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THE CRITICAL GROUP OF THE CONE OVER A SANDWICH GRAPH

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Abstract: We study the critical group of the cone over a sandwich graph. It is shown that this group can be described as the cokernel of a discrete Laplace operator. Furthermore, we prove that its exact structure can be determined by making use of a recurrence relation derived from the Laplace operator. This relation involves symmetric Laurent polynomials and provides a more efficient way to compute the group. The proposed method offers a powerful tool for analyzing invariants of graphs with complex combinatorial structure.

Keywords: critical group, circulant graph, discrete Laplacian, Smith normal form.

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1 Introduction

An important problem in algebraic graph theory is describing the structure of the *critical group* of a graph. This notion is closely related to discrete analogues of the classical Laplace operator. The discrete Laplacian naturally arises as a combinatorial version of its continuous counterpart, providing a useful tool for studying various graph invariants.

The significance of the critical group is not limited to the classical fact that its order equals the number of spanning trees in a graph [1]. Often, it also captures deeper structural properties, reflecting analogies with discrete versions of classical concepts from the theory of Riemann surfaces [1, 2], as well as connections to statistical physics [3, 4].

Although determining the exact structure of critical groups remains an open problem for many classes of graphs, certain families possess specific features that make them easier to analyze. Such special cases allow step-by-step progress toward understanding the general problem.

In Section 2, we define the main object of the study, namely the *cone graph* $\hat{G}$ constructed over a given sandwich graph G. The cone graph $\hat{G}$ is obtained by adding one new vertex connected to every vertex of G, thus forming a discrete analogue of a geometric cone with base G.

Several well-known graphs fit naturally into this sandwich graph construction. Important examples include the I-graph and the generalized Petersen graph (see, for example [5, 6]), and the generalized prism, which is also known as the cobordism of two circulant graphs [7].

Sandwich graphs form a special case of a more general construction known as *circulant foliation* [13, 14], a structure relevant to applications such as network design [15], quantum computing [16], chemistry [17], and number theory [18].

Section 2 also summarizes key known results. In particular, for cone graphs $\hat{G}$, there exists another naturally related group whose order equals the number of rooted spanning forests in the base graph G [19]. Understanding the structure of the discrete Laplacian for sandwich graphs thus gives essential insights into the critical group of $\hat{G}$ and allows us to develop computational methods that improve numerical efficiency, as shown in Section 3.

Classical results on critical groups, such as those in [1, 2, 8], are based on combinatorial and analytical ideas, but typically involve computing the Smith normal form of large Laplacian matrices [1, 8, 9, 10, 12] or using spectral methods [11]. These computations become difficult for graphs of large size or complex structure. Our approach, by contrast, uses the algebraic properties of circulant layers and relies on symmetric and bimonic Laurent polynomials. This allows us to express the critical group of the cone through a fixed-size matrix instead of a growing Laplacian matrix, which leads to a reduction in computational complexity.

We believe the results in this paper are new and may help to develop a more general understanding of circulant foliation graphs [13, 14].

2 Cone over a sandwich graph

Let us consider a classical path graph with vertices $v_1, v_2, \ldots, v_m$, characterized by the conditions $\deg(v_1) = \deg(v_m) = 1$ and $\deg(v_i) = 2$ for $i = 2, \ldots, m-1$, where $\deg(\cdot)$ denotes the vertex degree.

For given integers n and m, we introduce the concept of a *sandwich graph* G whose vertex set is defined as

$$V(G) = \{(k, v_i) \mid k = 0, 1, \ldots, n-1,\ i = 1, 2, \ldots, m\}.$$

In this graph, for each fixed i, vertices (k, v_i) together with integer parameters (called *jumps*) $s_{i,1}, s_{i,2}, \ldots, s_{i,k_i} \in \mathbb{Z}$ satisfying

$$1 \leq s_{i,1} < s_{i,2} < \ldots < s_{i,k_i} < \tfrac{n}{2},$$

form the circulant graph

$$G_i = C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}), \tag{1}$$

where each vertex (k, v_i) is adjacent to vertices

$$(k \pm s_{i,1}, v_i),\ (k \pm s_{i,2}, v_i),\ \ldots,\ (k \pm s_{i,k_i}, v_i) \mod n.$$

From this point of view, the circulant graphs play the role of *layers* placed at each vertex $v_1, v_2, \ldots, v_m$ of the path graph (see Fig. 1). Furthermore, if we fix k, then there is an edge connecting the vertices (k, v_i) and (k, v_{i+1}). Thus, for the sandwich graph G defined above, we say that G has m layers $G_1, \ldots, G_m$, each consisting of n vertices.

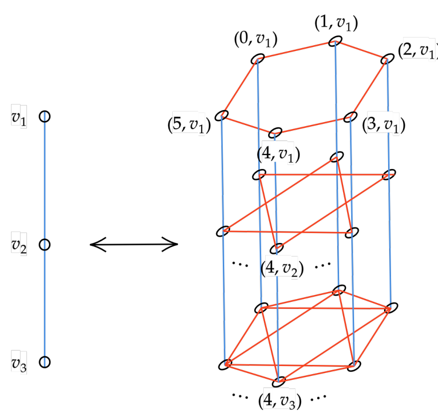

FIG. 1. From left to right: a path graph on 3 vertices and the corresponding sandwich graph G with $m = 3$ layers. From top to bottom, the layers are circulant graphs: $G_1 = C_6(1)$, $G_2 = C_6(2)$, and $G_3 = C_6(1, 2)$.

An important property of circulant graphs relevant to our study is their natural connection with Laurent polynomials [20]. Specifically, each circulant layer is associated with a Laurent polynomial

$$P_i(z) = 2k_i - \sum_{j=1}^{k_i} \left(z^{s_{i,j}} + z^{-s_{i,j}} \right), \quad i = 1, \dots, m, \quad j = 0, \dots, n-1, \tag{2}$$

which is *symmetric*, i. e. $P_i(z) = P_i(\frac{1}{z})$, and *bimonic*, meaning that the leading and trailing coefficients are equal to ± 1.

Finally, we construct the cone graph $\hat{G}$ over the sandwich graph G by adding one new vertex adjacent to all vertices of G. This provides a discrete analogue of a geometric cone with the graph G as its *base*.

Critical group of cone over a graph and Laplacian. Formally, for a connected graph G on n vertices, the critical group $\mathrm{K}(G)$ is the torsion part of the abelian group $\mathrm{coker}(\mathrm{L}(G))$ (see, for example [1]), where $\mathrm{L}(G)$ is the discrete Laplacian operator viewed as a homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$.

The discrete Laplacian is a discrete analogue of the classical Laplace operator, and it is represented by $n \times n$ matrix

$$\mathrm{L}(G) = D(G) - A(G),$$

where $D(G)$ is the diagonal degree matrix (whose diagonal entries are the degrees of vertices), and $A(G)$ is the adjacency matrix of G.

A fundamental property of the critical group is that its order equals the number of spanning trees of the graph [1]. The following Theorem 1 in [19] generalizes this classical fact, introducing a new group whose order equals the number of rooted spanning forests.

Theorem 1. *Let G be a connected graph on n vertices. Then the critical group of the cone $\hat{G}$ over the graph G is isomorphic to the cokernel of the linear operator $\mathrm{I}_n + \mathrm{L}(G)$, where $\mathrm{L}(G)$ is the discrete Laplacian of G, and I_n is the $n \times n$ identity matrix. More precisely,*

$$\mathrm{coker}(L(\hat{G})) \cong \mathrm{coker}(\mathrm{I}_n + \mathrm{L}(G)).$$

We refer to this group as the *forest group of a graph* $\mathrm{F}(G)$.

This result connects the critical group of the cone graph $\hat{G}$ with the operator $\mathrm{I}_n + \mathrm{L}(G)$, also known as the discrete Helmholtz operator [21]. It provides a discrete analogue of classical analytic concepts, thus extending continuous analytical methods into the discrete setting of combinatorial graph theory.

Theorem 1 shows that the cokernel of the Laplacian of the cone graph is a torsion group, as the operator $\mathrm{I}_n + \mathrm{L}(G)$ is nonsingular (has trivial kernel). Therefore, we have a decomposition

$$\mathrm{coker}(L(\hat{G})) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n},$$

where $(d_1, d_2, \dots, d_n)$ are *invariant factors* obtained from the Smith normal form of $L(\hat{G})$, satisfying $d_i \mid d_{i+1}$ for all $1 \leq i \leq n-1$. Consequently, we obtain the following representation of the forest group

$$\mathrm{F}(G) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}. \tag{3}$$

From a computational perspective, determining the Smith normal form of the discrete Laplacian $L(\hat{G})$ becomes challenging as the size of the graph grows, since the dimension of $L(\hat{G})$ increases with the number of vertices. However, the special algebraic structure of circulant graphs, encoded in their associated Laurent polynomials (2), provides a useful way to simplify the problem. The following Theorem 2 from [19] is formulated in a more general way for graphs G associated with arbitrary symmetric and bimonic Laurent polynomials.

Theorem 2. *Let $P(z)$ be a symmetric and bimonic Laurent polynomial with integer coefficients, and let $\mathcal{A}$ be the companion matrix associated with $P(z)$. Then*

$$\operatorname{coker}(\mathrm{L}(G)) \cong \operatorname{coker}(\mathcal{A}^n - I).$$

This result relies on a classical construction from the theory of polynomials, namely the *companion matrix*; see [22] for further details.

Theorem 2 allows us to replace the computation of the Smith normal form of the potentially large Laplacian matrix with the simpler computation of a fixed-size matrix $\mathcal{A}^n - I$, thus significantly reducing computational complexity.

3 Critical group of the cone over a sandwich graph

Let $A(G)$ denote the adjacency matrix of a sandwich graph G, and $D(X)$ denote the diagonal matrix with given independent variables $X = (x_v)$, indexed by the vertices. We define the matrix

$$L(G, X) = D(X) - A(G)$$

as the *generalized Laplacian matrix* of the graph G.

For convenience, let us denote the Laurent polynomials $L_i(z)$ defined for graphs (1) by

$$L_i = L_i(z) = P(z) + d_i + 1, \quad i = 1, \dots, m, \tag{4}$$

where $d_i = \deg(v_i)$ is the degree of vertex v_i in the path graph H.

Theorem 3. *Let $\hat{G}$ be the cone over the sandwich graph G with m circulant layers $G_1, \dots, G_m$, each on n vertices. Then the forest group $\mathrm{F}(G)$ is isomorphic to the cokernel of the linear operator $\mathcal{A}^n - I$, where $\mathcal{A}$ is the companion matrix of the symmetric and bimonic Laurent polynomial $D_m(z)$, which satisfies the recurrence relation*

$$D_m(z) = L_m(z) D_{m-1}(z) - D_{m-2}(z),$$

with initial conditions

$$D_{-1}(z) = 0, \quad D_0(z) = 1, \quad D_1(z) = L_1(z), \quad D_2(z) = L_1(z)L_2(z) - 1.$$

Proof. Let G be a sandwich graph. Consider its generalized Laplacian matrix

$$\mathrm{L}(G,X) = \begin{pmatrix} L_1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & L_2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & L_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & L_{m-1} & -1 \\ 0 & 0 & 0 & \cdots & -1 & L_m \end{pmatrix}, \tag{5}$$

where the entries $X = (L_i)$ are symmetric and bimonic Laurent polynomials with integer coefficients, satisfying $L_i(z) = L_i(\frac{1}{z})$.

As in the Introduction, let $\hat{G}$ denote the cone over G, and let $\mathrm{L}(G)$ denote the discrete Laplacian of its base graph. According to Theorem 1, the forest group of G is given by $\mathrm{F}(G) = \mathrm{coker}(\mathrm{I} + \mathrm{L}(G))$. Using the standard description of the cokernel as an abelian group of relations among n-periodic sequences, we obtain

$$\begin{aligned} &\langle x_i^1, x_i^2, \ldots, x_i^m \mid i \in \mathbb{Z} \mid \\ &\quad L_1 x_j^1 - x_j^2 = 0, \quad L_2 x_j^2 - x_j^1 - x_j^3 = 0, \quad \ldots, \quad L_m x_j^m - x_j^{m-1} = 0, \\ &\quad x_{j+n}^1 - x_j^1 = 0, \quad x_{j+n}^2 - x_j^2 = 0, \quad \ldots, \quad x_{j+n}^m - x_j^m = 0, \quad j \in \mathbb{Z}\rangle \end{aligned} \tag{6}$$

This is an abelian group generated by m infinite families of generators with $2m$ relations: m linear relations and m periodicity conditions.

By sequentially eliminating variables x_j^i for $i = 2, \ldots, m$, the system reduces to a single linear relation involving only the variables x_j^1. In particular, for $m = 2$ we have

$$(L_1 L_2 - 1) x_j^1 = 0,$$

and for $m = 3$, we obtain

$$(L_3 L_2 L_1 - L_3 - L_1) x_j^1 = 0.$$

In general, since each x_j^i is expressed linearly through x_j^{i-1} and x_j^{i+1}, the system collapses to a single linear recurrence relation

$$D_m x_j^1 = 0,$$

where the polynomial D_m satisfies the recurrence

$$D_m = L_m D_{m-1} - D_{m-2}, \tag{7}$$

with initial conditions

$$D_{-1} = 0, \; D_0 = 1, \; D_1 = L_1, \; D_2 = L_1 L_2 - 1. \tag{8}$$

Remark 1. *One can recognize that the recurrence relation (7) with initial conditions (8), coincides with the classical formula for the determinant of the tridiagonal matrix (5).*

Using the definition of $L_i(z)$ given in (4), it is straightforward to verify that the Laurent polynomial $D_m(z)$ satisfies the symmetry condition and is bimonic.

Since every x_j^i can be expressed through x_j^1, the periodicity condition for all variables is equivalent to the periodicity condition on x_j^1. Thus, the group (6) has the following presentation

$$\operatorname{coker}(I + L(H_n)) \cong \left\langle x_j^1 \mid D_m(z)\, x_j^1 = 0,\ x_{j+n}^1 - x_j^1 = 0,\ j \in \mathbb{Z} \right\rangle.$$

Applying Theorem 2, we immediately obtain the statement of the theorem. □

Proposition 1. *If the sandwich graph has at least one layer with zero jumps, i.e., if for some $i \in \{1, \dots, m\}$ we have $s_{i,1} = s_{i,2} = \cdots = s_{i,k_i} = 0$, then the polynomial $L_i(z)$ reduces to $L_i(z) = d_i + 1$. In this case, the Laurent polynomial $D_m(z)$ is not bimonic, and the statement of Theorem 3 no longer holds.*

Proof. This is because the isomorphism $\operatorname{coker}(L(\hat{G})) \cong \operatorname{coker}(\mathcal{A}^n - I)$ holds only when the associated Laurent polynomial is symmetric and bimonic, as follows from Theorem 2 (see [19]). □

Thus, the method based on Theorem 2 is limited to graphs that can be associated only with symmetric and bimonic Laurent polynomials.

4 Examples

The computational advantage provided by Theorem 3 can be stated as follows.

Remark 2. *Let G be a sandwich graph with $\nu = m \cdot n$ vertices, where m is the number of layers and n is the number of vertices in each circulant layer. Let $D_m(z)$ denote the symmetric and bimonic Laurent polynomial constructed recursively from the polynomials (2) associated with the layers, and let $\mathcal{A}$ be its companion matrix. Then the computation of the Smith normal form of the Laplacian matrix $\mathrm{L}(G) \in \mathbb{Z}^{\nu\times\nu}$ reduces to the computation of the Smith normal form of the matrix $\mathcal{A}^n - I$, whose size depends only on the parameters of the recurrence and remains independent of the number of vertices n.*

For clarity, we consider the special case of a sandwich graph G whose layers are identical and represented by simple cycles, i.e., $G_i = C_n(s_{i,1})$ with $s_{i,1} = 1$ for all $i = 1, \dots, m$. Here, m denotes the number of layers, and n is the number of vertices in each layer.

For example, if $m = 2$, then, by Theorem 3, the forest group $\mathrm{F}(G)$ is isomorphic to the cokernel of the linear operator $\mathcal{A}^n - I$, where $\mathcal{A}$ is the companion matrix corresponding to the Laurent polynomial $D_2(z) = 17 +$

$\frac{1}{z^2} - \frac{8}{z} - 8z + z^2$. Explicitly, the matrix $\mathcal{A}$ has the following form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 8 & -17 & 8 \end{pmatrix}. \tag{9}$$

The invariant factors were computed via Theorem 3 from the Smith normal form of $\mathcal{A}^n - I$, where $\mathcal{A}$ is defined in (9); see Table 1 for $n = 3, \dots, 10$ and $m = 2$, and Table 2 for $n = 3, \dots, 10$ and $m = 3, \dots, 6$. We do not list invariant factors equal to 1 in the tables, as they do not affect the resulting group structure.

Note for Table 1. Each cell in Table 1 lists vertically the invariant factors of the corresponding forest group (3). For clarity, each factor is written in its prime factorized form. For example, when $n = 3$ the invariant factors are $d_1 = 1, \quad d_2 = 1, \quad d_3 = 2^3 \cdot 3, \quad d_4 = 2^3 \cdot 3^2$. Hence, $\mathrm{F}(G) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_{72}$.

n / $m = 2$	3	4	5	6	7	8	9	10
	$2^3{\cdot}3$ $2^3{\cdot}3^2$	$3{\cdot}5$ $3{\cdot}5$ $3{\cdot}5{\cdot}7$	$11{\cdot}29$ $3{\cdot}11{\cdot}29$	2^3 2^3 $2^3{\cdot}3$ $2^3{\cdot}3^2{\cdot}5{\cdot}7$	$29{\cdot}139$ $3{\cdot}29{\cdot}139$	$3{\cdot}5{\cdot}7$ $3{\cdot}5{\cdot}7{\cdot}23$ $3{\cdot}5{\cdot}7{\cdot}23$	$2^3{\cdot}3^2{\cdot}19{\cdot}37$ $2^3{\cdot}3^3{\cdot}19{\cdot}37$	$5{\cdot}11{\cdot}19{\cdot}29$ $3{\cdot}5^2{\cdot}7{\cdot}11{\cdot}19{\cdot}29$

Table 1. Invariant factors of the forest group (prime factorized form) for a sandwich graph with two layers.

Additional numerical results are given in Table 2. This data was collected to investigate patterns arising when both the number of vertices per layer n and the total number of layers m are varied.

Note for Table 2. The structure of Table 2 is the same as in Table 1, except that here n indexes the rows and m indexes the columns.

$m \mapsto$ $n \downarrow$	3	4	5	6
3	2^2 $2^2\cdot5\cdot7$ $2^3\cdot5\cdot7$	$2^4\cdot3\cdot17$ $2^4\cdot3^2\cdot7\cdot17$	$2^2\cdot29\cdot41$ $2^2\cdot5\cdot11\cdot29\cdot41$	$2^3\cdot3$ $2^3\cdot3^2\cdot5\cdot7\cdot11$ $2^4\cdot3^3\cdot5\cdot7\cdot11$
4	2^2 $2^3\cdot3$ $2^3\cdot3^2$ $2^6\cdot3^2\cdot5$	$3\cdot5$ $3\cdot7\cdot23$ $3\cdot5\cdot7\cdot23\cdot47$	$3\cdot5\cdot11\cdot19\cdot29$ $3\cdot5^2\cdot11\cdot19\cdot29\cdot41$	$2^2\cdot3$ $2^4\cdot3^2\cdot5$ $2^4\cdot3^2\cdot5\cdot11$ $2^7\cdot3^2\cdot5\cdot7\cdot11\cdot23$
5	$11\cdot19\cdot41$ $2^3\cdot11\cdot19\cdot41$	$11\cdot29\cdot719$ $3\cdot7\cdot11\cdot29\cdot719$	$2^2\cdot3\cdot5$ $2^2\cdot3\cdot5^2\cdot11$ $2^2\cdot3\cdot5^2\cdot11\cdot31$ $2^2\cdot3\cdot5^2\cdot11\cdot31$	$11\cdot19\cdot29\cdot41\cdot661$ $2^4\cdot3^2\cdot11\cdot19\cdot29\cdot41\cdot661$
6	5 $2^2\cdot5$ $2^3\cdot5$ $2^3\cdot5\cdot7$ $2^3\cdot3\cdot5\cdot7$ $2^3\cdot3^2\cdot5\cdot7$	$2^4\cdot7$ $2^4\cdot7$ $2^4\cdot3\cdot7\cdot17$ $2^4\cdot3^2\cdot5\cdot7\cdot17\cdot47$	11 $11\cdot41$ $2^3\cdot5\cdot11\cdot19\cdot29\cdot41$ $2^3\cdot5^2\cdot11\cdot19\cdot29\cdot41$	2^3 $2^3\cdot3\cdot5$ $2^3\cdot3\cdot5$ $2^3\cdot3^2\cdot5\cdot7$ $2^3\cdot3^2\cdot5\cdot7\cdot11\cdot13$ $2^6\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot23$
7	$29\cdot71\cdot239$ $2^3\cdot29\cdot71\cdot239$	$7\cdot29\cdot139\cdot2113$ $3\cdot7^2\cdot29\cdot139\cdot2113$	$29\cdot71\cdot421\cdot8329$ $5\cdot11\cdot29\cdot71\cdot421\cdot8329$	$29\cdot71\cdot139\cdot239\cdot12781$ $2^4\cdot3^2\cdot29\cdot71\cdot139\cdot239\cdot12781$ $2^4\cdot3^3\cdot5\cdot7\cdot11$
8	$2^2\cdot7$ $2^4\cdot3\cdot7$ $2^4\cdot3^2\cdot7\cdot17$ $2^7\cdot3^2\cdot5\cdot7\cdot17$	5 5 5 $5\cdot7\cdot23$ $3\cdot5\cdot7\cdot23$ $3\cdot5\cdot7\cdot23\cdot47$	3 3 3 3 $3\cdot5\cdot7\cdot11\cdot19\cdot29\cdot31\cdot719$ $3\cdot5^2\cdot7\cdot11\cdot19\cdot29\cdot31\cdot41\cdot719$	2 2 2^2 $2^2\cdot7$ $2^3\cdot3\cdot7$ $2^5\cdot3^2\cdot5\cdot7\cdot23$ $2^5\cdot3^2\cdot5\cdot7\cdot11\cdot23\cdot47$ $2^8\cdot3^2\cdot5\cdot7\cdot11\cdot17\cdot23\cdot47$
9	2^2 $2^2\cdot5\cdot7\cdot19\cdot53\cdot199$ $2^3\cdot5\cdot7\cdot19\cdot53\cdot199$	$2^4\cdot3^2\cdot17\cdot19\cdot37\cdot8929$ $2^4\cdot3^3\cdot7\cdot17\cdot19\cdot37\cdot8929$	19 19 $2^2\cdot19\cdot29\cdot41\cdot109\cdot179\cdot251$ $2^2\cdot5\cdot11\cdot19\cdot29\cdot41\cdot109\cdot179\cdot251$	3 3 3 $2^3\cdot3^2$ $2^3\cdot3^3\cdot5\cdot7\cdot11\cdot19\cdot37\cdot53\cdot199\cdot829$ $2^4\cdot3^4\cdot5\cdot7\cdot11\cdot19\cdot37\cdot53\cdot199\cdot829$
10	11 11 $2^2\cdot5\cdot11\cdot19\cdot29\cdot41$ $2^5\cdot3\cdot5^2\cdot11\cdot19\cdot29\cdot41$	3 3 3 $3\cdot5\cdot7\cdot11\cdot19\cdot29\cdot31\cdot719$ $3\cdot5^2\cdot7\cdot11\cdot19\cdot29\cdot31\cdot41\cdot719$	5 $2\cdot5$ $2\cdot5^2$ $2^1\cdot5^2\cdot11$ $2^4\cdot3\cdot5^2\cdot11$ $2^4\cdot3\cdot5^2\cdot11$ $2^4\cdot3\cdot5^2\cdot11\cdot31$ $2^4\cdot3\cdot5^3\cdot11\cdot31\cdot41$	$11\cdot19\cdot29$ $11\cdot19\cdot29$ $2^3\cdot3\cdot5\cdot11\cdot19\cdot29\cdot41\cdot241\cdot661$ $2^6\cdot3^2\cdot5\cdot7\cdot11\cdot19\cdot29\cdot241\cdot661$

TABLE 2. Invariant factors of the forest group (prime factorized form) for a sandwich graph with $m = 3, 4, 5, 6$ layers.

Based on the data in these tables, we make the following observation.

Conjecture 1 (Lucas Divisibility)**.** *Let $p \geq 3$ be a prime and $m \geq 2$. For the sandwich graph G with m identical layers $C_p(1)$, and its cone $\hat{G}$, write*

$$\mathrm{F}(G) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_p} \quad (d_1 \mid \cdots \mid d_p).$$

Then the Lucas number *L_p divides the second largest invariant factor*

$$L_p \mid d_{p-1}.$$

Remark 3. *Numerical verification confirms the conjecture for all $2 \leq m \leq 6$ and $3 \leq p \leq 37$.*

However, the general pattern remains an open question.

Further research will be devoted to the study of the forest group for the cone over circulant foliation graphs [13, 14], which is a natural generalization of the sandwich graph considered in this work. It is planned to formulate corresponding restrictive conditions, similar to those in Proposition 1, and to apply the theory of voltage graphs (see [12]) for a more detailed structural analysis. Exact theoretical results on the invariant factors for such graphs will be obtained, taking into account the number of layers m and the number of vertices n in each layer. These results will make it possible to better understand the structure of the critical group for cones over graphs of arbitrary structure.

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