SOME PROPERTIES OF SOME SUBRINGS OF THE RING OF FORMAL POWER SERIES

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Abstract

It is well known thanks to the Borel's lemma that the range of the ring of smooth germs $\mathcal{E}_1$ under the Borel mapping is equal to the ring of formal power series $\mathbb{R}[[x]]$, in this paper we will show that if we consider a subring $\mathcal{C}_1$ of the ring $\mathcal{E}_1$ that contains the ring of the polynomials then the (x)-adic completion of $\mathcal{C}_1$ is the ring of formal power series $\mathbb{R}[[x]]$ and its range under the Borel mapping is dense under the (x)-adic topology, afterwards we will show the compactness of the inclusion mapping between some subrings of the ring of formal power series $\mathbb{R}[[x]]$.

1. **Introduction**

Let $\mathcal{E}_1$ denote the ring of germs at the origin in $\mathbb{R}$ of $\mathcal{C}^\infty$ functions in a neighborhood of $0 \in \mathbb{R}$ and $\mathbb{R}[[x]]$ the ring of formal series with real coefficients. If $f \in \mathcal{E}_1$, we denote by $\hat{f} \in \mathbb{R}[[x]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_1 \ni f \to \hat{f} \in \mathbb{R}[[x]]$ is called the Borel mapping. A subring $\mathcal{C}_1 \subseteq \mathcal{E}_1$ is called quasianalytic if the restriction of the Borel mapping to $\mathcal{C}_1$ is injective.
Firstly, we know by the Borel's lemma in [2] that there is a surjection from the ring $\mathcal{E}_1$ to the ring of formal power series $\mathbb{R}[[x]]$, in this paper we will show that if we consider a subring $\mathcal{C}_1$ of the ring $\mathcal{E}_1$ that contains the ring of the polynomials then the (x)-adic completion of $\mathcal{C}_1$ is the ring of formal power series $\mathbb{R}[[x]]$ and its range under the Borel mapping is dense under the (x)-adic topology. By [4], we have the compactness for the inclusions mappings between some subrings of the ring of smooth germs $\mathcal{E}_1$. In the last section we are going to show the

same for some subrings of the ring of formal power series $\mathbb{R}[[x]]$ which is the range of the ring $\mathcal{E}_1$ under the Borel mapping.

2. The (x)-adic topology over the rings $\mathcal{C}_1$.

Let's recall some facts about the m-adic topology.

Definition 2.1. Let R be a ring and m an ideal of its. The m-adic completion of the ring R is equal to the projective limit of R/m^n.

If the m-adic topology is separated (i-e $\bigcap_{i\geq 0} m^i = \{0\}$), then this topology is metrizable. Indeed, for all $x, y \in R$, let n be the largest integer such that $x - y \in m^n$, where $m^0 = R$. Then we define the metric $d(x,y) = \frac{1}{2^n}$ when n exists and 0 otherwise.

The (x)-adic completion of the polynomials ring $\mathbb{R}[x]$ is the ring of formal power series $\mathbb{R}[[x]]$.

The interested reader will find more information about this in a very readable form in [1], Chapter 10.

If $\mathcal{C}_1$ is a ring such that $\mathbb{R}[x] \subseteq \mathcal{C}_1 \subseteq \mathcal{E}_1$, then for each $f \in \mathcal{C}_1$, let $\hat{f}$ be the power series defined by:

$$\hat{f}(x) = \sum_{n\in\mathbb{N}} \frac{f^{(n)}(0)}{n!} x^n$$

Then the map $\phi : \mathcal{C}_1 \to \mathbb{R}[[x]]$, defined by $\phi(f) = \hat{f}$ for each $f \in \mathcal{C}_1$ is a ring homomorphism.

Example: If $\mathcal{C}_1$ is the ring of real analytic functions, then $\phi(f) = f$ for all $f \in \mathcal{C}_1$.

Proposition 2.2. *The completion of the ring $\phi(\mathcal{C}_1)$ for the (x)-adic topology is the ring $\mathbb{R}[[x]]$. Therefore this ring is dense in the ring $\mathbb{R}[[x]]$ for the same topology.*

Proof. By assumption the ring $\mathbb{R}[x]$ is contained in the ring $\mathcal{C}_1$, so $\mathbb{R}[x] \subseteq \phi(\mathcal{C}_1) \subseteq \mathbb{R}[[x]]$, as $\mathbb{R}[[x]]$ is the (x)-adic completion of the ring

$\mathbb{R}[x]$, the (x)-adic completion of the ring $\phi(\mathcal{C}_1)$ is also the ring of formal power series $\mathbb{R}[[x]]$.

□

Lemma 2.3. *Let R be a ring and m its ideal, put $J = \bigcap_{i\geq 0} m^i$, so the m-adic completion of the ring R is exactly the m/J-adic completion of the ring R/J up to isomorphism.*

Proof. Put $J = \bigcap_{i\geq 0} m^i$. Thanks to the third isomorphism theorem, for all $i \in \mathbb{N}$ there is an isomorphism $f_i : R/m^i \to (R/J)/(m^i/J)$. Let $(p_i : \breve{R} \to R/m^i)$ be the projective system corresponding to the m-adic completion $\breve{R}$ of R. Composing with f_i, we get a projective system $(f_i p_i : \breve{R} \to (R/J)/(m^i/J))$. Now, we can check easily that this satisfies the universal property of a projective limit ([3], chapter 5) (by using the corresponding properties of $(p_i : \breve{R} \to R/m^i)$). Thus $\breve{R}$ is the m/J-adic completion of R/J.

□

We can now state the main result of this section.

Proposition 2.4. *The completion of the ring $\mathcal{C}_1$ for the (x)-adic topology is the ring $\mathbb{R}[[x]]$ up to isomorphism.*

Proof. Put $J = \bigcap_{i\geq 0} (x)^i$. By the previous lemma the (x)-adic completion of the ring $\mathcal{C}_1$ is equal to the $(x)/J$-adic completion of the ring $\mathcal{C}_1/J$. We have that $\phi(\mathcal{C}_1)$ is isomorphic to the ring $\mathcal{C}_1/J$. So the (x)-adic completion of the ring $\phi(\mathcal{C}_1)$ is exactly the $(x)/J$-adic completion of the ring $\mathcal{C}_1/J$. So, by the proposition 2.2 and the lemma 2.3 the (x)-adic completion of the ring $\mathcal{C}_1$ is the ring $\mathbb{R}[[x]]$.

□

We say that the ring $\mathcal{C}_1$ is a quasianalytic ring if the Borel mapping $\phi : \mathcal{C}_1 \to \mathbb{R}[[x]]$, defined by $\phi(f) = \hat{f}$ for each $f \in \mathcal{C}_1$ is injective.

Equivalently, if the ring $\mathcal{C}_1$ satisfies the following implication for all $f \in \mathcal{C}_1$.

$$(f^{(j)}(0) = 0, j = 0, 1, ...) \Rightarrow (f = 0).$$

Thanks to the quasianaliticity, we will not distinguish between a germ and its range under the Borel mapping, therefore, we may assume that $\mathcal{C}_1 \subset \mathbb{R}[[x]]$.

We assume for the sequel that these rings satisfy the following property called the stability under monomial division.

Let $f \in \mathcal{C}_1$ and $f = x\hat{\varphi}$ where $\hat{\varphi} \in \mathbb{R}[[x]]$, then $\varphi \in \mathcal{C}_1$.

Remark 2.1. By the property of the stability under monomial division, the ring $\mathcal{C}_1$ is a principal domain.

Now we will give a criteria for the surjectivity of the Borel mapping $\wedge : \mathcal{C}_1 \to \mathbb{R}[[x]]$ under the assumption that the local ring $\mathcal{C}_1$ is closed under derivation.

Proposition 2.5. *If the Borel mapping $\wedge : \mathcal{C}_1 \to \mathbb{R}[[x]]$ is surjective, then the ring $\mathcal{C}_1$ is complete for the (x)-adic topology.*

Proof. Suppose that the mapping $\wedge$ is surjective, and let's take $f \in \mathcal{C}_1$ such that $\hat{f} \in (x^n)\mathbb{R}[[x]]$, where $n \in \mathbb{N}$, so if $f \notin (x^n)\mathcal{C}_1$, then there exists $k_0 \in [[0, n-1]]$ such that $f^{(k_0)}(0) \neq 0$, as the ring $\mathcal{C}_1$ is a local ring and closed under derivation, $f^{(k_0)}$ is invertible, so there exist $g \in \mathcal{C}_1$ such that $f^{(k_0)}g = 1$, so $\hat{f}^{(k_0)}\hat{g} = 1$ and therefore $\hat{f}^{(k_0)}(0) \neq 0$), which is a contradiction. It is clear that if $f \in (x^n)\mathcal{C}_1$, then $\hat{f} \in (x^n)\mathbb{R}[[x]]$, consequently, we have the equivalence $f \in (x^n)\mathcal{C}_1 \Leftrightarrow \hat{f} \in (x^n)\mathbb{R}[[x]]$, for all $n \in \mathbb{N}$, therefore, the mapping ϕ is a homeomorphism for the (x)-adic topology. As the mapping $\wedge$ is a linear homeomorphism and the ring $\mathbb{R}[[x]]$ is complete, we deduce that the ring $\mathcal{C}_1$ is also complete for the (x)-adic topology.

$\square$

3. Compactness of the inclusion mapping between some subrings of the ring of formal power series $\mathbb{R}[[x]]$.

Set $\Lambda_M := \{f = \sum_{n=0}^{\infty} a_n x^n, |a_n| \leq Ch^n M_n, \forall n \text{ pour un } C, h > 0\}$.

$\Lambda_{M,h} := \{f = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{R}[[x]], ||f||_{\Lambda,h} = \sup_{n\geq 0} |a_n/(h^n M_n)| < \infty\}$.

The space $\Lambda_{M,h}$ endowed with the norm $||f||_{\Lambda,h} = \sup_{n\geq 0} |a_n/(h^n M_n)|$ and $f = \sum_{n\geq 0} a_n x^n$ and $(M_n) \in (0, +\infty)$ is a Banach space.

Proposition 3.1. *The application inclusion $J : \Lambda_{M,h} \to \Lambda_{M,s}$ is compact for all $s > h > 0$.*

Proof. The application inclusion $J : \Lambda_{M,h} \to \Lambda_{M,s}$ is compact whenever $s > h > 0$. We are supposing that in each $\Lambda_{M,h}$, the Banach norm $||f||_h = \sup_{n\geq 0} |a_n/(h^n M_n)|$ is given, where $f = \sum_{n\geq 0} a_n x^n$ and $(M_n) \in (0, +\infty)$.

In order to show that J is a compact mapping, we should prove the following:

"Given a bounded sequence $(f_k) \subset \Lambda_{M,h}$, there exists a subsequence $(f_{m(j)})$ which is $||\cdot||_s$-converging to some $F \in \Lambda_{M,s}$".

With this aim, fix a sequence (f_k) as above. Then $f_k = \sum_{n\geq 0} a_{n,k} x^n$ and there is $A \in (0, +\infty)$ such that $|a_{n,k}| \leq Ah^n M_n$ for all $k \geq 1$ and all $n \geq 0$. In particular, taking $n = 0$, the sequence $(a_{0,k})_{k\geq 1}$ is bounded. By the Bolzano-Weierstrass theorem, this sequence possesses a convergent subsequence in $\mathbb{R}$, say $a_{0,k_j^0} \to b_0$. Now, the sequence $(a_{1,k})_{k\geq 1}$ is also bounded, and so its subsequence $(a_{1,k_j^0})_{j\geq 1}$ is bounded as well. It follows that it possesses a convergente subsequence in $\mathbb{R}$, say $a_{1,k_j^1} \to b_1$. With this procedure, and having contructed for a given $n \geq 0$ nested subsequences $(k_j^n) \subset (k_j^{n-1}) \subset \cdots \subset (k_j^0) \subset \mathbb{N}$ satisfying $a_{l,k_j^l} \to b_l$ as $j \to \infty$ $(l = 1, ..., n)$, one obtains by using the Bolzano-Weierstrass theorem a subsequence $(k_j^{n+1}) \subset (k_j^n)$ satisfying $a_{n+1,k_j^{n+1}} \to$ certain $b_{n+1} \in \mathbb{R}$. Consider the sequence $m(j) := k_j^j$ $(j \geq 1)$. This sequence is, eventually, a subsequence of each sequence $(k_j^n)_{j\geq 1}$ $(n = 0, 1, 2, ...)$. Therefore $a_{n,m(j)} \to b_n$ as $j \to \infty$ for each $n \geq 0$.

Next, we define $F := \sum_{n\geq 0} b_n x^n \in R[[x]]$:

On the one hand, we have $|a_{n,m(j)}| \leq AM_n h^n$ for all j, n.

Letting $j \to \infty$, we get $|b_n| \leq AM_n h^n$ for all n, from which one derives that $F \subset \Lambda_{M,h} \subset \Lambda_{M,s}$, so $F \in \Lambda_{M,s}$.

On the other hand, we are going to prove that $(f_{m(j)})$ converges to F in the norm $||\cdot||_s$ (and we would be done).

To this end, we must show that, given $\varepsilon > 0$, there is a positive integer j_0 such that $\sup_{n\geq 0} |a_{n,j} - b_n|/(M_n s^n) < \varepsilon$ for all $j \geq j_0$.
To this end, note that, due to the triangle inequality, we have

$|a_{n,m(j)} - b_n| \leq 2AM_n h^n$ for all n, j.

Now, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $2A(h/s)^n \to 0$ for all $n > n_0$ (because $h < s$). Then $|a_{n,m(j)} - b_n|/(M_n s^n) \leq 2A(h/s)^n < \varepsilon$ for all j and all $n > n_0$.

Finally, for each fixed $n \in \{0, 1, 2, ..., n_0\}$, the sequence
$(|a_{n,m(j)} - b_n|/(M_n s^n))_{j\geq 1}$ tends to 0. Hence there is j_n such that
$|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon$ for all $j \geq j_n$. If we set

$j_0 := \max\{j_1, ..., j_{n_0}\}$,

then we get $|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon$ for all $n = 0, 1, ..., n_0$ and all $j \geq j_0$. Putting all together, we consequently obtain

$|a_{n,m(j)} - b_n|/(M_n s^n) < \varepsilon$ for all $n \geq 0$ and all $j \geq j_0$.

But this tells us that $||f_{m(j)} - F||_s < \varepsilon$ for all $j \geq j_0$.
In other words, $f_{m(j)} \to F$ $(j \to \infty)$ in the norm $||\cdot||_s$, as required.

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