A characterization on the domatic polynomials

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Abstract

A subset $\boldsymbol{T \subseteq V}$ of a graph $\boldsymbol{G = (V, E)}$ is called a dominating set if every vertex in $\boldsymbol{V \setminus T}$ has atleast one neighbour in $\boldsymbol{T}$. A domatic partition of a graph $\boldsymbol{G}$ is a partition of $\boldsymbol{V}$ into disjoint dominating sets. The domatic number $\boldsymbol{d(G)}$ is the maximum number of disjoint dominating sets. Given a graph $\boldsymbol{G}$, the domatic polynomial $\boldsymbol{DP(G, x)}$ is defined as $\boldsymbol{DP(G, x) = \sum_{j=1}^{d(G)} dp(G, j)x^j}$, where $\boldsymbol{dp(G, j)}$ represents the number of domatic partition of $\boldsymbol{G}$ with size $\boldsymbol{j}$. We explore the domatic polynomial for several well-known graphs, including cycle graphs, complete bipartite graphs, star graphs, fan graphs, complete graphs and path graphs. Additionally, we study the domatic polynomials for the join and corona products of two graphs. Finally, we provide a characterization on the domatic polynomials.

Keywords: domination, domatic partition, domatic number, domatic polynomial.

MSC Classification: 05C69

1 Introduction

We consider a simple graph $G = (V, E)$ of order $|V| = n$. The set $N_G(q) = \{p|pq \in E(G)\}$ is called the open neighborhood of a vertex $p \in V$ and the set $N_G[q] = N_G(q) \cup \{q\}$ is called the closed neighbourhood of $q \in V$. In G, a subset $T \subseteq V$ is called a *dominating set* if $N_G[T] = V(G)$, or every vertex in $V \setminus T$ has atleast one neighbor in T. The minimum cardinality of a dominating set in G is represented by the domination number $\gamma(G)$. For more details on domination, we refer [5–7]. The concept of the domination polynomial $D(G,x)$ of a graph G (which is the generating function for the number of dominating sets of G) was defined by Alikhani and Peng [2] in 2009.

A *domatic partition* of a graph G is a partition of vertex set into disjoint dominating sets. The domatic number $d(G)$ is the maximum size of a domatic partition of a graph G. Cockayne and Hedetniemi [11] introduced the domatic number of a graph G. More details on the domatic number can be seen in [10–13].

Graph polynomials are a well-developed field that can be utilized to analyze graph properties. Recently study the number of the domatic partitions of a graph has attracted the attention of researchers. Let state the definition of the domatic polynomial of a graph G.

Definition 1. [1] *Let $\mathcal{DP}(G,j)$ be the family of domatic partitions of a graph G with cardinality j, and let $dp(G,j) = |\mathcal{DP}(G,j)|$. Then the domatic polynomial $DP(G,x)$ of G is defined as*

$$DP(G,x) = \sum_{j=1}^{d(G)} dp(G,j)x^j,$$

where $d(G)$ is the domatic number of G.

The graph $G = G_1 \circ G_2$, which is made up of one copy of G_1 and $|V(G_1)|$ copies of G_2, is the corona of two graphs G_1 and G_2, where the k-th vertex of G_1 is adjacent to every vertex in the k-th copy of G_2, see [4]. In particular, the graph created from a copy of G, where a new vertex u' and a pendant edge uu' are added for each vertex $u \in V(G)$, is known as the corona $G \circ K_1$. The join $G_1 \vee G_2$ of two graphs with $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$ is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

In section 2, we find the results of the domatic polynomial of the cycle graphs C_n, path graphs P_n, complete bipartite graphs $K_{m,n}$, star graph S_n, and graphs of the form $G \circ K_1$. In Section 3, we discuss six graph operations and characterize the domatic polynomials to determine the structure of the graph.

2 Domatic polynomial for certain graphs

In this section, we begin to explore and establish some properties of the domatic polynomial of the cycle graphs, path graphs, star graphs, complete bipartite graphs, and fan graphs. To calculate the domatic number and domatic polynomial of a graph G, we first need to establish the following two results:

Theorem 1. [3] *For any graph G, $d(G) \leq \delta + 1$, where δ is the minimum degree of G.*

Lemma 1. [1] *If G is a connected graph, then $d(G) \geq 2$.*

The following results can be easily proved using the definitions of domatic number and domatic polynomial of graph G.

Proposition 1. *Let G be a trivial or null graph. Then $d(G) = 1$, and $DP(G,x) = x$.*

Proposition 2. *(i) The domatic number of the cycle C_n is*

$$d(C_n) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{otherwise} \end{cases}.$$

(ii) $DP(C_n, x) = x + a_2x^2 + a_3x^3$, *where,*

$$a_2 = dp(C_n, 2), \quad a_3 = 1 \text{ if } n \equiv 0 \pmod 3, \text{ and } a_3 = 0 \text{ if } n \not\equiv 0 \pmod 3$$

Proposition 3. *(i) The domatic number of the complete bipartite graph* $K_{m,n}$, *for* $m, n \geq 2$ *is*

$$d(K_{m,n}) = \begin{cases} m & \text{if } m = n \\ 2 & \text{otherwise} \end{cases}.$$

(ii)

$$DP(K_{m,n}, x) = \begin{cases} a_0x^n + a_1x^{n-1} + \ldots + a_nx & \text{if } m = n \\ x + (2^{m+n-1} - 2^n - 2^m + 3)x^2 & \text{otherwise} \end{cases},$$

where $a_0, a_1, \ldots, a_n$ *are coefficients that count the number of corresponding partitions.*

Proposition 4. [1] *Let* G *be a path graph* P_n. *Then* $d(G) = 2$, *and*

$$DP(P_n, x) = x + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} x^2, \qquad \forall n \geq 2.$$

Now, we compute the domatic polynomial of the join of two graphs. Initially, we give an example for calculating the domatic polynomial of the join of two graphs. Further, we compute the domatic polynomial of the corona product of two graphs, say K_1 and for any graph G.

Example 1. *The Join graph,* $P_6 \vee K_1$ *has seven domatic partition of* 3 *disjoint dominating sets,* 31 *domatic partition of 2 disjoint dominating sets and one domatic partition of 1 dominating set. So its domatic polynomial is* $DP(P_6 \vee K_1, x) = 7x^3 + 31x^2 + x$.

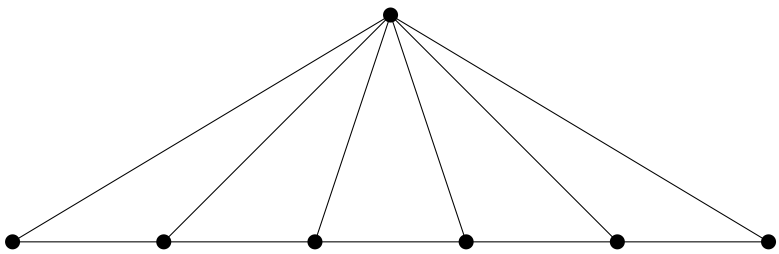

Fig. 1: $P_6 \vee K_1$

Remark 1. *For C_n where $n = 3, 4$ and 5, we have*

$$DP(C_n \vee K_1, x) = DP(C_n, x)DP(K_1, x) + n\, DP(C_n, x) - x(ax + (n - 1)),$$

where $a = 3, 2$ and 7 for $n = 3, 4$ and 5 respectively.

Theorem 2. *Let G be a star graph $S_{n+1} = \bar{K_n} \vee K_1$. Then $d(G) = 2$, and $DP(G, x) = x^2 + x$.*

The following theorem, as proven in [1], using the concept of weak 2-coloring of a graph. However, we have established the same result using the domatic partition of a graph.

Theorem 3. *[1] Let G be a graph of order n, Then*

$$DP(G \circ K_1, x) = 2^{n-1}x^2 + x.$$

Proof. To obtain the number of domatic partition of a graph of size two, we should enumerate the dominating sets S whose its complement S^c is a dominating set, too. In the graph of the form $G \circ K_1$ the number of this kind of set is the half of the number of dominating sets of $G \circ K_1$ of size n, i.e., $\frac{1}{2}d(G \circ K_1, n)$. But it is easy to see that $d(G \circ K_1, n) = 2^n$ (see [2]) and so $dp(G \circ K_1) = 2^{n-1}$. □

3 Domatic polynomial and some graph operations

In this section, we give the construction of the six graph operations which will be used in proving the following theorems. Suppose that graph $V(G)$ has k-partite sets, say $A_1, A_2, \ldots, A_k$ with $|V(G)| = n$. Let $v \in A_i$, then we say the label of a vertex v, is $\text{lab}(v) = A_i$. The cardinality of A_i may not be equal to each other. A minimum of two vertices is required for $\mathcal{A}$-operation, while at least three vertices are necessary for the remaining five operations. Now, we define the following graph operations.

- $\mathcal{A}$-operation: Let u_1, v_1 and v_2 be vertices with $\text{lab}(u_1) = A_1$ and $\text{lab}(v_1) = A_2 = \text{lab}(v_2)$. Now, join the vertices u_1, v_1 and v_1, v_2 to get a new graph. If we repeat p times $\mathcal{A}$- operation, the new graph can be represented as $G(\mathcal{A})$.
- $\mathcal{B}$-operation: Let u_1, v_1 and v_2 be vertices with $\text{lab}(u_1) = A_1$ and $\text{lab}(v_1) = A_2 = \text{lab}(v_2)$. Now, join the edge u_1v_1, v_1v_2 and v_2u_1 to get a new graph. If we repeat $\mathcal{B}$-operation followed by connecting the vertices in a sequential order, the new graph can be represented as $G(\mathcal{B})$.
- $\mathcal{C}$-operation: Let u_1, v_1 and v_2 be vertices with $\text{lab}(u_1) = A_1$ and $\text{lab}(v_1) = \text{lab}(v_2) = A_2$. Now, join the edges u_1v_1 and u_1v_2 to get a new resultant graph. If we repeat p times $\mathcal{C}$-operation, the new graph can be represented as $G(\mathcal{C})$.
- $\mathcal{D}$-operation: Let u_1, u_2 and v_1 be vertices with $\text{lab}(u_1) = \text{lab}(u_2) = A_1$ and $\text{lab}(v_1) = A_2$, such that $|A_1| = n - 1$ and $|A_2| = 1$. Now, join the edge u_1v_1, and u_2v_1 to get a new graph. If we repeat p times $\mathcal{D}$-operation followed by connecting the vertices in sequential order, the new graph can be represented as $G(\mathcal{D})$.

- $\mathcal{E}$-operation: Let u_1, v_1, w_1 and x_1 be vertices with lab$(u_1) = A_1$, lab$(v_1) = A_2$ and lab$(w_1) = A_3$ and lab$(x_1) = A_4$. Join the edges of the four partite sets in such a way that each vertices of partite set is connected to all remaining $n-1$ vertices to get a new graph. If we repeat p times $\mathcal{E}$-operation, the new graph can be represented as $G(\mathcal{E})$.
- $\mathcal{F}$-operation: Let u_1, u_2, v_1, v_2, w_1 and w_2 be the vertices with lab(u_1) = A_1 = lab(u_2), lab(v_1) = A_2 = lab(v_2) and lab(w_1) = A_3 = lab(w_2). Now, join the first vertex of each partite set (u_1, v_1, w_1) to the second vertex of every partite set (u_2, v_2, w_2), ensuring each pair from different partite sets is connected, i.e., we get new edges, $E_1 = \{u_1u_2, u_1v_2, u_1w_2, v_1u_2, v_1v_2, v_1w_2, w_1u_2, w_1v_2, w_1w_2\}$. If we repeat p times $\mathcal{F}$-operation, the new graph can be represented as $G(\mathcal{F})$.

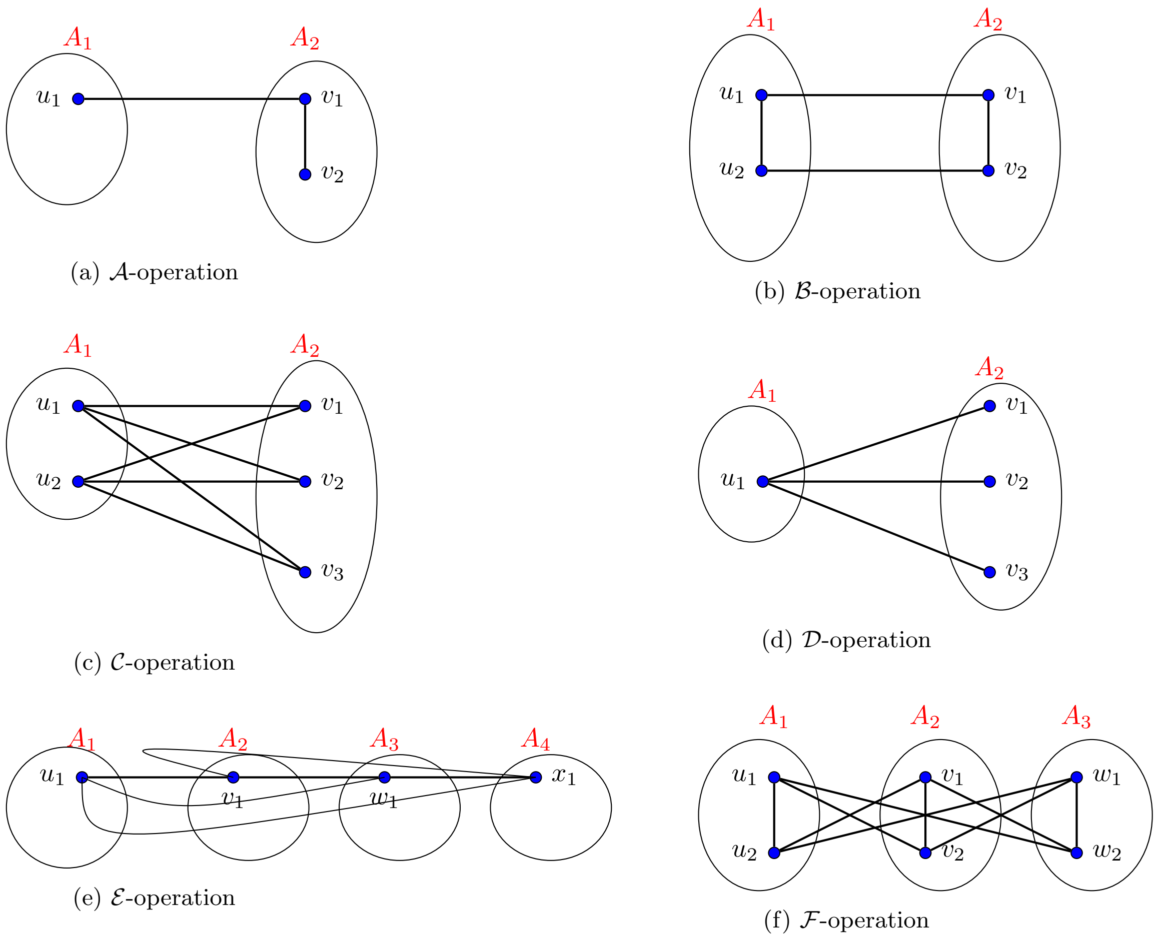

Fig. 2: Six graph operations

Theorem 4. *$DP(G,x)$ is a monomial iff G is either a trivial graph or a null graph.*

Proof. Let $DP(G,x)$ be a monomial. Then $DP(G,x)$ has only one term, i.e., $V(G)$ itself is a domatic partition. So, we have two cases:
Case 1: If G is connected then G is a trivial graph.
Case 2: If G is disconnected then G is a null graph.
The converse is supported by Proposition 1. □

Theorem 5. *The domatic polynomial $DP(G,x)$ is quadratic iff G is obtained by using one of the four operations: $\mathcal{A}$-operation, $\mathcal{B}$-operation (if $n \not\equiv 0 \pmod 3$), $\mathcal{C}$-operation, or $\mathcal{D}$-operation.*

Proof. Let $DP(G,x)$ be a quadratic polynomial. Then $d(G) = 2$, i.e., $V(G)$ can be split into two dominating sets, say, A_1 & A_2. Now, we have the following domatic partitions.

I: If n is even, then $|A_1| = \frac{n}{2} = |A_2|$, say $|A_1| = p$ and $|A_2| = q$. If necessary, $|A_1| = p + i = p_i$ and $|A_2| = n - p_i$.

II: If n is odd, then $|A_1| = \lceil \frac{n}{2} \rceil = p$ and $|A_2| = n - p$. Similarly, $|A_1| = p + i = p_i$ and $|A_2| = n - p_i$ (if necessary).

III: The sizes of A_1 and A_2 will have $n - 1$ and 1, respectively.

Now, we'll prove the necessary conditions in 4 cases:

Case 1: Let G be a graph with two partite sets, $A_1 = \{u_1, u_2, u_3, \ldots, u_p\}$ and $A_2 = \{v_1, v_2, v_3, \ldots, v_q\}$ such that $\text{lab}(u_i) = A_1$ and $\text{lab}(v_i) = A_2$. The sizes of A_1 and A_2 will depend on n which is either even or odd. Now, apply $\mathcal{A}$-operation on partite sets A_1 and A_2 alternately, we get the new edges, $E = \{u_1v_1, v_1u_2, \ldots, v_{p-1}u_p, u_pv_q\}$. In other words, $\mathcal{A}$-operation on (A_1, A_2) is equivalent to P_n. Consequently, $G(\mathcal{A}) = P_n$, $\forall n \geq 2$.

Case 2: Let G be a graph with two partite sets, $A_1 = \{u_1, u_2, u_3, \ldots, u_p\}$ and $A_2 = \{v_1, v_2, v_3, \ldots, v_q\}$ such that $\text{lab}(u_i) = A_1$ and $\text{lab}(v_i) = A_2$. Let us assume that sizes of A_1 and A_2 follows I and II. Now, apply $\mathcal{B}$-operation on partite sets A_1 and A_2, we get the new edges, $E = \{u_1v_1, v_1v_2, v_2v_3, \ldots, v_{q-1}, v_q, v_qu_p, \ldots, u_2u_1\}$. Consequently, $\mathcal{A}$-operation on (A_1, A_2) is equivalent to C_n, i.e., $G(\mathcal{B}) = C_n$ for $n \neq 3k$. In particular, we get $C_4 = K_{2,2}$.

Case 3: Let G be a graph with two partite sets, $A_1 = \{u_1, u_2, \ldots, u_p\}$ and $A_2 = \{v_1, v_2, \ldots, v_q\}$ such that $\text{lab}(u_i) = A_1$ and $\text{lab}(v_i) = A_2$. Similarly, we assume that sizes of A_1 and A_2 satisfy I and II. Now, apply $\mathcal{C}$-operation on partite sets A_1 and A_2, we get the new edges, $E = \{u_1v_1, u_1v_2, \ldots, u_1v_q, u_2v_1, u_2v_2, \ldots, u_2v_q, \ldots, u_pv_1, u_pv_2, \ldots, u_pv_q\}$. Consequently, $\mathcal{C}$-operation on (A_1, A_2) is equivalent to $K_{m,n}$, $m \neq n$, i.e., $G(\mathcal{C}) = K_{m,n}$ for $m \neq n$.

Case 4: Let G be a graph with two partite sets, $A_1 = \{u_1, u_2, \ldots, u_{p-1}\}$ and $A_2 = \{v_1\}$ such that $\text{lab}(u_i) = A_1$ and $\text{lab}(v_1) = A_2$, which is satisfied III. Then, we have two sub-cases:
Sub-case 1: Apply $\mathcal{D}$-operation on partite sets A_1 and A_2, we get the new edges,

$E = \{v_1u_1, v_1u_2, \ldots, v_1u_{p-1}\}$. Consequently, $\mathcal{D}$-operation on (A_1, A_2) is equivalent to S_n, i.e., $G(\mathcal{D}) = S_n$ for $n \geq 4$.
Sub-case 2: Now, let us assume that the partite set A_1 is not a null graph. Then, join every vertices of partite set A_2 to every vertices of the partite set A_1, which is the corona operation of A_1 and A_2, i.e., $G = H \circ K_1$.

The converse is supported by Propositions 2, 3, 4 and Theorem 3. □

Theorem 6. *$DP(G, x)$ is a cubic polynomial iff G is obtained by using $\mathcal{B}$-operation (if $n = 3k,\ k \in \mathbb{N}$).*

Proof. Let $DP(G, x)$ be a cubic polynomial. Then $d(G) = 3$, i.e., $V(G)$ can be split into three dominating sets with $|A_1| = |A_2| = |A_3| = \frac{n}{3}$ such that $\text{lab}(u_i) = A_1$, $\text{lab}(v_i) = A_2$ and $\text{lab}(w_i) = A_3$. Now, apply $\mathcal{B}$-operation on partite sets A_1, A_2 and A_3 alternately, we get the new edges, $E_1 = \{u_1v_1, v_1w_1, \ldots, w_ru_1\}$. In other words, $\mathcal{B}$-operation on (A_1, A_2, A_3) is equivalent to C_n. Consequently, $G \cong C_n$ for $n = 3k, k \in \mathbb{N}$. In particular, $C_3 = K_3$, and $C_6 = K_{3,3}$.

The converse is supported by Propositions 2 and 3. □

Theorem 7. *$DP(G, x)$ be an n^{th} degree polynomial iff G is obtained by using one of the two operations: $\mathcal{E}$-operation, or $\mathcal{F}$-operation.*

Proof. Let $DP(G, x)$ be an n^{th} degree polynomial. Then $d(G) = n$, i.e., $V(G)$ can be split into n dominating sets. Now, we'll prove the theorem in two parts.

1st Part: When $n = 2$, we get $G = K_2$ (see proof of Theorem 5). When $n = 3$, we get $G = K_3$ (see proof of Theorem 6). When $n = 4$, then $d(G) = 4$, i.e., $V(G)$ can be split into four dominating sets with $|A_1| = |A_2| = |A_3| = |A_4| = 1$ such that $\text{lab}(u_1) = A_1$, $\text{lab}(v_1) = A_2$, $\text{lab}(w_1) = A_3$ and $\text{lab}(x_1) = A_4$. Now, apply $\mathcal{E}$-operation on (A_1, A_2, A_3, A_4), we get the new edges, $E_1 = \{u_1v_1, u_1w_1, u_1x_1, v_1w_1, v_1x_1, w_1x_1\}$. Consequently, $G \cong K_4$. Similarly, when $n \geq 5$, repeat p times $\mathcal{E}$-operations, the new graph can be represented as $G(\mathcal{E}) = K_n$.

2nd Part: When $n = 2$, we get $G = K_{2,2}$ (see proof of Theorem 5). When $n = 3$, we get $G = K_{3,3}$ (see proof of Theorem 6). When $n = 4$, then $d(G) = 4$, i.e., $V(G)$ can be split into four dominating sets with $|A_1| = |A_2| = |A_3| = |A_4| = 2$ such that $\text{lab}(u_i) = A_1$, $\text{lab}(v_i) = A_2$, $\text{lab}(w_i) = A_3$ and $\text{lab}(x_i) = A_4$, where $1 \leq i \leq 2$. Now, apply $\mathcal{F}$-operations on partite sets (A_1, A_2, A_3, A_4), we get the new edges, $E_1 = \{u_1u_2, u_1v_2, u_1w_2, u_1x_2, v_1u_2, v_1v_2, v_1w_2, v_1x_2, w_1u_2, w_1v_2, w_1w_2, \ldots, x_1w_2, x_1x_2\}$. Consequently, $G \cong K_{4,4}$ for $m = n = 4$. Similarly, when $n \geq 5$, repeat p times $\mathcal{F}$-operations, the new graph can be represented as $G(\mathcal{F}) = K_{m,n}$ for $m = n$.

Conversely, suppose that G is a complete graph K_n. Then, from [8], we have

$$DP(K_n, x) = \sum_{j=1}^{n} dp\begin{Bmatrix} n \\ j \end{Bmatrix} x^j,$$

where $dp\left\{{n \atop j}\right\}$ is the number of j-domatic partitions of K_n.
The other part is supported by Propositions 3. Therefore, the theorem is proved. □

4 Conclusion

In this article, we have explored the domatic polynomial $DP(G, x)$ of cycle graphs, path graphs, complete bipartite graphs, star graphs, complete graphs and fan graphs. We have also extended the study of domatic polynomials to operations on graphs, including the join and corona products. Additionally, we have constructed six operations to obtain the graph structure of domatic polynomials, thereby characterizing these polynomials through graph operations. In general, these results offer new insights into the structure and partitioning behavior of graphs and open avenues for further research in graph theory.

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