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ITALIAN DOMINATION NUMBER OF SOME SNARK FAMILY

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Abstract: For a graph $G = (V, E)$, an Italian dominating function (IDF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either v is adjacent to a vertex assigned 2 under f, or v is adjacent to at least two vertices assigned 1 under f. The weight of an Italian dominating function is the $\sum_{v \in V} f(v)$, and the minimum weight of a Italian dominating function f is the Italian domination number. Determining the Italian domination number of a graph is a well discussed NP-complete problem. Finding an exact value of Italian domination number is NP-complete even for bipartite graphs. In this work we identifies the Italian domination number of some families of snarks such as flower snarks, Loupekine snarks, Goldberg snarks, Watkin snark and Szekeres snark.

Keywords: Domination, Italian domination number, Roman domination, Regular graphs, Snark.

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1 Introduction

All graphs studied here are simple, finite and undirected. For additional notations and terminologies not given here we refer to [1, 12, 27]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ of order $n = |V(G)|$ and edge set $E(G)$. The open neighborhood $N(v)$ of a vertex v consists of the vertices adjacent to v, while the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of v ($deg(v)$) is the cardinality of its open neighborhood. The maximum degree of G is the number $\Delta(G) = \max\{deg(v) : v \in V(G)\}$. A graph in which all of its vertices are of degree three is called a cubic graph. A dominating set is a set $D \subseteq V(G)$ such that any vertex of G is in D or has a neighbor in D. The minimum cardinality of a domination set of G is called the domination number of G, denoted by $\gamma(G)$.
The domination of graphs has been extensively researched in the literature [12, 13, 14], resulting in different variations and generalizations. One of these variations, with historical significance, came to be called Roman domination. Stewart [21] introduced the concept of Roman domination on graphs, which was further developed by Cockayne et al. [6]. This concept was motivated by the military problem faced by Emperor Constantine the Great in the fourth century A.D., when the Roman Empire was under attack. The Italian domination is another significant variant of Roman domination that was first put forward by Chellali et al. [4] as Roman $\{2\}$- domination. Based on the defense plan of Roman empire, every location that lacks a legion must have a surrounding locations with two legion or at least two neighboring locations that each carry one legion. Formally, Italian-dominating function $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$ either there is a vertex $u \in N(v)$ with $f(u) = 2$, or at least two vertices $u, w \in N(v)$ with $f(u) = f(w) = 1$. The weight of an Italian dominating function is the $\sum_{v \in V} f(v)$ and the minimum weight of an Italian dominating function f is the Italian domination number, denoted by $\gamma_I(G)$. Figure 1 illustrates an Italian dominating function assigned for the Petersen graph.

Italian domination number was connected with other domination parameters by Chellali et al. [4] and also showed that Italian domination is NP-complete for bipartite graphs. Henning et al. [15] characterize the trees T with $\gamma(T) + 1 = \gamma_I(T)$ and $\gamma_I(T) = 2\gamma(T)$ in 2017. Subsequently, in 2021 various results based on binary operations in Italian domination are presented, particularly for Cartesian products and rooted products [8, 16]. Graphs with restricted maximum degree are frequently taken into consideration in the study of most graph parameters. The Italian domination number has been calculated for connected graphs with a maximum degree of two, which are either paths or cycles [4]. The next stage is to explore graphs with a maximum degree of three. So far, the Italian domination number for a cubic graph has only been found for generalized Petersen graphs [9]. Being snarks are cubic graphs and other forms of domination have been researched, no one has explored the Italian domination number of this particular graph classes. This motivated

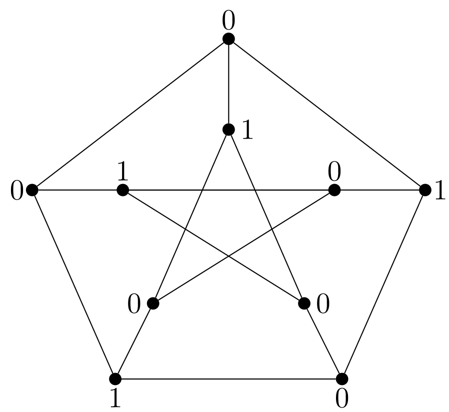

FIG. 1. IDF assigned on Petersen graph with $w(f) = 4$.

us to explore the Italian domination number of snark families. In 1880, Tait demonstrated that the Four-Color Theorem is identical to the statement that every bridgeless cubic planar graph has a 3-edge coloring [23], beginning off the study of snarks. Snarks are non-Hamiltonian and remain interesting in graph theory because they serve as minimal counter examples to certain well-known conjectures.
This study discusses the Italian domination over certain members of the snark family. In Section 2, we present some preliminary findings. In Section 3, we describe our key findings on the Italian domination number of snark families such as flower snarks, Goldberg snarks, Loupekine snarks, Watkin snark, and Szekeres snark.

2 Preliminary results

In this section, we illustrate some basic bounds and theorems on Italian domination number of a graph.

Theorem 1. [4] *If G is a connected graph of order n and maximum degree $\Delta(G) = \Delta$, then $\gamma_I(G) \geq \frac{2n}{(\Delta+2)}$.*

Theorem 2. [4] *For every graph $G, \gamma(G) \leq \gamma_I(G) \leq 2\gamma(G)$.*

Theorem 3. [4] *For the classes of paths P_n and cycles C_n, $\gamma_I(P_n) = \lceil \frac{n+1}{2} \rceil$ and $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$.*

Proposition 1. [24] *If G is a graph of order n, then $\gamma_I(G) = n$ if and only if $\Delta(G) \leq 1$.*

Proposition 2. [24] *If G is a graph of order $n \geq 2$, then $\gamma_I(G) = 2$ if and only if $\Delta(G) = n - 1$ or there exist two different vertices u and v such that $N(u) \cap N(v) = V(G) \backslash \{u, v\}$.*

3 Snark

The study of snark graphs were initiated in early 1880's from a classical problem in graph theory namely the Four-Colour theorem. The equivalence of Four-Colour theorem with the fact that every bridgeless cubic graph is 3-colourable highlights the importance of the family of snark graphs. The Petersen graph is considered as the smallest snark graph. In 1975 Isaacs found an infinite family of snarks namely Flower snarks [17]. The Szekeres snark was the other prominent one, discovered by George Szekeres in 1973 [22]. Further in 1976, F. Loupekine proposed a construction of two infinite families of snarks, using subgraphs of other known snarks. Although developed by Loupekine, this method was originally published by R. Isaacs [18]. In 1981, Goldberg added his contribution to the infinite family of snarks which is named after him as Goldberg snark [10]. Another well known snark on 50 vertices is the Watkins snark discovered by John J. Watkins in 1989 [25]. The articles [2, 5, 7, 11, 19, 20] provide further details on the definition and properties of the snark family. Driven by its physical importance and motivated by the interesting counter examples on snark graphs available in the literature, we have obtained the Italian domination number of some families of snark.

To make this work easier to understand, let $f(v_1, v_2, v_3, \cdots, v_n) = (1, 0, 1, \cdots, 1)$ denotes the Italian dominating function f assigns $1, 0, 1, \ldots, 1$ to the vertices $v_1, v_2, v_3, \cdots, v_n$ respectively.

3.1. Flower Snarks. The flower snarks F_n on $4n$ vertices, where n is odd can be constructed with the following steps. First consider n copies of the star graph on 4 vertices. Denote the central vertex of each star A_i and the outer vertices as B_i, C_i and D_i, resulting a disconnected graph on $4n$ vertices with $3n$ edges A_iB_i, A_iC_i and A_iD_i for $1 \leq i \leq n$. Then construct the n-cycle $(B_1 \cdots B_n)$ which adds n edges. Finally construct the $2n$-cycle $(C_1 ... C_n D_1 ... D_n)$ which adds $2n$ edges. Figure 2 shows the flower snark F_5 constructed using 5 copies of star graph.

Theorem 4. *Let F_n be a flower snark, where n is odd. Then $\gamma_I(F_n) = 2n-1$.*

Proof. Let F_n be a flower snark with $4n$ vertices, where n is odd, which can be constructed as described above. Consider an IDF, $f : V \to \{0, 1, 2\}$ defined as $f(B_1, B_2, B_3, \ldots, B_n) = (0, 1, 0, \ldots, 0)$, $f(C_1, C_2, \ldots, C_n, D_1, D_2, \ldots, D_n) = (0, 1, 0, \ldots, 0, 0, 1, 0, \ldots, 0)$ and $f(A_1, A_2, A_3, \ldots, A_n) = (1, 0, 1, \ldots, 1)$. Then $w(f) = \lfloor \frac{n}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = 2n-1$, n is odd. This is the IDF that provides the minimal weight, because when considering the cycle of length n by Theorem 3, $\gamma_I(C_n) = \lceil \frac{n}{2} \rceil$, however here the IDF allocates 1 to just $\lfloor \frac{n}{2} \rfloor$ vertices. Furthermore, F_n is a 3-regular, with all vertices assigned 1 being adjacent to only the vertices assigned 0. Hence, $\gamma_I(F_n) = 2n - 1$. □

3.2. Goldberg Snarks. Goldberg snark G_n are recursive structures generated by the basic block graph B_n. The vertex and edge set of B_n is defined

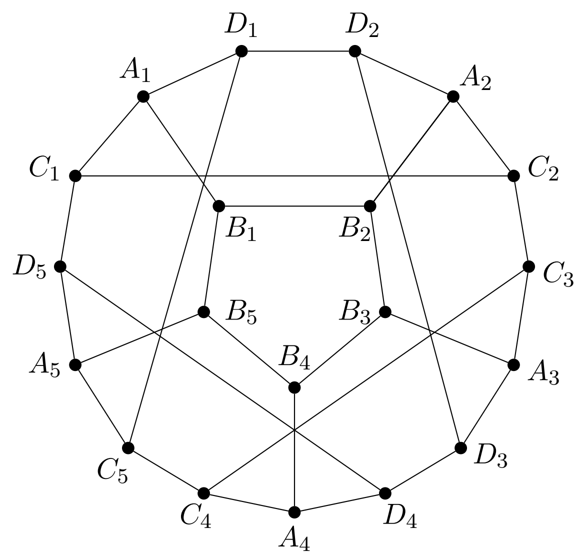

FIG. 2. Flower snark F_5.

as $V(B_n) = \{a_n, b_{n0}, b_{n1}, c_{n0}, c_{n1}, u_n, v_n, w_n\}$, $E(B_n) = \{a_nv_n, v_nw_n, v_nu_n, u_nb_{n0}, b_{n0}b_{n1}, b_{n1}w_n, w_nc_{n0}, u_nc_{n1}, c_{n0}c_{n1}\}$. The graph in Figure 3(a) is the basic block graph B_n. For n odd, $n \geq 3$ graph G_n is obtained from n copies of B_n by adding a set of link edges E_{nj}, where $E_{nj} = \{c_{n1}c_{j0}, b_{n1}b_{j0}, a_na_j\}, j = n+1$. The Goldberg snark G_3 shown in Figure 3(b) is obtained from adding link edges on basic block graphs B_1, B_2 and B_3.

Theorem 5. *Let G_n be a Goldberg snark, where n is odd. Then $\gamma_I(G_n) = 3n + \lfloor \frac{n}{2} \rfloor$.*

Proof. Let G_n be a Goldberg snark and f be an IDF on $V(G_n)$. In the graph, $B = \{b_{10}, b_{11}, b_{20}, b_{21}, \ldots, b_{n0}, b_{n1}\}$ and $C = \{c_{10}, c_{11}, c_{20}, c_{21}, \ldots, c_{n0}, c_{n1}\}$ forms a $2n$-cycle. So by the Theorem 3, f can assign 0 and 1 alternatively to the vertices in B and C. In the following step, $f(u_i) = 0$, $f(w_i) = 0$, and $f(v_i) = 1$, for $1 \leq i \leq n$. Note that so far all the vertices labeled 1 are adjacent to only the vertices labeled 0, through which it is not possible to reduce the weight $w(f) = 3n$ further. To meet the IDF condition, we must label the remaining vertices $A = \{a_1, a_2, \ldots, a_n\}$ that form a n-cycle with 0 and 1 in alternate order. Thus the Italian dominating function defined above is the function that offers the least weight because each vertex labeled 1 is adjacent to only at most one vertex labeled 1. Hence $\gamma_I(G_n) = 3n + \lfloor \frac{n}{2} \rfloor$. □

3.3. Loupekine snarks. LP_1-snarks are an infinite family of Loupekine snarks whose construction is presented below. Let k be an odd positive integer. A k-LP_1-snark is constructed from $k \geq 3$ sub graphs called blocks B_i for $1 \leq i \leq k$, obtained from the Petersen graph. Figure 4 (a) illustrates an arbitrary block B_i with its vertices named x_i, u_i, w_i, v_i, y_i, commonly

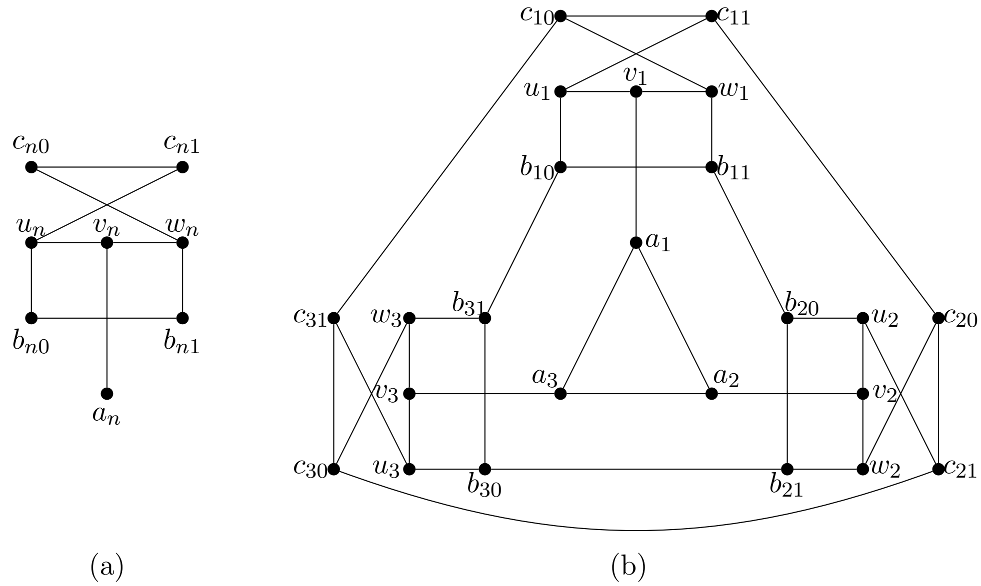

FIG. 3. (a) Basic block B_n. (b) Goldberg snark G_3 constructed from B_1, B_2 and B_3.

called border vertices. For all $i \in \{1, \ldots, k\}$, the border vertices v_i and y_i of block B_i are linked to the border vertices u_{i+1} and x_{i+1} of block B_{i+1} (indices taken modulo k) by edges called linking edges. The linking edges can be $\{v_i x_{i+1}, y_i u_{i+1}\}$ or $\{v_i u_{i+1}, y_i x_{i+1}\}$, but not both. The resulting graph, denoted by G_B, is the block subgraph of k-LP_1-snark. Figure 5 with out dotted edges shows an example of G_B constructed using the linking edges $\{v_i u_{i+1}, y_i x_{i+1}\}$.
Notice that G_B has exactly k vertices with degree two : $w_1, \ldots w_k$. Let G_C be a central subgraph of k-LP_1-snark with exactly k vertices of degree one, namely $z_1, \cdots, z_k$, and all the other vertices of degree three such that each connected component of G_C is isomorphic to one of the graphs K_2 and S_3, with K_2 being a complete graph with two vertices, and S_3 a star with three vertices of degree one (see Figure 4 (b)). For each i, with $1 \leq i \leq k$, identify vertices w_i and z_i, thus concluding the construction of a k-LP_1-snark.

Theorem 6. *Let k-LP_1 be a Loupekine Snark for $k \geq 3$ and k is odd. Then $\gamma_I(k\text{-}LP_1) = 3k$.*

Proof. Let k-LP_1 be a Loupekine Snark with $7k + 1$ vertices, developed from subgraph G_B with linking edges $\{v_i u_{i+1}, y_i x_{i+1}\}$ and central graph G_C as defined in the construction. Consider the subgraph G_B and an Italian dominating function f on $V(G_B)$. Since $X = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$ and $U = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ form a two disjoint $2k$ cycle, by Theorem 3, f can assign 0 and 1 alternatively to the vertices both in X and Y. For

FIG. 4. (a) Basic Block B_i (b) A graph G_c with seven pendant vertices.

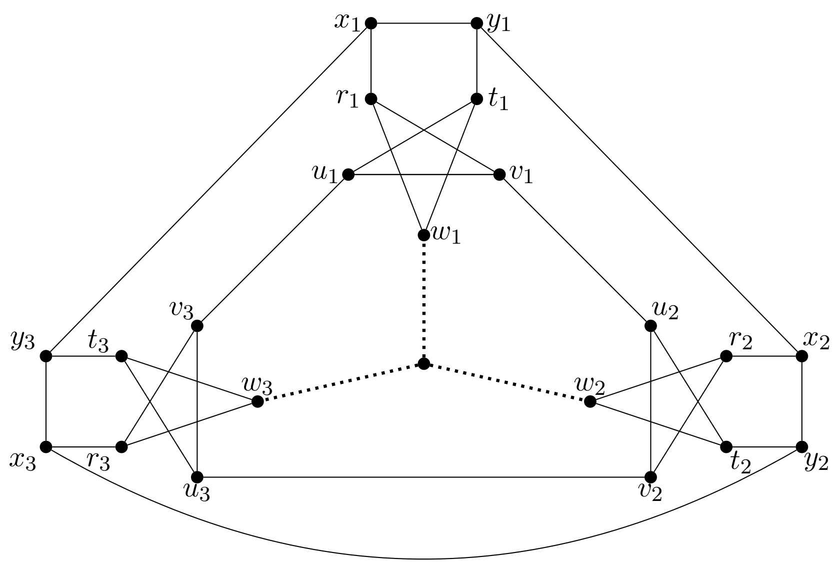

FIG. 5. Loupekine snark 3-LP_1.

all $1 \leq i \leq k$, both vertices r_i and t_i are adjacent to w_i, and just one vertex assigned 1 in cycles. When f assigns 1 to w_i and 0 to r_i and t_i, for all $1 \leq i \leq k$, all vertices assigned 0 are adjacent to exactly two vertices assigned 1. Hence, the total weight $w(f) = 3k$ is minimal for the subgraph G_B. Finally to form k-LP_1 Loupekine Snark the vertex w_i is identified with z_i, for all $1 \leq i \leq k$. Then the IDF f defined above can be extended to k-LP_1 Loupekine Snark, by assigning 0 to the central vertices of star mentioned in G_c, since as k is odd, at least one set of any three w_i is adjacent to the central vertex of star. Hence minimum weight of an Italian dominating function on k-LP_1 Loupekine Snark is $3k$, where $k \geq 3$ and k is odd. □

3.4. Watkin Snark. Watkins snark is a snark on 50 vertices and 75 edges constructed using the 5 copies of basic block shown in Figure 6. Watkin snark constructed from basic block graph B_i, for $1 \leq i \leq 5$ is illustrated in Figure 7.

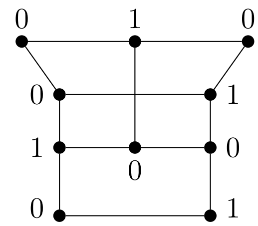

FIG. 6. An IDF assigned on basic block of Watkin.

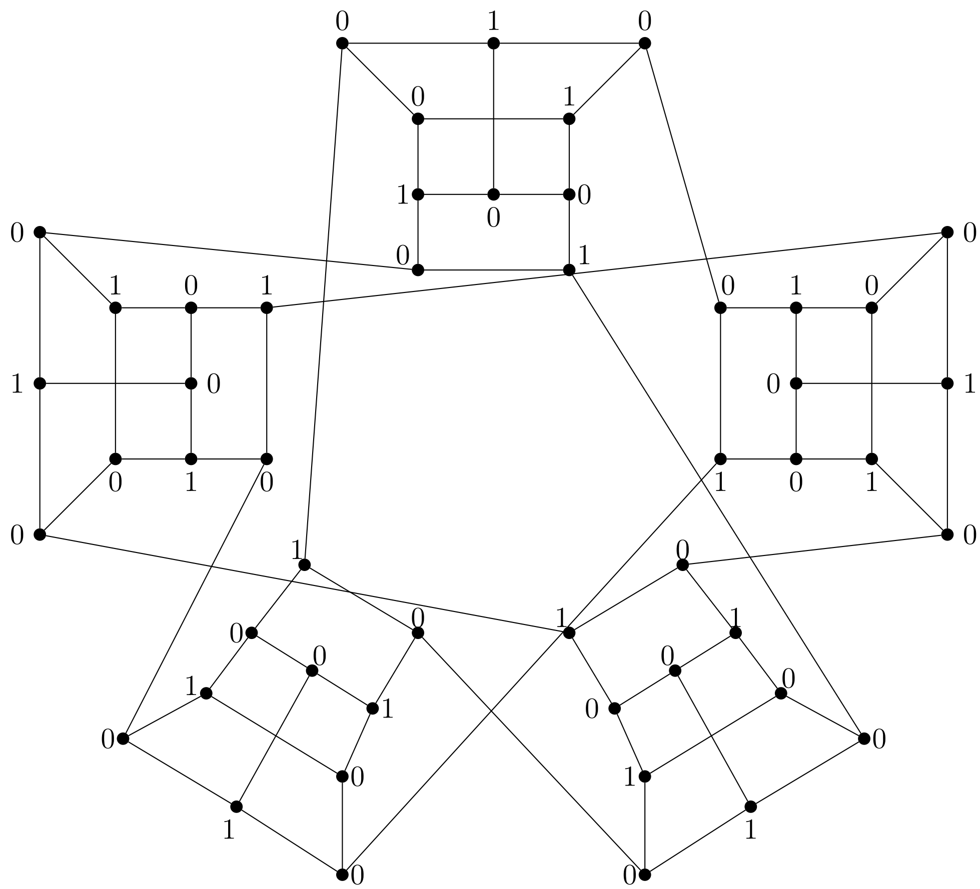

FIG. 7. Watkins Snark

Theorem 7. *Let W be the Watkin snark. Then $\gamma_I(W) = 20$.*

Proof. Let W be the Watkin snark with 50 vertices and B be the basic block used to construct the Watkin snark. Let f be an Italian domination function assigned for the vertices of each block B_i, for $1 \leq i \leq 5$, as shown in Figure 6. While examining each block, the IDF condition fails for only one two-degree vertex labeled 0. However, they are connected to the vertex assigned 1 during the formation of Watkin snark (see Figure 7). Therefore, $\gamma_I(W) \leq 5w(f) = 20$. According to Theorem 1, $\gamma_I(G) \geq \frac{2n}{\Delta+2} = \frac{100}{5} = 20$. Hence $\gamma_I(G) = 20$. □

3.5. Szekeres Snark. Szekeres snark is a snark with 50 vertices and 75 edges. Figure 8 shows an IDF assigned on the Szekeres graph.

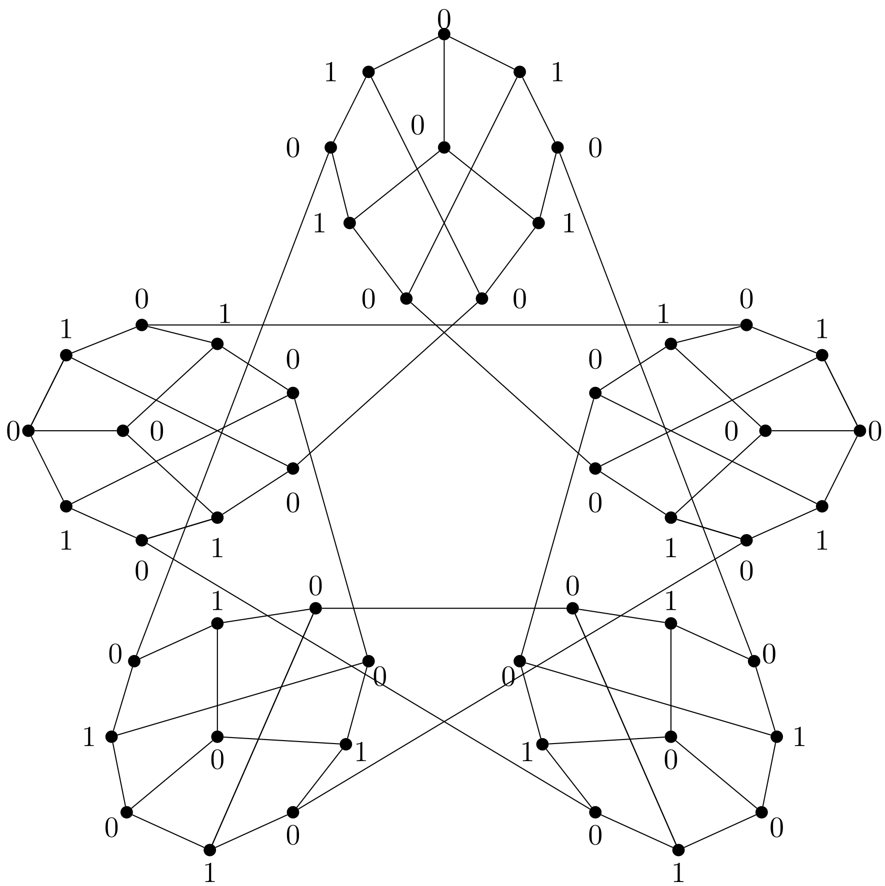

FIG. 8. Szekeres Snark

Theorem 8. *Let G be the Szekeres snark. Then $\gamma_I(G) = 20$.*

Proof. Let G be the Szekeres snark with 50 vertices and f be the IDF for Snekers snark as assigned in Figure 8. Then $\gamma_I(G) \leq w(f) = 20$. According to Theorem 1, $\gamma_I(G) \geq \frac{2n}{\Delta+2} \geq 20$. Hence $\gamma_I(G) = 20$. □

4 Conclusion and future scope

In this paper we found the exact values of Italian domination number of some families of snark. In the near future, one can explore the Italian domination number of other classes of graphs.

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