = \{9, 7, 2, 6\},\ \mathcal{X}_3 = \{3, 8\},\ \mathcal{X}_4 = \{10, 4\},\ \mathcal{X}_5 = \{5\}$$

and we set $\vartheta\big|_{\mathcal{X}_i} = i$ (see Remark 1). The second partition can be introduced as follows:

$$\mathcal{X}'_1 = \{1\},\ \mathcal{X}'_2 = \{6, 2\},\ \mathcal{X}'_3 = \{9, 7, 3\},\ \mathcal{X}'_4 = \{10, 8, 4\},\ \mathcal{X}'_5 = \{5\}$$

with $\vartheta'\big|_{\mathcal{X}'_i} = i$. Let the ground field be $\mathbb{F}_2 = \{0, 1\}$. We look at the incidence algebra $\mathcal{A} = \mathcal{A}_{10}(\preceq, \mathbb{F}_2)$. Item 1 of Theorem 1 provides two upper bounds:

$$\ell(\mathcal{A}) \leq 0 + 2 + 1 + 1 + 0 + 5 - 1 = 8 \quad \text{for } \vartheta,$$
$$\ell(\mathcal{A}) \leq 0 + 1 + 1 + 1 + 0 + 5 - 1 = 7 \quad \text{for } \vartheta'.$$

Note that both estimations are better than the bound $n - 1 = 9$ from Theorem 2. So this example also illustrates how Item 1 of Proposition 2 can be applied.

4 The case of complete multipartite posets

We proceed to prove the second item of Theorem 1. The main result of the section is Proposition 3, which provides the lower bound for the length under some restriction. We need to introduce a technical notion.

Definition 9. Let $q \in \mathbb{N}$ be a power of a prime and $n \in \mathbb{N}$. A pair (n, q) is called *suitable* if there exist such subset $S = \{D_1, \dots, D_m\} \subseteq D_n(\mathbb{F}_q)$ and such monomials $f_1, \dots, f_n \in \mathbb{F}_q[x_1, \dots, x_m]$ of m commuting variables that:

1. S generates $D_n(\mathbb{F}_q)$;
2. $\ell(S) = \ell(D_n(\mathbb{F}_q))$;
3. the set of matrices $\{f_i(D_1, \dots, D_m) \mid i : \deg\ f_i \le k\}$ constitutes a basis of $\mathcal{L}_k(S)$ for each $k = 0, \dots, \ell(S)$;
4. $f_1 \equiv 1$ and $\deg f_n = \ell(S)$;
5. $\{f_i(D_1, \dots, D_m)\}_{i=1}^n$ is a basis of $D_n(\mathbb{F})$;
6. the matrix $G = \mathcal{F}(f_1(D_1, \dots, D_m), \dots, f_n(D_1, \dots, D_m))$ is invertible and the last row of G^{-1} contains no zero (see Definition 7).

Concrete examples of suitable pairs will be obtained in Section 5.

Proposition 3. *Consider the incidence algebra*

$$\mathcal{A} = \left(\begin{array}{c|c|c|c} D_{\mathcal{W}_1}(\mathbb{F}_q) & M_{\mathcal{W}_1\times\mathcal{W}_2}(\mathbb{F}_q) & \dots & M_{\mathcal{W}_1\times\mathcal{W}_h}(\mathbb{F}_q) \\ \hline O & D_{\mathcal{W}_2}(\mathbb{F}_q) & \dots & M_{\mathcal{W}_2\times\mathcal{W}_h}(\mathbb{F}_q) \\ \hline \vdots & \ddots & \ddots & \vdots \\ \hline O & O & \dots & D_{\mathcal{W}_h}(\mathbb{F}_q) \end{array}\right) \subseteq M_n(\mathbb{F}_q)$$

over the field $\mathbb{F}_q$ with $n, h \ge 2$. Assume that $(\mathcal{W}_2, q), \dots, (\mathcal{W}_{h-1}, q)$ are suitable pairs. Then $\ell(\mathcal{A}) \ge \gamma$, where $\gamma = h - 1 + \sum_{i=1}^h \ell(D_{\mathcal{W}_i}(\mathbb{F}_q))$.

For convenience of the reader, a short plan of the proof is given.

[*Step* 1.] Construct a generating subset $\mathbf{S} \subseteq \mathcal{A}$.
[*Step* 2.] Construct an auxiliary subset $\Phi \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$. Prove that the linear span $\langle\Phi\rangle$ coincides with $J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$.
[*Step* 3.] Construct another subset $\Upsilon \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$. Prove that $\Phi \subset \langle\Upsilon\rangle$. Hence, $\langle\Upsilon\rangle = J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$ according to Step 2.
[*Step* 4.] Prove that some elements of Υ are zero.
[*Step* 5.] Show that $|\Upsilon \setminus \{O_n\}| < \dim(J(\mathcal{A}))$. Then by Step 3, we obtain that $\dim(J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})) < \dim(J(\mathcal{A}))$. Thus $J(\mathcal{A}) \nsubseteq \mathcal{L}_{\gamma-1}(\mathbf{S})$ and so $\mathcal{L}_{\gamma-1}(\mathbf{S}) \ne \mathcal{A}$. From this, $\ell(\mathcal{A}) \ge \ell(\mathbf{S}) \ge \gamma$.

Following our plan we need to begin with the set $\mathbf{S}$.

Construction 1. *For each $j = 2, \dots, h-1$, introduce the set of matrices $S_j = \{D_1^{(j)}, \dots, D_{m_j}^{(j)}\} \subseteq D_{\mathcal{W}_j}(\mathbb{F}_q)$ and monomials $\{f_1^{(j)}, \dots, f_{\mathcal{W}_j}^{(j)}\}$ of m_j commuting variables as in Definition 9. Also, consider*

$$G_j = \mathcal{F}(f_1^{(j)}(D_1^{(j)}, \dots, D_{m_j}^{(j)}), \dots, f_{\mathcal{W}_j}^{(j)}(D_1^{(j)}, \dots, D_{m_j}^{(j)})) \in M_{\mathcal{W}_j}(\mathbb{F}_q). \quad (4)$$

In addition, for $j \in \{1, h\}$, *we define* $S_j = \{D_1^{(j)}, \ldots, D_{m_j}^{(j)}\} \subseteq D_{\mathcal{W}_j}(\mathbb{F}_q)$ *and monomials* $\{f_1^{(j)}, \ldots, f_{\mathcal{W}_j}^{(j)}\}$ *of* m_j *commuting variables which satisfy at least Conditions* $1-5$ *of Definition 9. This is possible according to* [6, Lemma 2.1].

For each $j = 1, \ldots, h-1$, *we define a matrix* $A_{j,j+1}$ *of the size* $\mathcal{W}_j \times \mathcal{W}_{j+1}$ *as follows. Let all entries of* $A_{1,2}$ *equal* 1. *For convenience, we will refer to each column of* $A_{1,2}$ *as* $\left(z_1^{(1)}, \ldots, z_{\mathcal{W}_1}^{(1)}\right)^t = (1, \ldots, 1)^t$. *Next, for* $2 \leq j \leq h-1$ *we make all columns of* $A_{j,j+1}$ *the same and equal the last row of the matrix* G_j^{-1}. *In other words,*

$$\begin{pmatrix} (A_{j,j+1})_{1,1} \\ \vdots \\ (A_{j,j+1})_{\mathcal{W}_j,1} \end{pmatrix} = \ldots = \begin{pmatrix} (A_{j,j+1})_{1,\mathcal{W}_{j+1}} \\ \vdots \\ (A_{j,j+1})_{\mathcal{W}_j,\mathcal{W}_{j+1}} \end{pmatrix} = \begin{pmatrix} z_1^{(j)} \\ \vdots \\ z_{\mathcal{W}_j}^{(j)} \end{pmatrix} = (G_j^{-1})^t \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \qquad (5)$$

Hence, we have

$$\begin{pmatrix} z_1^{(j)} & \ldots & z_{\mathcal{W}_j}^{(j)} \end{pmatrix} G_j = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}, \quad j = 2, \ldots, h-1. \qquad (6)$$

It is important to emphasize that any entry of each matrix $A_{j,j+1}$ *does not equal zero by Item* 6 *of Definition 9, that is*

$$z_\alpha^{(j)} \neq 0, \qquad j = 1, \ldots, h-1, \quad \alpha = 1, \ldots, \mathcal{W}_j. \qquad (7)$$

For each $j = 1, \ldots, h-1$, *we define* $\widehat{A}_{j,j+1} \in \mathcal{A}$ *to be such block matrix that all blocks are zero except for the* $(j, j+1)$*-block which equals* $A_{j,j+1}$. *Similarly, for all* $j = 1, \ldots, h$ *and* $i_j = 0, \ldots, m_j$, *let* $\widehat{D}_{i_j}^{(j)} \in \mathcal{A}$ *be such block matrix that all blocks are zero except for* (j,j)*-block, which is equal to* $D_{i_j}^{(j)}$ *if* $i_j \neq 0$ *and the identity matrix* $I_{\mathcal{W}_j}$ *if* $i_j = 0$. *Let* $\widehat{S}_j = \{\widehat{D}_0^{(j)}, \widehat{D}_1^{(j)}, \ldots, \widehat{D}_{m_j}^{(j)}\}$, $j = 1, \ldots, h$. *Eventually, we introduce*

$$\mathbf{S} = \{\widehat{A}_{1,2},\ \widehat{A}_{2,3},\ \ldots,\ \widehat{A}_{h-1,h}\} \cup \widehat{S}_1 \cup \widehat{S}_2 \cup \ldots \cup \widehat{S}_h \subseteq \mathcal{A}. \qquad (8)$$

This is the end of our construction.

In fact, the constructed set generates our algebra.

Lemma 3. *The set* $\mathbf{S}$ *generates the algebra* $\mathcal{A}$.

Proof. The set S_j generates the algebra $D_{\mathcal{W}_j}(\mathbb{F}_q)$. The structure of matrices $\widehat{D}_0^{(j)} \in \widehat{S}_j$ implies that the set $\bigcup_{j=1}^h \widehat{S}_j$ generates $D_{\mathcal{W}_1}(\mathbb{F}_q) \oplus \ldots \oplus D_{\mathcal{W}_h}(\mathbb{F}_q) = D_n(\mathbb{F}_q)$. Hence, the set $\bigcup_{j=1}^h \widehat{S}_j$ satisfies Item 1 of [11, Theorem] applied to the algebra $D_n(\mathbb{F}_q)$. Since $\bigcup_{j=1}^h \widehat{S}_j \subseteq \mathbf{S}$, the set $\mathbf{S}$ satisfies Item 1 of [11, Theorem] applied to $\mathcal{A}$. Equations (2), (7) ensure that the set $\{\widehat{A}_{j,j+1}\}_{j=1}^{h-1}$ fulfils Item 2 of [11, Theorem] applied to $\mathcal{A}$. Thus, $\mathbf{S}$ generates $\mathcal{A}$. □

Next, we will estimate the length of $\mathbf{S}$. For this, the space $J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(S)$ will be examined. The following construction introduces a special subset $\Phi \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(S)$.

Construction 2. *For each $j = 1, \dots, h$ and $k \in \mathbb{N}$, we define*

$$\widehat{S}_j^k = \underbrace{\widehat{S}_j \cdot \ldots \cdot \widehat{S}_j}_{k \ times} = \{\widehat{D}_{t_1}^{(j)} \cdot \ldots \cdot \widehat{D}_{t_k}^{(j)} \mid t_1, \dots, t_k \in \{1, \dots, m_j\}\},$$

where numbers t_i are allowed to repeat. Also, let $\widehat{S}_j^0 = \{I_n\}$. Then we set $\widehat{S}_j^{\leq k} = \bigcup_{i=0}^{k} \widehat{S}_j^i$. After that, for all $s = 1, \dots, h-1$, $i = 1, \dots, h-s$, and $k_i, \dots, k_{i+s} \in \mathbb{N}_0$, we introduce the set $\Phi_{s,i,k_i,\dots,k_{i+s}}$ which consists of all possible products $\left(\prod_{j=i}^{i+s-1} \widehat{C}^{(j)} \widehat{A}_{j,j+1}\right) \cdot \widehat{C}^{(i+s)}$, where each $\widehat{C}^{(j)} \in \widehat{S}_j^{\leq k_j}$. It remains to define

$$\Phi \ = \ \bigcup_{s=1}^{h-1} \ \bigcup_{i=1}^{h-s} \bigcup \{\Phi_{s,i,k_i,\dots,k_{i+s}} \mid s + \sum_{u=i}^{i+s} k_u \leq \gamma - 1, \ k_u \in \mathbb{N}_0\}, \qquad (9)$$

where γ is from Proposition 3.

We need to check some basic properties of the set Φ.

Lemma 4. *We have $\Phi \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$ and $\langle \Phi \rangle = J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$.*

Proof. According to Construction 2 we obtain the inclusion $\Phi_{s,i,k_i,\dots,k_{i+s}} \subseteq J(\mathcal{A}) \cap \mathcal{L}_{s+k_i+\ldots+k_{i+s}}(\mathbf{S})$. Hence, Equation (9) implies that $\Phi \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$. The equality $\langle \Phi \rangle = J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$ follows from these observations:

1) a matrix $H \in \mathcal{L}_{\gamma-1}(\mathbf{S})$ has zero diagonal iff H is a linear combination of such products from the set $\mathbf{S}^{\gamma-1} = \mathbf{S} \cdot \ldots \cdot \mathbf{S}$ ($\gamma - 1$ times) that include at least one matrix $\widehat{A}_{j,j+1}$;

2) $\widehat{A}_{i,i+1} \cdot \widehat{A}_{j,j+1} = O_n$ if $i + 1 \neq j$;

3) if $j \neq i$, then $\widehat{D}_{i_j}^{(j)} \widehat{A}_{i,i+1} = O_n$ for all $i_j = 0 \dots, m_j$;

4) if $j \neq i+1$, then $\widehat{A}_{i,i+1} \widehat{D}_{i_j}^{(j)} = O_n$ for all $i_j = 0 \dots, m_j$. □

Our next goal is to construct another subset $\Upsilon \subseteq \mathcal{A}$, which is "smaller" than Φ in some sense as we will see later.

Construction 3. *We will use the monomials $f_1^{(j)}, \dots, f_{W_j}^{(j)}$ from Construction 1. For all $s = 1, \dots, h-1$, $i = 1, \dots, h-s$, and also $d_j = 1, \dots, \mathcal{W}_j$ with $j = i, \dots, i+s$, we consider the matrix*

$$B_{s,i,d_i,\dots,d_{i+s}} = \left(\prod_{j=i}^{i+s-1} f_{d_j}^{(j)}(\widehat{D}_1^{(j)}, \dots, \widehat{D}_{m_j}^{(j)}) \cdot \widehat{A}_{j,j+1} \right) \cdot$$
$$\cdot f_{d_{i+s}}^{(i+s)}(\widehat{D}_1^{(i+s)}, \dots, \widehat{D}_{m_{i+s}}^{(i+s)}). \quad (10)$$

Then we can define Υ by

$$\Upsilon = \bigcup_{s=1}^{h-1} \ \bigcup_{i=1}^{h-s} \bigcup \{B_{s,i,d_1,\dots,d_{i+s}} \mid s + \sum_{j=i}^{i+s} \deg f_{d_j}^{(j)} \leq \gamma - 1, \ 1 \leq d_j \leq \mathcal{W}_j\}, \ (11)$$

where γ is from Proposition 3.

It turns out that the linear span of Υ is equal to $J(\mathcal{A})\cap\mathcal{L}_{\gamma-1}(\mathbf{S})$.

Lemma 5. *The inclusion $\Upsilon\subseteq J(\mathcal{A})\cap\mathcal{L}_{\gamma-1}(\mathbf{S})$ holds. Moreover, $\Phi\subseteq\langle\Upsilon\rangle$ and so $\langle\Upsilon\rangle = J(\mathcal{A})\cap\mathcal{L}_{\gamma-1}(\mathbf{S})$.*

Proof. According to Equation (10) we have

$$B_{s,i,d_i,\ldots,d_{i+s}}\in J(\mathcal{A})\cap\mathcal{L}_{s+\deg f^{(i)}_{d_i}+\ldots+\deg f^{(i+s)}_{d_{i+s}}}(\mathbf{S}).$$

Hence, $\Upsilon\subseteq J(\mathcal{A})\cap\mathcal{L}_{\gamma-1}(\mathbf{S})$ by Equation (11).

Let us show that $\Phi\subseteq\langle\Upsilon\rangle$. Consider an arbitrary matrix X from Φ. Construction 2 implies that X belongs to some set $\Phi_{s,i,k_i,\ldots,k_{i+s}}$ and so we can write

$$X=\left(\prod_{j=i}^{i+s-1}\widehat{C}^{(j)}\widehat{A}_{j,j+1}\right)\cdot\widehat{C}^{(i+s)}, \tag{12}$$

where $\widehat{C}^{(j)}\in\widehat{S}_j^{\le k_j}$, $s\in\{1,\ldots,h-1\}$, $i\in\{1,\ldots,h-s\}$ and

$$s+k_i+\ldots+k_{i+s}\le\gamma-1. \tag{13}$$

Since each $\widehat{C}^{(j)}\in\widehat{S}_j^{\le k_j}$, we can write $\widehat{C}^{(j)}=\widehat{D}^{(j)}_{t_1}\cdot\ldots\cdot\widehat{D}^{(j)}_{t_{s_j}}$ for some possibly repeating indices $t_1,\ldots,t_{s_j}\in\{1,\ldots,m_j\}$ and $s_j\le k_j$. If $s_j=0$, we have $\widehat{C}^{(j)}=I_n$, but according to block structure of matrices and Equation (12) we temporarily assume that $s_j=1$ (later it will return to zero value) and $\widehat{C}^{(j)}=\widehat{D}^{(j)}_0$, where

$$\widehat{D}^{(j)}_0=O_{\mathcal{W}_1}\oplus\ldots\oplus O_{\mathcal{W}_{j-1}}\oplus I_{\mathcal{W}_j}\oplus O_{\mathcal{W}_{j+1}}\oplus\ldots\oplus O_{\mathcal{W}_h}. \tag{14}$$

By definition of $D^{(j)}_{i_j}$ (see Construction 1) we have

$$\widehat{C}^{(j)}=O_{\mathcal{W}_1}\oplus\ldots\oplus O_{\mathcal{W}_{j-1}}\oplus D^{(j)}_{t_1}\cdot\ldots\cdot D^{(j)}_{t_{s_j}}\oplus O_{\mathcal{W}_{j+1}}\oplus\ldots\oplus O_{\mathcal{W}_h}.$$

Note that $D^{(j)}_{t_1}\cdot\ldots\cdot D^{(j)}_{t_{s_j}}$ is a product of length $s_j\le k_j$ and each matrix $D^{(j)}_{t_i}$ belongs to the generating set S_j of the algebra $D_{\mathcal{W}_j}(\mathbb{F}_q)$. Hence, we have $D^{(j)}_{t_1}\cdot\ldots\cdot D^{(j)}_{t_{s_j}}\in\mathcal{L}_{k_j}(S_j)$. However, Construction 1 and Item 3 of Definition 9 ensure that the set

$$\{f^{(j)}_{d_j}(D^{(j)}_1,\ldots,D^{(j)}_{m_j})\mid d_j:\ \deg\ f^{(j)}_{d_j}\le k_j\}$$

is a basis of the space $\mathcal{L}_{k_j}(S_j)$. Then we can consider the decomposition on this basis

$$D^{(j)}_{t_1}\cdot\ldots\cdot D^{(j)}_{t_{s_j}}=\sum_{\substack{d_j\in\{1,\ldots,\mathcal{W}_j\}:\\ \deg\ f^{(j)}_{d_j}\le k_j}}\lambda^{(j)}_{d_j}\cdot f^{(j)}_{d_j}(D^{(j)}_1,\ldots,D^{(j)}_{m_j})$$

for some coefficients $\lambda_{d_j}^{(j)}$ from the ground field $\mathbb{F}_q$. Recall that $f_1^{(j)} \equiv 1$, so $f_1^{(j)}(D_1^{(j)}, \ldots, D_{m_j}^{(j)}) = I_{\mathcal{W}_j}$. Returning to $n \times n$ matrices we obtain

$$\widehat{C}^{(j)} = \widehat{D}_{t_1}^{(j)} \cdot \ldots \cdot \widehat{D}_{t_{s_j}}^{(j)} = \sum_{\substack{d_j \in \{2, \ldots, \mathcal{W}_j\}: \\ \deg\ f_{d_j}^{(j)} \leq k_j}} \lambda_{d_j}^{(j)} \cdot f_{d_j}^{(j)}(\widehat{D}_1^{(j)}, \ldots, \widehat{D}_{m_j}^{(j)}) + \lambda_1^{(j)} \widehat{D}_0^{(j)},$$

where we used Equation (14). Next, we substitute the above expression for $\widehat{C}^{(j)}$ into Equation (12):

$$X = \prod_{j=i}^{i+s-1} \left(\left(\sum_{\substack{d_j \in \{2, \ldots, \mathcal{W}_j\}: \\ \deg\ f_{d_j}^{(j)} \leq k_j}} \lambda_{d_j}^{(j)} \cdot f_{d_j}^{(j)}(\widehat{D}_1^{(j)}, \ldots, \widehat{D}_{m_j}^{(j)}) + \lambda_1^{(j)} \widehat{D}_0^{(j)} \right) \widehat{A}_{j,j+1} \right) \cdot$$
$$\cdot \left(\sum_{\substack{d_{i+s} \in \{2, \ldots, \mathcal{W}_{i+s}\}: \\ \deg\ f_{d_{i+s}}^{(i+s)} \leq k_{i+s}}} \lambda_{d_{i+s}}^{(i+s)} \cdot f_{d_{i+s}}^{(i+s)}(\widehat{D}_1^{(i+s)}, \ldots, \widehat{D}_{m_{i+s}}^{(i+s)}) + \lambda_1^{(i+s)} \widehat{D}_0^{(i+s)} \right).$$

Then we expand brackets taking into account the relations $\widehat{D}_0^{(j)} \widehat{A}_{j,j+1} = \widehat{A}_{j,j+1} \widehat{D}_0^{(j+1)} = \widehat{A}_{j,j+1}$, which follow from block structure of the matrices. These relations also ensure that we can replace the matrix $\widehat{D}_0^{(j)}$ with the identity matrix $I_n = f_1^{(j)}(\widehat{D}_1^{(j)}, \ldots, \widehat{D}_{m_j}^{(j)})$ everywhere in the above formula. So if we previously replaced $s_j = 0$ with $s_j = 1$, we can now return it to $s_j = 0$. After that, we obtain

$$X = \sum_{(d_i, \ldots, d_{i+s}) \in \Delta} \mu_{d_i, \ldots, d_{i+s}} \left(\prod_{j=i}^{i+s-1} f_{d_j}^{(j)}(\widehat{D}_1^{(j)}, \ldots, \widehat{D}_m^{(j)}) \widehat{A}_{j,j+1} \right) \cdot f_{d_{i+s}}^{(i+s)}(\widehat{D}_1^{(i+s)}, \ldots, \widehat{D}_m^{(i+s)})$$

for scalars $\mu_{d_i, \ldots, d_{i+s}}$ from the ground field $\mathbb{F}_q$, some of which may equal zero. The set Δ consists of such tuples $(d_i, \ldots, d_{i+s})$ that $1 \leq d_j \leq \mathcal{W}_j$ and

$$\deg\ f_{d_i}^{(i)} + \ldots + \deg\ f_{d_{i+s}}^{(i+s)} \leq k_i + \ldots + k_{i+s}. \tag{15}$$

By Equation (10), the above expression for X can be rewritten as

$$X = \sum_{(d_i, \ldots, d_{i+s}) \in \Delta} \mu_{d_i, \ldots, d_{i+s}} \cdot B_{s,i,d_i, \ldots, d_{i+s}}.$$

Combining this relation with Equations (11), (13), (15) we obtain $X \in \langle \Upsilon \rangle$. Since X is an arbitrary matrix from Φ, we deduce $\Phi \subseteq \langle \Upsilon \rangle$. It remains to apply Lemma 4 in order to deduce that

$$\langle \Upsilon \rangle = J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(S).$$

This completes the proof of the lemma. □

Our next step is to show that some elements of the set Υ equal zero.

Lemma 6. *In terms of Construction 1, we have*

$$\widehat{A}_{j-1,j} f^{(j)}_{d_j}(\widehat{D}^{(j)}_1, \ldots, \widehat{D}^{(j)}_{m_j}) \widehat{A}_{j,j+1} = O_n, \quad j = 2, \ldots, h-1, \quad d_j \neq \mathcal{W}_j.$$

Proof. The statement is equivalent to

$$A_{j-1,j} f^{(j)}_{d_j}(D^{(j)}_1, \ldots, D^{(j)}_{m_j}) A_{j,j+1} = O_{\mathcal{W}_{j-1}\times\mathcal{W}_{j+1}} \quad j = 2, \ldots, h-1, \;\; d_j \neq \mathcal{W}_j.$$

By Equation (5), these relations can be rewritten in terms of rows of the matrix $A_{j-1,j}$ and columns of the matrix $A_{j,j+1}$ as follows

$$\begin{pmatrix} z^{(j-1)}_\alpha & z^{(j-1)}_\alpha & \ldots & z^{(j-1)}_\alpha \end{pmatrix} \cdot f^{(j)}_{d_j}(D^{(j)}_1, \ldots, D^{(j)}_{m_j}) \cdot \begin{pmatrix} z^{(j)}_1 \\ \vdots \\ z^{(j)}_{\mathcal{W}_j} \end{pmatrix} = 0$$

for all $\alpha = 1, \ldots, \mathcal{W}_{j-1}, \;\; j = 2, \ldots, h-1, \;\; d_j \neq \mathcal{W}_j$. Equation (7) allows to divide both sides by $z^{(j-1)}_\alpha$

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix} \cdot f^{(j)}_{d_j}(D^{(j)}_1, \ldots, D^{(j)}_{m_j}) \cdot \begin{pmatrix} z^{(j)}_1 \\ \vdots \\ z^{(j)}_{\mathcal{W}_j} \end{pmatrix} = 0.$$

Since all $f^{(j)}_{d_j}(D^{(j)}_1, \ldots, D^{(j)}_{m_j})$ are diagonal matrices, these equations can be rewritten in terms of the map $\mathcal{F}$ (see Definition 7)

$$\begin{pmatrix} z^{(j)}_1 & \ldots & z^{(j)}_{\mathcal{W}_j} \end{pmatrix} \cdot R^{(j)} = \begin{pmatrix} 0 & \ldots & 0 \end{pmatrix}, \tag{16}$$

$$R^{(j)} = \mathcal{F}(f^{(j)}_1(D^{(j)}_1, \ldots, D^{(j)}_{m_j}), \ldots, f^{(j)}_{\mathcal{W}_j-1}(D^{(j)}_1, \ldots, D^{(j)}_{m_j}), O_{\mathcal{W}_j\times\mathcal{W}_j})$$

These relations are immediate consequences of Equations (4), (6) if all elements of the last column of G_j were replaced by zeroes. □

The previous lemma implies an estimation for the cardinality $|\Upsilon|$.

Lemma 7. *We have* $|\Upsilon \setminus \{O_n\}| < \dim J(\mathcal{A})$.

Proof. Consider Equations (10), (11). Lemma 6 implies that for any $s \geq 2$, the matrix $B_{s,i,d_i,\ldots,d_{i+s}} = O$ when $d_j \neq \mathcal{W}_j$ for some $j \in \{i+1, \ldots, i+s-1\}$. It follows that any matrix from $\Upsilon \setminus \{O\}$ has type $B_{s,i,d_i,\mathcal{W}_{i+1},\ldots,\mathcal{W}_{i+s-1},d_{i+s}}$ for $1 \leq s \leq h-1$. Moreover, Equation (11) and Item 4 of Definition 9 ensures that $B_{h-1,1,\mathcal{W}_1,\mathcal{W}_2\ldots,\mathcal{W}_h} \notin \Upsilon$ since

$$h - 1 + \deg f^{(1)}_{\mathcal{W}_1} + \ldots + \deg f^{(h)}_{\mathcal{W}_h} = \\ h - 1 + \ell(D_{\mathcal{W}_1}(\mathbb{F}_q)) + \ldots + \ell(D_{\mathcal{W}_h}(\mathbb{F}_q)) = \gamma > \gamma - 1.$$

Next, we define the set of tuples

$$\Lambda = \bigcup_{1\le i<j\le h} \{(d_i, d_j, i, j) \mid d_i = 1, \dots, \mathcal{W}_i,\ d_j = 1, \dots, \mathcal{W}_j\}$$

and the map $\nu : \Lambda \setminus \{(\mathcal{W}_1, \mathcal{W}_h, 1, h)\} \to \Upsilon$, which sends the tuple (d_i, d_j, i, j) to $B_{j-i,i,d_i,\mathcal{W}_{i+1},\dots,\mathcal{W}_{j-1},d_j}$. Then

$$|\Upsilon \setminus \{O_n\}| \le |\operatorname{Im}\nu| \le |\Lambda \setminus \{(\mathcal{W}_1, \mathcal{W}_h, 1, h)\}| < |\Lambda| = \sum_{1\le i<j\le h} \mathcal{W}_i \cdot \mathcal{W}_j = \sum_{1\le i<j\le h} \dim M_{\mathcal{W}_i\times\mathcal{W}_j}(\mathbb{F}_q) = \dim J(\mathcal{A})$$

since $J(\mathcal{A}) \subseteq \mathcal{A}$ is a subspace of matrices with the zero diagonal. □

We are ready now to complete the proof of Proposition 3.

Proof of Proposition 3. Consider the subset $\mathbf{S} \subseteq \mathcal{A}$ from Construction 1. Then Lemma 3 ensures that $\mathbf{S}$ generates the algebra $\mathcal{A}$. We will examine the subspace $J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S}) \subseteq \mathcal{A}$. According to Lemmas 5, 7, there exists such subset $\Upsilon \subseteq J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$ that its linear span coincides with $J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})$ and $|\Upsilon \setminus \{O_n\}| < \dim J(\mathcal{A})$. It follows that $\dim(J(\mathcal{A}) \cap \mathcal{L}_{\gamma-1}(\mathbf{S})) < \dim J(\mathcal{A})$. Hence, $J(\mathcal{A}) \nsubseteq \mathcal{L}_{\gamma-1}(\mathbf{S})$ and so $\mathcal{L}_{\gamma-1}(\mathbf{S}) \ne \mathcal{A}$ in particular. Then by definition of the length of an algebra, we obtain $\ell(\mathcal{A}) \ge \ell(\mathbf{S}) \ge \gamma$. This completes the proof of the proposition □

5 Suitable pairs of numbers

Section 5 is devoted to suitable pairs of numbers (see Definition 9). The main result of the section is Proposition 4, which gives sufficient conditions for the pair of natural numbers to be suitable.

Lemma 8. *If q is a power of a prime number and $n \le q$, then the pair (n, q) is suitable.*

Proof. Since $n \le q$, there exists at least n distinct elements of the field $\mathbb{F}_q$, say, $\alpha_1, \dots, \alpha_n$. Then we set $D = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. So the degree of the minimal polynomial of the matrix D equals n. Hence, $\{D\}$ generates the algebra $D_n(\mathbb{F}_q)$ and $\ell(\{D\}) = n - 1 = \ell(D_n(\mathbb{F}_q))$, see Equation (1). So the set $S = \{D\}$ and monomials $\{f_i(x) = x^{i-1}\}_{i=1}^{n-1}$ satisfy all Conditions $1, 2, 4, 5$ of Definition 9. Note that Condition 3 follows directly from the definition of the space $\mathcal{L}_k(S)$. Also, Condition 6 can be obtained from Lemma 1 since $\mathcal{F}(f_1(D), \dots, f_n(D)) = \mathbf{V}(\alpha_1, \dots, \alpha_n)$. □

Now we can prove a sufficient condition for the pair of numbers to be suitable.

Proposition 4. *Let q be a power of a prime number. Assume that a natural number n can be represented as $n = kq^r$ with $r \in \mathbb{N}_0$ and $k \in \{1, \dots, q-1\}$. Then the pair (n, q) is suitable.*

Proof. If $r = 0$, then the statement follows from Lemma 8. Henceforth, we shall assume that $r > 0$. There are two cases.

[*Case* 1.] Let $k \geq 2$. We will use a construction from [13]. Denote all elements of the field $\mathbb{F}_q$ by $\alpha_1, \ldots, \alpha_q$. Consider $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_q) \in D_q(\mathbb{F}_q)$ and also $P = \mathrm{diag}(\alpha_1, \ldots, \alpha_k) \in D_k(\mathbb{F}_q)$. Next, we introduce the following Kronecker products of matrices:

$$D_j = I_q^{\otimes(j-1)} \otimes D \otimes I_q^{\otimes(r-j)} \otimes I_k, \quad j = 1, \ldots, r$$
$$D_{r+1} = I_q^{\otimes r} \otimes P.$$

Denote $S = \{D_1, \ldots, D_{r+1}\}$. Then for any $i_1, \ldots, i_r \in \{0, \ldots, q-1\}$, and $i_{r+1} \in \{0, \ldots, k-1\}$, we consider the monomial $x_1^{i_1} \ldots x_{r+1}^{i_{r+1}}$. The set of all such monomials can be lexicographically ordered. So using only one index $i = 1, \ldots, n$ we can enumerate all the monomials from the least to the greatest one. Denote the i-th monomial by f_i. We will show that the set of matrices S and monomials $\{f_i\}_{i=1}^n$ fulfil all Conditions 1-6 of Definition 9.

1) Note that $D_n(\mathbb{F}_q) = D_q(\mathbb{F}_q)^{\otimes r} \otimes D_k(\mathbb{F}_q)$. The sets $\{D\}$, $\{P\}$ generate $D_q(\mathbb{F}_q)$ and $D_k(\mathbb{F}_q)$, respectively. Therefore, Condition 1 follows from [13, Lemma 5.1].

2) The relation $n = kq^r$, Equation (1), and [13, Lemma 5.1] imply that

$$\ell(S) = r \cdot \ell(D_q(\mathbb{F}_q)) + \ell(D_k(\mathbb{F}_q)) = (q-1)[\log_q n] + [q^{\{\log_q n\}}] - 1 = \ell(D_n(\mathbb{F}_q)).$$

3) Note that the set $\Theta_k = \{f_i(D_1, \ldots, D_{r+1}) \mid i : \ \deg \ f_i \leq k\}$ coincides with all possible products of not greater than k matrices from the set S. Hence, $\langle \Theta_k \rangle = \mathcal{L}_k(S)$ dy definition of space $\mathcal{L}_k$. Let us show that the set Θ_k is linearly independent. In fact, the sets $\{I_q = D^0, D, D^2, \ldots, D^{q-1}\}$ and $\{I_k = P^0, P, P^2, \ldots, P^{k-1}\}$ are bases of the algebras $D_q(\mathbb{F}_q)$ and $D_k(\mathbb{F}_q)$, respectively. Then by the properties of tensor product, we obtain that the matrices $D^{i_1} \otimes \ldots \otimes D^{i_r} \otimes P^{i_{r+1}}$ with $i_1, \ldots, i_r \in \{0, \ldots, q-1\}$ and $i_{r+1} \in \{0, \ldots, k-1\}$ constitute a basis of the algebra $D_n(\mathbb{F}_q) = D_q(\mathbb{F}_q)^{\otimes r} \otimes D_k(\mathbb{F}_q)$. From this, Θ_k is linearly independent as a subset of a linearly independent set.

4) We have $f_1 = x_1^0 \ldots x_{r+1}^0 = 1$ and also $\deg f_n = \deg(x_1^{q-1} \ldots x_r^{q-1} x_{r+1}^k) = r(q-1) + k = \ell(S)$ as in the proof of Condition 2.

5) Condition 5 is a partial case of Condition 3 when $k = \ell(S)$.

6) First we notice that the matrix G is nonsingular according to Condition 5. Next, applying Lemma 2 we can deduce that

$$G = \mathcal{F}(I_q, D, \ldots, D^{q-1})^{\otimes r} \otimes \mathcal{F}(I_q, P, \ldots, P^{k-1}) = \\ = \mathbf{V}(\alpha_1, \ldots, \alpha_n)^{\otimes r} \otimes \mathbf{V}(\alpha_1, \ldots, \alpha_k)$$

Then

$$G^{-1} = (\mathbf{V}(\alpha_1, \ldots, \alpha_n)^{-1})^{\otimes r} \otimes \mathbf{V}(\alpha_1, \ldots, \alpha_k)^{-1}.$$

Lemma 1 ensures that both matrices $\mathbf{V}(\alpha_1,\ldots,\alpha_n)^{-1}$, $\mathbf{V}(\alpha_1,\ldots,\alpha_k)^{-1}$ contains no zeros in the last row. Therefore, the matrix G^{-1} have no zero in the last row as well.

[*Case* 2.] Let $k = 1$. Then all arguments are literally the same with the only difference that now $S = \{I_q^{\otimes(i-1)} \otimes D \otimes I_q^{\otimes(r-i)}\}_{i=1}^r$ and monomials are $x_1^{i_r}\ldots x_r^{i_r}$ for $i_1,\ldots,i_r \in \{0,\ldots,q-1\}$. □

6 Completion of proof of the main theorem

In this section, we prove Item 2 of Theorem 1 and also Corollary 1. First we need to combine the results from two previous sections.

Theorem 3. *Let $(\mathbf{n},\preceq,\vartheta)$ be a complete h-partite poset, $n,h\geq 2$. Consider the incidence algebra $\mathcal{A}=\mathcal{A}_n(\preceq,\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. Assume that the Witney numbers $\mathcal{W}_2$, $\mathcal{W}_3$, …, $\mathcal{W}_{h-1}$ can be represented as $\mathcal{W}_j = k_j q^{r_j}$ for some $r_j\in\mathbb{N}_0$, $k_j\in\{1,\ldots,q-1\}$. Then*

$$\ell(\mathcal{A}) = \ell(D_{\mathcal{W}_1}(\mathbb{F}_q)) + \ldots + \ell(D_{\mathcal{W}_h}(\mathbb{F}_q)) + h - 1.$$

Proof. The proof of of Item 1 of Theorem 1 implies that there exists another partial order $\preceq'$ on $\mathbf{n}$ such that $(\mathbf{n},\preceq)\cong(\mathbf{n},\preceq')$ and $\mathcal{A}'=\mathcal{A}_n(\preceq',\mathbb{F}_q)$ has the block-triangular form as in Proposition 3.

Denote $\gamma = \ell(D_{\mathcal{W}_1}(\mathbb{F}_q)) + \ldots + \ell(D_{\mathcal{W}_h}(\mathbb{F}_q)) + h - 1$. All the pairs $(\mathcal{W}_2,q)$, $(\mathcal{W}_3,q)$, …, $(\mathcal{W}_{h-1},q)$ are suitable by Proposition 4. So $\ell(\mathcal{A}')\geq\gamma$ by virtue of Proposition 3. Therefore, $\ell(\mathcal{A}')=\gamma$ according to Item 1 of Theorem 1. It remains to note that $\ell(\mathcal{A})=\ell(\mathcal{A}')$ because $\mathcal{A}\cong\mathcal{A}'$. □

The previous theorem provides necessary examples, which show sharpness of the upper bound from Theorem 1.

Proof of Item 2 of Theorem 1. If the field $\mathbb{F}$ is infinite, Equation (1) implies that $\ell(D_{x_i}(\mathbb{F})) = x_i - 1$ and so the statement follows from Theorem 2. Assume further that $\mathbb{F}=\mathbb{F}_q$.

For each $i = 1,\ldots,m$, define $r_i = [\log_q x_i']$ and $k_i = \left[\dfrac{x_i'}{q^{[\log_q x_i']}}\right] = [q^{\{\log_q x_i'\}}]$. Then there exist such numbers $r_{0,i}\in\{0,\ldots,q^{r_i}-1\}$ that each x_i' can be represented as $x_i' = k_i\cdot q^{r_i} + r_{0,i}$. We introduce the quantities

$$\mathcal{W}_i = k_i\cdot q^{r_i},\quad N=\sum_{i=1}^m \mathcal{W}_i,\quad M = n - N = \sum_{i=1}^m r_{0,i}.$$

Then for the set $\mathbf{N}=\{1,\ldots,N\}$, consider a complete m-partite poset $(\mathbf{N},\widetilde{\preceq},\widetilde{\vartheta})$ with the sequence of the Whitney numbers $(\mathcal{W}_1,\ldots,\mathcal{W}_m)$. Denote by $\mathcal{B}=\mathcal{A}_N(\widetilde{\preceq},\mathbb{F}_q)$ the corresponding incidence algebra.

If $M = 0$, that is, $r_{0,i}=0$ for all i, then we have $\mathcal{W}_i = x_i$ and $N = n$. Then it suffices to set $\preceq' = \widetilde{\preceq}$, $\vartheta'=\widetilde{\vartheta}$ and the statement of Item 2 follows from Theorem 3.

Henceforth, we shall assume that $M \neq 0$, i.e. $r_{0,i} \neq 0$ for at least one i. Then we put $\preceq' = \widetilde{\preceq} \cup \{(j,j) \mid j = N+1, \ldots, N+M\}$. So the corresponding incidence algebra $\mathcal{A}_n(\preceq', \mathbb{F}_q) = \mathcal{B} \oplus D_M(\mathbb{F}_q)$. We need to introduce the function $\vartheta' : \mathbf{n} \to \mathbf{m}$ as required in Item 2. Let $r_{0,i_1}, \ldots, r_{0,i_d}$ be precisely all nonzero numbers from the family $\{r_{0,i}\}_{i=1}^m$ and $1 \leq i_1 < \ldots < i_d \leq m$. Then for each $u = 1, \ldots, d$ we introduce the set $\mathcal{Y}_u$ as follows

$$\mathcal{Y}_1 = \{N+1, \ldots, N+r_{0,i_1}\},\ \mathcal{Y}_u = \{N+\sum_{j=1}^{u-1} r_{0,i_j}+1, \ldots, N+\sum_{j=1}^{u} r_{0,i_j}\},\ u \geq 2.$$

Thus, we obtain the partition $\mathbf{n} = \mathbf{N} \cup \mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_d$ with $|\mathcal{Y}_u| = r_{0,i_u}$ and we can define

$$\vartheta'(t) = \left[\begin{array}{ll} \vartheta(t) & \text{if } t \in \mathbf{N}, \\ i_u & \text{if } t \in \mathcal{Y}_u. \end{array} \right.$$

The function ϑ' is strictly order-preserving and surjective since ϑ is so.

According to Item 2 it is necessary to check that $\vartheta'^{-1}(i) = x'_i$ for $i \in \mathbf{n}$. Indeed, if i belongs to $\{i_1, \ldots, i_d\}$, say $i = i_u$, we have $|\vartheta'^{-1}(i)| = |\vartheta^{-1}(i)| + |\mathcal{Y}_u| = \mathcal{W}_i + r_{0,i} = k_j \cdot q^{r_i} + r_{0,i} = x'_i$. If i does not belong to $\{i_1, \ldots, i_d\}$, then $|\vartheta'^{-1}(i)| = |\vartheta^{-1}(i)| = \mathcal{W}_i = k_j \cdot q^{r_i} = x'_i$ because $r_{0,i} = 0$.

Let us calculate the length of $\mathcal{A}' = \mathcal{A}_n(\preceq', \mathbb{F}_q)$. Then [16, Theorem 2] and Theorem 3 ensure that $\ell(\mathcal{A}') \geq \ell(\mathcal{B}) = \ell(D_{\mathcal{W}_1}(\mathbb{F}_q)) + \ldots + \ell(D_{\mathcal{W}_m}(\mathbb{F}_q)) + m - 1$. However, by Equation (1),

$$\ell(D_{\mathcal{W}_i}(\mathbb{F}_q)) = (q-1)[\log_q \mathcal{W}_i] + [q^{\{\log_q \mathcal{W}_i\}}] - 1 = (q-1)r_i + k_i - 1 = \\ = (q-1)[\log_q x'_i] + [q^{\{\log_q x'_i\}}] - 1 = \ell(D_{x'_i}(\mathbb{F}_q)).$$

Thus, we obtain that $\ell(\mathcal{A}') \geq \ell(D_{x'_1}(\mathbb{F}_q)) + \ldots + \ell(D_{x'_h}(\mathbb{F}_q)) + h - 1$. The inverse inequality follows from Item 1 of Theorem 1. $\square$

Proof of Corollary 1. Apply Item 1 of Theorem 1. According to Proposition 1 it is possible to introduce at least one mapping ϑ for $m = h$. Since each preimage $\vartheta^{-1}(i)$ is an antichain, we have $x_i = |\vartheta^{-1}(i)| \leq w$. Then monotonicity of the function $\phi(x) = \ell(D_x(\mathbb{F}))$ (see [13, Theorem 3.2]) implies that $\ell(D_{x_i}(\mathbb{F})) \leq \ell(D_w(\mathbb{F}))$. Consequently,

$$\ell(\mathcal{A}) \leq \ell(D_{x_1}(\mathbb{F})) + \ldots + \ell(D_{x_h}(\mathbb{F})) + h - 1 \leq (\ell(D_w(\mathbb{F})) + 1) \cdot h - 1.$$

Let us prove the second part of the corollary. We define $r = [\log_q w]$, $k = \left[\dfrac{w}{q^{[\log_q w]}}\right] = [q^{\{\log_q w\}}]$, $\widetilde{w} = kq^r$, so $0 \leq w - \widetilde{w} \leq q^r - 1$. Consider the complete h-partite poset $(\mathbf{n}, \preceq', \vartheta')$ with the Witney numbers $\mathcal{W}_1 = w$, $\mathcal{W}_2 = \mathcal{W}_3 = \ldots = \mathcal{W}_{h-1} = \mathcal{W}_h = \widetilde{w}$. Note that the width of $(\mathbf{n}', \preceq')$ coincides with $\max\limits_{j=1,\ldots,h} \mathcal{W}_j = w$. Moreover, the corresponding incidence algebra $\mathcal{A} = \mathcal{A}_n(\preceq, \mathbb{F})$ satisfies conditions of Theorem 3. Therefore $\ell(\mathcal{A}) = \ell(D_w(\mathbb{F})) + (h-1)\ell(D_{\widetilde{w}}(\mathbb{F})) + h - 1$. However, $r = [\log_q w] = [\log_q \widetilde{w}]$ and also $k =$

$\left[\dfrac{w}{q^{[\log_q w]}}\right] = \left[\dfrac{\widetilde{w}}{q^{[\log_q \widetilde{w}]}}\right]$. It follows that $\ell(D_w(\mathbb{F})) = \ell(D_{\widetilde{w}}(\mathbb{F}))$ by Equation (1). Hence, we obtain $\ell(\mathcal{A}) = (\ell(D_w(\mathbb{F})) + 1) \cdot h - 1$ completing the proof. □

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References

[1] Y.A. Al'pin, K.D. Ikramov, *Reducibility theorems for pairs of matrices as rational criteria*, Linear Algebra Appl., **313** (2000), 155–161.
[2] G. Agnarsson, *On multipartite posets*, Discrete Math., **308** (2008), 5284–5288.
[3] A.E. Guterman, M.A. Khrystik, O.V. Markova, *On the lengths of group algebras of finite abelian groups in the modular case*, J. Algebra Appl. **21**:6 (2022), 2250117.
[4] A.E. Guterman, D.K. Kudryavtsev, *Upper bounds for the length of non-associative algebras*, J. Algebra **544** (2020), 483–497.
[5] A.E. Guterman, T.J. Laffey, O.V. Markova, H. Šmigoc, *A resolution of Paz's conjecture in the presence of a nonderogatory matrix*, Linear Algebra Appl., **543** (2018), 234–250.
[6] N.A. Kolegov, *On the lengths of matrix incidence algebras with radicals of square zero*, Linear Multilinear Algebra **71**:10 (2023), 1736–1753.
[7] E. Krot-Sieniawska, *On incidence algebras description of cobweb posets*, Preprint, 2008.
[8] A.K. Kwaśniewski, M. Dziemiańczuk, *Cobweb posets — recent results*, Preprint, 2011.
[9] W.E. Longstaff, *H-minimum spanning lengths and an extension to Burnside's theorem on irreducibility*, Bull. Aust. Math. Soc., **104**:1 (2021), 78–93.
[10] W.E. Longstaff, *Irreducible families of complex matrices containing a rank-one matrix*, Bull. Aust. Math. Soc., **102**:2 (2020), 226–236.
[11] W.E. Longstaff, P. Rosenthal, *Generators of matrix incidence algebras*, Australas. J. Combin., 22 (2000), 117–21.
[12] L.A Mirsky, *Dual of Dilworth's decomposition theorem*, Am. Math. Mon., 78 (1971), 876–877.
[13] O.V. Markova, *Upper bound for the length of commutative algebras*, Sb. Math., **200**:(11–12) (2009), 1767–1787.
[14] O.V. Markova, *Length computation of matrix subalgebras of special type*, J. Math. Sci. (N.Y.), **155**:6 (2008), 908–931.
[15] Markova, O.V. *Length function and simultaneous triangularization of matrix pairs*, J Math Sci. (N.Y.), **272** (2023), 566–573.
[16] O.V. Markova, *On the length of the algebra of upper-triangular matrices*, Russian Math. Surveys, **60**:5 (2005), 984–985.
[17] O.V. Markova, C. Martínez, R.L. Rodrigues, *Algebras of length one*, J. Pure Appl. Algebra **226** (2022), 106993.
[18] C. Miguel, *Lengths of cyclic algebras and commutative subalgebras of quaternion matrices*, Linear Algebra Appl., **628** (2021), 130–139.
[19] B. Schröder, *Ordered Sets*, Springer International Publishing, 2016.
[20] Ya. Shitov, *An improved bound for the lengths of matrix algebras*, Algebra Number Theory, **13**:6 (2019), 1501–1507.
[21] E. Spiegel, C.J. O'Donnel, *Incidence algebras*, Marcel Dekker, New-York, 1997.
[22] C. Winter, *Poset Ramsey number $R(P, Q_n)$. I. Complete multipartite posets*, Order, **41** (2024), 391–399.

Nikita Antonivich Kolegov
Lomonosov Moscow State University,
119991, Moscow, Russia;

Moscow Center for Fundamental and Applied Mathematics,
119991, Moscow, Russia
Email address: na.kolegov@yandex.ru