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NECESSARY CONDITION FOR ISOMORPHISM OF $GBS(n,1)$ GROUPS WITH NON-TRIVIAL CENTER

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Abstract: A finitely generated group G_n that acts on a tree T such that all edge stabilizers are infinite cyclic groups and all vertex stabilizers are free Abelian groups of rank n will be called a *generalized Baumslag–Solitar group of type (n,1)* ($GBS(n,1)$ group). In this paper we find a criterion for such groups to have a non-trivial center and prove that if $n \geqslant 3$ and two such groups with non-trivial center are isomorphic, then the corresponding $GBS(1,1)$ groups must also be isomorphic.

Keywords: generalized Baumslag–Solitar group, isomorphism problem, group with non-trivial center.

1 Introduction

By the Bass-Serre theorem, every $GBS(n,1)$ group G_n can be represented as $\pi_1(\mathbb{A})$ – the fundamental group of graph of groups $\mathbb{A}$ [1] all of whose edge groups are infinite cyclic, and all of whose vertex groups are free Abelian groups of rank n. The $GBS(1,1)$ group is called *generalized Baumslag–Solitar group (GBS group)*.

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The isomorphism problem for such groups can be posed as follows: to determine algorithmically when two given graph of groups define isomorphic groups. The isomorphism problem for $GBS(n,1)$ groups even for $n=1$ has been solved only in some special cases and remains a difficult open problem. A survey of results on this problem can be found, for example, in [2].

The fundamental group $\pi_1(\mathbb{A})$ of graph of groups can be defined by generating set and defining relations (see, for example, [1] or [3]). If v is a vertex of A, then we denote generators of the corresponding vertex group by $v_1, v_2, \ldots, v_n$. Each edge e of graph A connecting vertices u and v defines elements u_e and v_e – generators of images of embeddings of the corresponding edge group into endpoint vertex groups of edge e. Let $\overline{A}$ denote the graph obtained from A by identifying e and $\overline{e}$. The maximal subtree R of the graph $\overline{A}$ defines a representation of the group $\pi_1(\mathbb{A})$

$$\left\langle \begin{array}{l} v_1, \ldots, v_n, v \in V(\overline{A}), \\ t_e, e \in E(\overline{A}) \setminus E(R) \\ \end{array} \;\middle|\; \begin{array}{l} v_e = u_e, e \in E(R), \\ v_i v_j = v_j v_i, 1 \leqslant i \neq j \leqslant n, v \in V(\overline{A}), \\ t_e^{-1} v_e t_e = u_e, e \in E(\overline{A}) \setminus E(R) \end{array} \right\rangle .$$

An element of the group $G_n \cong \pi_1(\mathbb{A})$ is called *elliptic* if it stabilizes a vertex of the tree T, otherwise it is called *hyperbolic*. Elements $v_1^{k_1} \cdot v_2^{k_2} \cdot \ldots \cdot v_n^{k_n}, k_1, k_2, \ldots, k_n \in \mathbb{Z}, v \in V(A)$ are elliptic. Elements $t_e, e \in E(\overline{A}) \setminus E(R)$ are hyperbolic. Each elliptic element of the group G_n is conjugate to a suitable vertex group.

Let t be a hyperbolic generator of the group G_n. If there exists an non-trivial elliptic element g and non-zero integers k, l such that $t^{-1} g^k t = g^l$, then we define $\Delta(t) = \frac{k}{l}$. It is not difficult to show that such a definition is correct. We prove a criterion for $GBS(n,1)$ groups to have a non-trivial center.

Lemma 1. *$GBS(n,1)$ group has a non-trivial center if and only if all edge groups intersect non-trivially and $\Delta(t) = 1$ for every hyperbolic generator t.*

Remark 1. *If the $GBS(n,1)$ group G_n has a non-trivial center, then one can choose vertex generators $v_1, v_2, \ldots, v_n$ for every vertex v so that the edge subgroups are generated by powers of $v_1, v \in V(A)$. In this case, the subgroup G of the group G_n generated by $v_1, v \in V(A)$ and hyperbolic generators is a GBS group and the centers of G and G_n coincide. We will call such a subgroup G of the group G_n the corresponding GBS group.*

Main result is a necessary condition for the isomorphism of $GBS(n,1)$ groups with non-trivial center.

Theorem 1. *Let G_n and H_n be two $GBS(n,1)$ groups with non-trivial center and $n \geqslant 3$. If G_n and H_n are isomorphic, then their corresponding GBS groups G and H are also isomorphic.*

It is known [4] that the isomorphism problem is algorithmically decidable for GBS groups with non-trivial center.

2 Auxiliary statements

Proof. Let us prove Lemma 1. For $n = 1$ lemma is true [5, prop. 2.5]. Let $n \geqslant 2$. If $1 \neq z \in Z(G_n)$ and G_n acts on the tree T, then for every vertex $v \in V(T)$ and for every $a \in Stab_{G_n}(v)$ we have

$$a(zv) = z(av) = zv.$$

Therefore, $Stab_{G_n}(v) \subseteq Stab_{G_n}(zv)$. Thus, if we choose in the stabilizer of a vertex v an element a that stabilizes only this vertex v, then we get that $zv = v$. Note that the existence of such an element follows from the fact that the intersection of the vertex stabilizers is either trivial or cyclic, and the edge stabilizers are isomorphic to $\mathbb{Z}^n, n \geqslant 2$. This shows that z stabilizes all vertices of T and hence lies in every vertex group. Therefore, all vertex groups intersect non-trivially and edge groups intersect non-trivially, the center lies in this intersection.

Let t be a hyperbolic generator. Since every non-trivial element z of the center lies in the intersection of edge groups, then $t^{-1}zt = z$. Therefore $\Delta(t) = 1$.

Let us prove the converse. Let $1 \neq z$ lie in the intersection of all edge groups, then z commutes with all vertex elements. Moreover, $t^{-1}zt = z$ for all hyperbolic generators due to the fact that $\Delta(t) = 1$ and z lies in the corresponding edge groups. Hence, z lies in the center of G. $\square$

Integer powers of generators $v_1, v \in V(A)$ of the group G_n mentioned in Remark 1 will be called elements of I *type*. Non-trivial elements from $\langle v_2, \ldots, v_n\rangle, v \in V(A)$ will be called elements of II *type*. Finally, non-zero integer powers of hyperbolic generators will be called elements of III *type*. The sets of such elements will be denoted by I_A, II_A and III_A respectively.

Next we will study $GBS(n,1)$ groups with non-trivial center. Remark 1 allows us to represent the graph of groups $\mathbb{A}$ using a labeled graph $\mathbb{A} = (A, \lambda)$, where A is a graph and $\lambda\colon E(A) \to \mathbb{Z} \setminus \{0\}$ are labels on edges. If v is the origin of an edge e, then $v_e = v_1^{\lambda(e)}$. We will write that the edge e is a (λ, μ)-edge, understanding that the labels on the geometric edge $\{e, \overline{e}\}$ are equal to λ and μ.

Figure 1 shows *slide* – a transformation of a labeled graph. Slide does not change the fundamental group of a labeled graph.

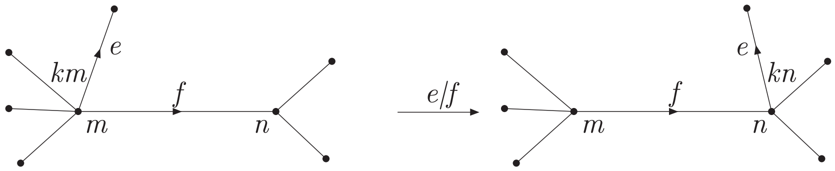

FIG. 1. Slide e/f.

Remark 2. *If a* $GBS(n,1)$ *group* G_n *has a non-trivial center and is represented by a labeled graph* $\mathbb{A}$*, then using slides one can obtain a labeled graph* $\widehat{\mathbb{A}}$ *such that* $G_n \cong \pi_1(\mathbb{A}) \cong \pi_1(\widehat{\mathbb{A}})$ *and the label 1 on an edge occurs in* $\widehat{\mathbb{A}}$ *only in the following cases:*

1. *near an isolated vertex,*
2. *on a (1,1)-edge, one of whose vertices is isolated,*
3. *on a (1,1)-loop.*

The set of isolated vertices with labels 1 of the labeled graph $\widehat{\mathbb{A}}$ will be denoted by $V_{out}(\widehat{\mathbb{A}})$, and the rest by $V_{inn}(\widehat{\mathbb{A}})$.

Remark 3. *Given a generator* v_1 *of* I *type. It is impossible to extract a root from* v_1 *in the group* $\pi_1(\widehat{\mathbb{A}})$ *if and only if* $v \in V_{inn}(\widehat{\mathbb{A}})$ *or* v *is an isolated endpoint of (1,1)-edge.*

Proof. Suppose that v_1 does not have a root and $v \in V_{out}(\widehat{\mathbb{A}})$ is an isolated endpoint of $(1,k)$-edge, $k \neq 1$. Then there exists a vertex $u \in V_{inn}(\widehat{\mathbb{A}})$ such that $v_1 = u_1^k$, contradiction.

The converse. Let $v \in V_{inn}(\widehat{\mathbb{A}})$ or v be an isolated endpoint of (1,1)-edge and there exist $g \in G$ and $k \neq 1$ such that $g^k = v_1$. Then g is an elliptic element and since a power of II type element cannot become an I type element, g is conjugate to a I type element.

Then there exists $h \in G, \alpha \in \mathbb{Z}\setminus\{0\}$ and $u \in V(\widehat{\mathbb{A}})$ such that $g = h^{-1}u_1^{\alpha}h$. Therefore, $v_1 = h^{-1}u_1^{k\alpha}h$. If $u = v$, then $\alpha k = 1$. This contradicts the choice of k.

Therefore $u \neq v$ and v_1 lies in some edge subgroup with $h^{-1}u_1^{k\alpha}h$. Since $v \in V_{inn}(\widehat{\mathbb{A}})$, v_1 can only lie in the edge subgroup of an $(1,1)$-edge with one isolated vertex or $(1,1)$-loop. In particular, this means that $|\alpha k| = 1$. $\square$

3 Main results

Lemma 2. *Let* G_n *and* H_n *be two* $GBS(n,1)$ *groups with non-trivial center, represented by labeled graphs* $\widehat{\mathbb{A}}$ *and* $\widehat{\mathbb{B}}$*, respectively. If* $n \geqslant 3$ *and* $\varphi\colon G_n \to H_n$ *is an isomorphism, then the images of* I *type elements are conjugate to* I *type elements of* H_n*. The isomorphism induces a bijection of the sets* $V_{inn}(\widehat{\mathbb{A}})$ *and* $V_{inn}(\widehat{\mathbb{B}})$*.*

Proof. Under isomorphism, the center $\langle z_G\rangle$ of G_n is mapped to the center $\langle z_H\rangle$ of H_n. Without loss of generality, we can assume that $\varphi\colon z_G \mapsto z_H$. Denote by

$$SQ(z_G) = \{g \in G_n \mid \exists k \in \mathbb{N} : g^k = z_G\},$$

we similarly define $SQ(z_H)$. Then the restriction of φ to $SQ(z_G)$ is a bijection of $SQ(z_G)$ and $SQ(z_H)$.

Since the power of a hyperbolic element is a hyperbolic element, we obtain that $SQ(z_G)$ and $SQ(z_H)$ consist of elliptic elements. Furthermore, every I

type generator v_1 of G lies in $SQ(z_G)$ and every g in $SQ(z_G)$ is conjugate to the power of a suitable v_1.

Note that it is impossible to extract a root from $v_1, v \in V_{inn}(\widehat{\mathbb{A}})$ in G_n by Remark 3, and the isomorphism respects this property. Let $CESQ(z_G)$ be the set of such classes of conjugate elements from $SQ(z_G)$, that it is impossible to extract a root from any representative. The group isomorphism induces a bijection on conjugacy classes. Therefore, φ induces a bijection of the sets $CESQ(z_G)$ and $CESQ(z_H)$.

Finally we see that there is a bijection of the set $V_{inn}(\widehat{\mathbb{A}})$ and $CESQ(z_G)$. For this, we establish a mapping

$$\zeta\colon V_{inn}(\widehat{\mathbb{A}}) \to CESQ(z_G)$$

by the rule

$$\zeta\colon v \mapsto v_1^G.$$

If $v \neq u \in V_{inn}(\widehat{\mathbb{A}})$, then the corresponding vertices of T lie in different orbits of the action of G. Consequently, v_1 and u_1 cannot be conjugate. Hence, ζ is an injection, and from Remark 3 it follows that ζ is a surjection. □

Lemma 3. *Let G_n and H_n be two $GBS(n,1)$ groups with non-trivial center and $n \geqslant 3$. If $\varphi\colon G_n \to H_n$ is an isomorphism, then the images of II type elements of G_n are conjugate to II type elements of H_n and the images of III type elements of G_n do not lie in the normal closure of the elliptic elements of H_n.*

Proof. Let g be a hyperbolic element of G_n with respect to its action on the tree T. If $h \in C_{G_n}(g)$ and $\overrightarrow{g}$ is the axis of g [3], then

$$g(h\overrightarrow{g}) = h(g\overrightarrow{g}) = h\overrightarrow{g},$$

therefore $h\overrightarrow{g} \subseteq \overrightarrow{g}$. Applying the same arguments to $h^{-1} \in C_{G_n}(g)$ yields $h^{-1}\overrightarrow{g} \subseteq \overrightarrow{g}$. Consequently, $h\overrightarrow{g} = \overrightarrow{g}$. This means that there exist integers k and l such that the actions of h^k and g^l on $\overrightarrow{g}$ coincide. Therefore

$$h^k g^{-l} \in Stab_{G_n}\overrightarrow{g} = \bigcap_{v \in V(\overrightarrow{g})} Stab_{g_n} v \subseteq \langle g_e \rangle,$$

where g_e is the generator of the stabilizer of some edge of the tree $\overrightarrow{g}$. Therefore $h^k = a_h \cdot g^l$ for a suitable $a_h \in \langle g_e \rangle$. Since $h \in C_{G_n}(g)$, so does a_h. It follows that $h^{km} = a_h^m \cdot g^{lm}$ for every integer m.

Since $a_h \in \langle g_e \rangle$, there exists an integer power s such that $a_h^s \in \langle z_{G_n} \rangle$. Thus, we have shown that every element h in the centralizer g of a hyperbolic element in the group G_n has the following property: there exists a power r such that $h^r \in \langle g \rangle \times \langle z_G \rangle$ and the group $\langle g \rangle \times \langle z_G \rangle$ is a free Abelian group of rank 2.

If a is a II type element, then $\mathbb{Z}^n \leq C_{G_n}(a)$ and since $n \geqslant 3$, it is clear that there is h in the centralizer of a such that no power of h lies in $\langle a \rangle \times \langle z_G \rangle$. This means that the image of a II type element cannot be hyperbolic and cannot be conjugate to a I type element of H_n due to the injectivity of φ

and Lemma 2. This means that the image of a II type element is conjugate to a II type element. Consequently, the normal closure of elliptic elements under isomorphism goes over to the normal closure of elliptic elements. □

Proof. Proof of Theorem 1. Let G_n and H_n be represented by labeled graphs $\widehat{\mathbb{A}}$ and $\widehat{\mathbb{B}}$. Denote the normal closures of II type elements in groups G_n and H_n by N_A and N_B, respectively. Then

$$G \cong G_n/N_A, H \cong H_n/N_B.$$

We define homomorphisms ψ_A and ψ_B from G_n to G and from H_n to H, respectively, as canonical homomorphisms with kernels N_A and N_B. We prove that the restriction of $\psi_B \circ \phi$ to G is the desired isomorphism.

Injectivity. Let $g \in G$ and $\phi(g) \in Ker\psi_B = N_B$. Then $g = \phi^{-1}(\phi(g)) \in N_A$ by Lemma 3, but N_A intersects G only at 1.

Surjectivity. Let $w_1 \in I_B$. Then for some $g \in G_n$ and $u_1 \in I_A$ we have $\phi(u_1^g) = w_1$. Since u_1^g and $u_1^{\psi_A(g)}$ are conjugate,

$$\phi(u_1^{\psi_A(g)}) = w_1^s$$

for a suitable $s \in H_n$. Note that

$$\psi_A(u_1^g) = \psi_A(u_1^{\psi_A(g)}).$$

Therefore $u_1^g \cdot (u_1^{\psi_A(g)})^{-1} \in N_A = Ker\psi_A$. Consequently,

$$\phi(u_1^g \cdot (u_1^{\psi_A(g)})^{-1}) = w_1 \cdot (w_1^s)^{-1} \in N_B = Ker\psi_B.$$

Whence it follows that $\psi_B(w_1^s) = \psi_B(w_1) = w_1$. Hence, $u_1^{\psi_A(g)} \in G$ and

$$\psi_B(\phi(u_1^{\psi_A(g)})) = \psi_B(w_1^s) = w_1.$$

Therefore, $\langle I_B \rangle$ is contained in the image of the restriction of $\psi_B \circ \phi$ to G.

Similarly for III_B. Let $t \in III_B$. Then for some $g \in \langle\langle I_A, II_A \rangle\rangle$ and $1 \neq r \in \langle III_A \rangle$ we have $\phi(rg) = t$. Then

$$\phi(r\psi_A(g)) = \phi(rg \cdot g^{-1}\psi_A(g)) = th$$

for suitable $h \in \langle\langle I_B, II_B \rangle\rangle$. Note that

$$\psi_A(rg) = \psi_A(r\psi_A(g)).$$

Therefore, $rg(r\psi_A(g))^{-1} \in N_A = Ker\psi_A$. Consequently,

$$\phi(rg(r\psi_A(g))^{-1}) = t(th)^{-1} \in N_B = Ker\psi_B.$$

Whence it follows that $\psi_B(th) = \psi_B(t) = t$. Hence, $r\psi_A(g) \in G$ and

$$\psi_B(\phi(r\psi_A(g))) = \psi_B(th) = t.$$

Therefore, $\langle III_B \rangle$ is contained in the image of the restriction of $\psi_B \circ \phi$ to G. □

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