On Graphs From Algebraic Structures Using Arithmetic Functions

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A B S T R A C T

The study of graphs associated with algebraic structures and their related problems is an important area in algebraic graph theory. Several classes of such graphs and their associated properties have been extensively studied by various mathematicians. In this paper, we propose a unified approach to defining graphs associated with algebraic structures using arithmetic functions by introducing a new class of graphs. We study the basic properties and structural parameters of this new class. Furthermore, we show that many graphs defined using algebraic structures in the existing literature are special cases of this new graph. We conclude our study by studying the reduced form of this new class of graphs.

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1 Introduction

Algebraic graph theory is a fascinating branch of mathematics that integrates Graph Theory, Abstract Algebra and Linear Algebra. The interpretation of graph theoretic notion to algebraic structures helps in determining structural properties of algebraic structures. This approach has been successfully applied in various branches of algebra, including group theory, semigroup theory, ring theory, vector space and so on

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[3, 4, 6, 10, 14, 19, 25, 29, 31]. One of the most significant study in graph associated with algebraic structures are the study of graphs originating from groups. The convergence of group theory and graph theory in abstract algebra has a rich history, dating back to the early 18^{th} century. The pioneering work of Arthur Cayley, who introduced Cayley graphs [11] in 1878, marked the beginning of a fruitful intersection of these two fields. Since then, Cayley graphs have been extensively studied and research on their properties and applications remains an active area of investigation [1, 2, 15, 26], with new discoveries and insights continuing to emerge. Graphs associated with algebraic structures offer a valuable tool for uncovering their distinctive characteristics. In this context, power graphs have been a subject of intense study in recent decades, providing insights into the underlying algebraic properties. The power graph of a finite group was introduced by Kelarev and Quinn [24] in 2000. They defined the directed version of a power graph of a group G, denoted by $P(G)$. Motivated by this, Chakrabarty *et al.* [12] defined the undirected power graph $P(G)$.

Definition 1. [27] The power graph $P(G)$ of a finite group G, is a simple undirected graph with vertex set G and two vertices $g_1, g_2 \in G$ are adjacent in $P(G)$ if and only if $g_1 \neq g_2$ and $g_1^m = g_2$ or $g_2^m = g_1$, $m \in \mathbb{N}$.

The concept of power graphs has led to significant advancements in the study of groups, yielding remarkable results [8, 9, 13]. Furthermore, this concept has also been successfully applied to other algebraic structures, such as rings and semigroups, revealing new insights and importance in these areas [27, 33, 36]. Thereafter, in 2012 the coprime graph of a finite group was introduced by Ma and Feng [28]. They defined the coprime graph of a group G, denoted by $\Gamma(G)$

Definition 2. [17] The coprime graph of a group G, denoted by Γ_G, is a simple graph whose vertices are elements of G and two elements $g_1 \neq g_2$ are adjacent if and only if gcd $(|g_1|, |g_2|) = 1$.

For more works we can refer [20, 30, 35]. In [34], its shown that in the definition of coprime graph we can use the concept of Euler totient function for finding the adjacency. Similar idea is applied in the Mobius function graph of finite groups, which has recently become a focus of intense study due to its connections with various mathematical disciplines.

Definition 3. *[22] The Mobius function graph $M(G)$ of a finite group G is a simple graph whose vertex set is same as the elements of G and any two distinct vertices g_1, g_2 are adjacent in $M(G)$ if and only if $\mu(|g_1||g_2|) = \mu(|g_1|)\mu(|g_2|)$, where μ is the Mobius function.*

Ongoing research is actively investigating various aspects of the Mobius function graph $M(G)$, including its connectivity indices, spectral properties and graph coloring [21, 22]. An in-depth review of existing literature on these topics, we identifies a significant gap that, there is currently no comprehensive graph that encompasses all algebraic structures. To address this challenge, we propose a new framework. In this paper, our goal is to create a generalized graph that encompasses various algebraic structures. As a part of this idea we put forward **Generalized Arithmetic Function Graph**. Throughout this paper, we consider three classes of arithmetic functions, which are defined as

Definition 4. *[5, 23] A function $f : \mathbb{N} \to \mathbb{R}$ is said to be* ***multiplicative*** *if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $gcd(m, n) = 1$. An arithmetic function f is said to be* ***weakly multiplicative*** *if f is not identically zero and $f(mn) = f(m)f(n)$ if and only if $(m, n) = 1$ and an arithmetic function f is said to be* ***completely multiplicative*** *if $f(mn) = f(m)f(n)$ for every $m, n \in \mathbb{N}$.*

This paper is organized as follows. In Section 1, introduces the Generalized Arithmetic Function Graph, a novel graph construct and demonstrates how various graphs connecting different algebraic structures are encompassed by this concept. Additionally, we explore the Complementary Generalized Arithmetic Function Graph. Section 2 presents a reduced version of these graphs and we discuss some of their properties. For a review of the basic concepts, readers are referred to [5, 7, 16, 18, 32, 37]

2 Main Result

In this section we introduce a new class of graphs, where elements from any algebraic structure are represented as vertices. Each element is associated with a unique positive integer and adjacency is determined by an arithmetic function. Under this framework its found that various established graphs are special cases of our proposed graph structure. In the subsequent section, we will work on the construction and properties of these graphs. We also demonstrate their connections to existing graph theories.

Definition 5. *Let G be a finite algebraic structure and associated with each element of $g \in G$ assume that there exist a non-negative integer $i(g)$. Let ϕ be an arithmetic function. Then, the generalized arithmetic function graph of a finite structure G is a graph whose vertices are elements of G itself and any two elements $g, g' \in G$ are adjacent if and only if $\phi(i(g)i(g')) = \phi(i(g))\phi(i(g'))$. The graph is denoted by $\mathcal{G}_{\phi(G)}$.*

Associated with each different pair of functions (ϕ, i) there exist a unique Arithmetic (ϕ, i) graph. Using this definition we can obtain many classes of graphs associated with algebraic structures which are available in literature, such as Mobius function graph, coprime graph, power graph.

Remark 6. *The Mobius function graph is obtained from Definition 5 by considering G as a group and the pair (μ, i) where μ is the Mobius function and $i(g) = |g| =$ order of the element.*

Remark 7. *The coprime graph can be obtained from Definition 5, by considering G as a group and the pair (ϕ, i) where ϕ as any weakly multiplicative function and $i(g) = |g| =$ order of the element.*

Remark 8. *Using the Definition 5, we can obtain the power graph by considering G as a group and the following pair (ϕ, i). Let ϕ be a multiplicative function and for any pair of elements $a, b \in G$, the function $i_a(b)$(analogously $i_b(a)$) is defined as*

$$i_a(b) = \begin{cases} 1, & \text{if } a = b \\ m, & \text{if } a^m = b, a^n \neq b \text{ for } 0 < n < m, m, n \in \mathbb{N} \\ 0, & \text{Otherwise.} \end{cases}$$

Now, for every pair of elements $(a, b) \in G \times G$, if $a^m = b$, then

$$\phi(i_a(a)i_a(b)) = \phi(m) = \phi(i_a(a)) \times \phi(i_a(b)) = \phi(1) \times \phi(m)$$

Therefore, a and b are adjacent in $\mathcal{G}_{\phi(G)}$. Analogously if $a = b^n$, then $\phi(i_b(a)i_b(b)) = \phi(i_b(a))\phi(i_b(b))$. Now, when $a \neq b^m$ for any m, then $\phi(i_a(a)i_a(b)) = \phi(0)$ is not defined, also, $\phi(i_a(a))\phi(i_a(b)) = \phi(1)\phi(0)$ is also undefined. Therefore, there is no adjacency relation when $a \neq b^m$.

We also propose a complementary version of this graph as follows,

Definition 9. *Let G be a finite algebraic structure and associated with each element of $g \in G$ assume that there exist a non-negative integer $i(g)$. Let ϕ be an arithmetic function. Then, the complementary generalized arithmetic function graph of a finite structure G is a graph whose vertices are elements of G itself and any two elements $g, g' \in G$ are adjacent if and only if $\phi(i(g)i(g')) \neq \phi(i(g))\phi(i(g'))$. The graph is denoted by $\mathcal{G}^c_{\phi(G)}$.*

It has been observed that even slight changes to algebraic structures, integers associated with elements and the arithmetic functions can make a significant impact on the resulting graphs. The following examples establishes that by taking G as group, $\phi = \mu$ and $i(g) = |g|$ in generalized arithmetic function graph we will get the Mobius function graph explained in Definition 3.

Example 10. Consider the cyclic group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ under the operation addition modulo 4. Here $i(0) = |0| = 1$, $i(1) = |1| = 4$, $i(2) = |2| = 2$ and $i(3) = |3| = 4$. Take $\phi = \mu$, the Mobius function. Now, $\mu(i(0)i(1)) = \mu(1.4) = 0$ and $\mu(i(0))\mu(i(1)) = \mu(1)\mu(4) = 0$. Hence, $\mu(i(0)i(1)) = \mu(i(0))\mu(i(1))$. Therefore vertices 0 and 1 are adjacent. Similarly, for any two vertices g, g' in $\mathcal{G}_{\mu(\mathbb{Z}_4)}$, $\mu(i(g)i(g')) = \mu(i(g))\mu(i(g'))$. The corresponding $\mathcal{G}_{\mu(\mathbb{Z}_4)}$ is shown in Figure 1(a). The complementary generalized arithmetic function graph $\mathcal{G}^c_{\mu(\mathbb{Z}_4)}$ as per Definition 9 is shown in Figure 1(b).

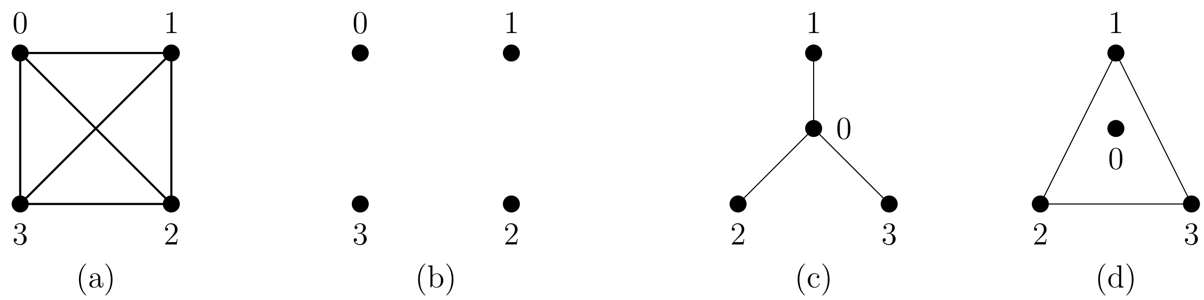

Figure 1: The arithmetic and complementary arithmetic function graphs described in Example 10 and Example 11.

Next we like to illustrate that if we change the function ϕ as Euler totient function in the above example, then the resulting generalized arithmetic function graph is equivalent to coprime graph explained in Definition 2.

Example 11. Since Euler totient function is multiplicative, $\phi(i(g)i(g')) = \phi(i(g))\phi(i(g'))$ whenever $\gcd(i(g), i(g')) = 1$, here $i(g) = |g|$. It is clear that $\gcd(i(0), i(g)) = 1$ for all $g \in Z_4 - \{0\}$. Hence identity vertex is adjacent to all other vertices. And in all other cases they are non adjacent since $\gcd(i(1), i(2)) \neq 1$, $\gcd(i(1), i(3)) \neq 1$, $\gcd(i(2), i(3)) \neq 1$. The corresponding $\mathcal{G}_{\phi(\mathbb{Z}_4)}$ is shown in Figure 1(c). And the complementary graph $\mathcal{G}^c_{\phi(\mathbb{Z}_4)}$ as per Definition 9 is shown in Figure 1(d)

The following example illustrates that applying Remark 8 yields the power graph as described in Definition 1.

Example 12. *Consider the dihedral group* $D_6 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$*. Let* g *be the non identity element in* D_6*. Clearly* $\phi(i(e)i(g)) = \phi(i(e))\phi(i(g))$*. Next take* $g_1 = x$ *and* $g_2 = x^2$ *then* $g_1^2 = g_2$ *, Here*

$$\phi(i_{g_1}(g_1)i_{g_1}(g_2)) = \phi(2) = \phi(1)\phi(2) = \phi(i_{g_1}(g_1))\phi(i_{g_1}(g_2)).$$

By taking $g_1 = x$ *and* $g_2 = x^3$ *then* $g_1^3 = g_2$ *, Here*

$$\phi(i_{g_1}(g_1)i_{g_1}(g_2)) = \phi(3) = \phi(1)\phi(3) = \phi(i_{g_1}(g_1))\phi(i_{g_1}(g_2)).$$

Also if $g_1 = x^2$ *and* $g_2 = x^3$ *then* $g_1 = g_2^2$ *, Here*

$$\phi(i_{g_2}(g_1)i_{g_2}(g_2)) = \phi(2) = \phi(2)\phi(1) = \phi(i_{g_2}(g_1))\phi(i_{g_2}(g_2)).$$

In all these scenarios g_1 *and* g_2 *are adjacent. Conversely, in all other cases, it can be observed that* $g_1 \neq g_2^m$*. Figure 2 , (a) and (b) represents arithmetic function graph* $\mathcal{G}_{\phi(D_6)}$ *and its complement graph* $\mathcal{G}^c_{\phi(D_6)}$ *respectively.*

In Example 13 we take field as the algebraic structure, $\phi = \mu$ and $i(a) = |a|$ we can construct two graphs corresponding to two group operations in the field.

Example 13. Consider the field $F = (\mathbb{Z}_5, +, \times)$. Here we take $i(a) = |a|$ for the a in the field. Then $i(0) = 1$, $i(1) = i(2) = i(3) = i(4) = 5$ for $(F, +)$. Take $\phi = \mu$, the Mobius function. While constructing $\mathcal{G}_{\mu(F)}$, $\{0, 1, 2, 3, 4\}$ act as its vertex set. Now applying the condition of arithmetic function graph on vertices. We get Figure 3 (a).
While considering its multiplication operation in field. In $\mathcal{G}_{\mu(F)}$, $\{1, 2, 3, 4\}$ act as its vertex set, with $i(1) = 1$, $i(2) = i(3) = 3$ and $i(4) = 2$. Now applying the condition of arithematic function graph on vertices. We get Figure 3 (b).

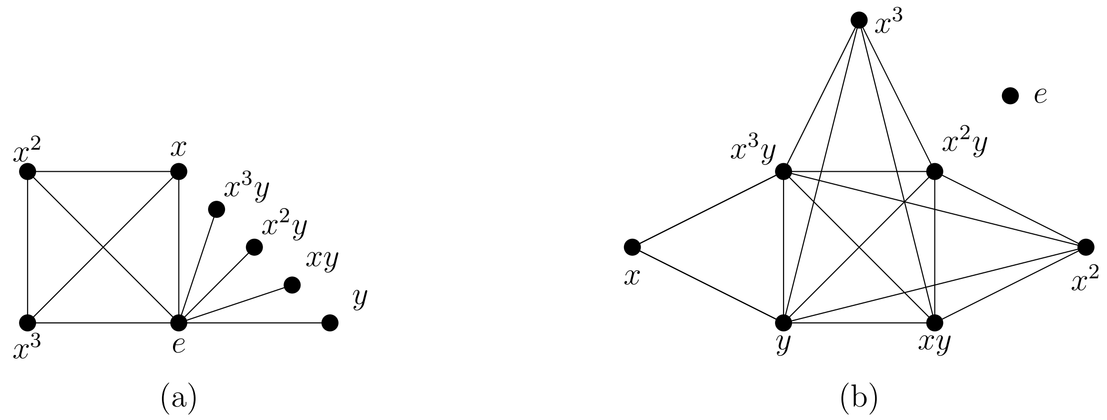

Figure 2: The arithmetic and complementary arithmetic function graphs described in Example 12.

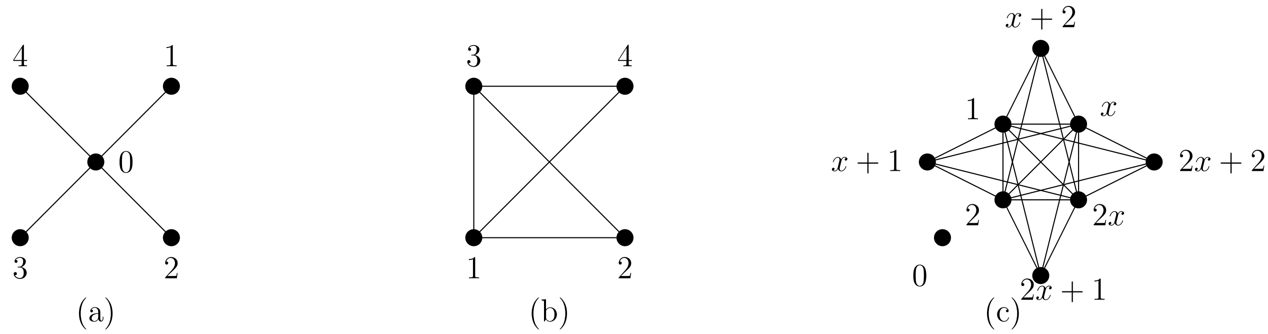

Figure 3: The arithmetic function graphs described in Example 13 and Example 14.

In the upcoming example we take vector space as the structure. The arithmetic function graph so obtained is same as linear dependence graph in [29]

Example 14. *Consider the vector space* $V = \mathbb{Z}_3[x]/ < x^2 + 1 >$ *over* $\mathbb{Z}_3$*. The space have 9 elements* $\{0, 1, 2, x, x+1, x+2, 2x, 2x+1, 2x+2\}$*. Let* S *be a basis of the vectors space* V *Now, define* $i(a) = min|S'|$ *where* $S' \subseteq S, a \in Span(S')$ *or* $i(a)$ *is the order of the minimum spanning subset of the basis which spans* a*. In this case,* $S = \{1, x\}$ *is the basis. Clearly* $i(1) = i(2) = i(x) = i(2x) = 1$ *,* $i(x+1) = i(x+2) = i(2x+1) = i(2x+2) = 2$*. When we take* ϕ *as the Euler's totient function, corresponding* $\mathcal{G}_{\phi(V)}$ *is shown in Figure 3 (c).*

Remark 15. *Let* V *be the vector space of polynomials of degree at most* n *over* $\mathbb{F}_p$*. The generalized arithmetic function graph whose vertices correspond to the polynomials in* V *and the function* $i(a)$ *represents the minimum number of elements needed to span the* *polynomial a, contains a clique.*

Next, we discuss some basic properties of generalized arithmetic function graph and complementary generalized arithmetic function graph.

Theorem 16. *Let G be any finite algebraic structure. If ϕ is completely multiplicative function then $\mathcal{G}_{\phi(G)}$ is complete graph.*

Corollary 17. *For any finite algebraic structure G, with ϕ being a completely multiplicative function, then $\mathcal{G}^c_{\phi(G)}$ is totally disconnected.*

Theorem 18. *For the pair (ϕ, i), where ϕ is a multiplicative function and $i(a) = |a|$ in any finite algebraic structure G, the following statements are true,*

(a.) $\mathcal{G}_{\phi(G)}$ is connected.

(b.) $\{e\}$ is the dominating set with $\Delta(\mathcal{G}_{\phi(G)}) = n - 1$

(c.) Diam$(\mathcal{G}_{\phi(G)}) \leq 2$.

(d.) Girth is either 3 or infinity

Corollary 19. *For the pair (ϕ, i), where ϕ is a multiplicative function and $i(a) = |a|$ in any finite algebraic structure G, then $\mathcal{G}^c_{\phi(G)}$ id disconnected.*

Theorem 20. *For a fixed pair (ϕ, i), if the algebraic structures G_1 and G_2 are such that $G_1 \cong G_2$, then $\mathcal{G}_{\phi(G_1)} \cong \mathcal{G}_{\phi(G_2)}$*

Proof. Given $G_1 \cong G_2$. Let $f : G_1 \to G_2$ be an isomorphism from G_1 to G_2. Then for any two elements $g, g' \in G_1$ there exist unique elements $f(g), f(g') \in G_2$. Now, define a graph isomorphism $\Phi : \mathcal{G}_{\phi(G_1)} \to \mathcal{G}_{\phi(G_2)}$ as $\Phi(g) = f(g)$, clearly Φ is a bijection. Also, for any arithmetic function ϕ, $\phi(i(g)i(g')) = \phi(i(f(g))i(f(g'))) = \phi(i(\Phi(g))i(\Phi(g')))$. Thus, a, b are adjacent in $\mathcal{G}_{\phi(G_1)}$ if and only if $f(a), f(b)$ are adjacent in $\mathcal{G}_{\phi(G_2)}$. Therefore, Φ preserves adjacency, thus the corresponding generalized arithmetic function graphs are $\mathcal{G}_{\phi(G_1)}$ and $\mathcal{G}_{\phi(G_2)}$ are isomorphic. □

Remark 21. *Converse of Theorem 20 may not be true. Consider cyclic group $\mathbb{Z}_8$ and non cyclic Quarternion group Q_8. if we consider the pair (ϕ, i) where ϕ is a completely multiplicative function and $i(g) = |g|$, then the graphs $\mathcal{G}_{\phi(\mathbb{Z}_8)}$ and $\mathcal{G}_{\phi(Q_8)}$ are complete, hence isomorphic. But the groups are not isomorphic.*

Now, let's explore the generalized arithmetic function graph on a group G, where we define $i(g) = |g|$, the order of the element g. Throughout the next set of theorems, we will discuss some fundamental properties of this function.

Theorem 22. *Let ϕ be a multiplicative function such that $\phi(r^2) \neq \phi(r)^2$, for any prime r. Let G be a finite group of order p^k, with $p > 2$ is prime and $k > 2$. Then for $1 \leq m, n \leq k$*

(a.) $\mathcal{G}_{\phi(G)}$ is star if and only if $\phi(p^{m+n}) \neq \phi(p^m)\phi(p^n)$

(b.) If $\phi(p^{m+n}) = \phi(p^m)\phi(p^n)$ then $\mathcal{G}_{\phi(G)}$ is a complete split graph.

Proof. (a.) Consider the group G, with $|G| = p^k$. Clearly, any non identity element of G has order p^i, $1 \leq i \leq k$. Also given that ϕ is multiplicative, $\phi(1) = 1$ and so vertex corresponding to identity element e is adjacent to all other vertices in $\mathcal{G}_{\phi(G)}$. Now, for any two non-identity elements $g, g' \in G$, $i(g) = p^i$, $i(g') = p^j$ for some $1 \leq i, j \leq k$, by our assumption $\phi(p^{i+j}) \neq \phi(p^i)\phi(p^j)$. Therefore g and g' are not adjacent in $\mathcal{G}_{\phi(G)}$. Thus, $\mathcal{G}_{\phi(G)}$ is a star.

Conversely, suppose $\mathcal{G}_{\phi(G)}$ is star. Let u be the vertex adjacent to all the vertices in the star. It follows that no two vertices $g \neq u, g' \neq u$ are adjacent. Now, identity element is of order 1, therefore, $\phi(i(e)i(g)) = \phi(i(g)) = \phi(i(e))\phi(i(g))$. Thus, $u = e$. Since $|G| = p^k$, every non-identity element is of order $p^i, 1 \leq i \leq k$. Now, every non-identity elements are not adjacent, $\phi(p^{m+n}) \neq \phi(p^m)\phi(p^n)$. Therefore, the result.

(b.) Suppose $\phi(p^{m+n}) = \phi(p^m)\phi(p^n)$. We can partition the vertex set of $\mathcal{G}_{\phi(G)}$ as $\{e\}$, $A_1 = \{x \in G : |x| = p\}$ and $A_2 = \{y \in G : y \neq e,\ |y| \neq p\}$. For any $g, h \in A_1$ and $g \neq h$, by definition of ϕ , $\phi(r^2) \neq \phi(r)^2$. Thus g is not adjacent to h in $\mathcal{G}_{\phi(G)}$ and hence the subgraph induced by A_1 in $\mathcal{G}_{\phi(G)}$ is totally disconnected.

For any $g', h \in A_2$, with $|g'| = p^i$ and $|h| = p^j$, $i \neq j$, $2 \leq i, j \leq k$.

$$\phi(|g'||h|) = \phi(p^i.p^j) = \phi(p^i)\phi(p^j) = \phi(|g'|)\phi(|h|)$$

Therefore g' is adjacent to h in $\mathcal{G}_{\phi(G)}$. Hence, the sub-graph of induced by A_2 in $\mathcal{G}_{\phi(G)}$ is complete.

Next we take any $g \in A_1$ and $g' \in A_2$, with $|g| = p$ and $|g'| = p^i$, where $2 \leq i \leq k$.

$$\phi(|g||g'|) = \phi(p.p^i) = \phi(p)\phi(p^i) = \phi(|g|)\phi(|g'|)$$

Therefore g is adjacent to g' in $\mathcal{G}_{\phi(G)}$. Hence, every element in A_1 is adjacent to every element of A_2.

The identity element $\{e\}$ is adjacent to all the vertices of $\mathcal{G}_{\phi(G)}$. From the above arguments, $\{e\} \cup A_2$ forms a clique in $\mathcal{G}_{\phi(G)}$ and A_1 forms a completely disconnected graph whose vertices are adjacent to every vertex of $\{e\} \cup A_2$. Thus we can conclude that $\mathcal{G}_{\phi(G)}$ is a complete split graph.

□

Theorem 23. *Let G be a finite group of prime order $p > 2$ and ϕ be a multiplicative function. Then,*

(a.) $\mathcal{G}_{\phi(G)}$ is star if and only if $\phi(p^2) \neq \phi(p)^2$

(b.) $\mathcal{G}_{\phi(G)}$ is a complete graph if and only if $\phi(p^2) = \phi(p)^2$

Proof. (a.) It directly follows from Theorem 22.

(b.) Since $|G| = p$, in Theorem 22, $A_2 = \phi$. Therefore, $\mathcal{G}_{\phi(G)}$ is complete. Conversely assume that $\mathcal{G}_{\phi(G)}$ is complete. Since $|G| = p$, every non-identity element must be of order p. Now, since $\mathcal{G}_{\phi(G)}$ is complete, for any two elements $g, g' \in G, g \neq e, g' \neq e$, g is adjacent to g'. Therefore

$$\phi(i(g), i(g')) = \phi(p^2) = \phi(i(g))\phi(i(g'))$$

Therefore, the result follows.

□

Theorem 24. *Let G be a non-cyclic group of order $2p$ where p is an odd prime and ϕ be a multiplicative function. Then $\mathcal{G}_{\phi(G)}$ is the complete split graph $CS(2p, 2p-2)$ if $\phi(r^2) \neq \phi(r)^2, r = 2, p$ and G has exactly one element of order 2.*

Proof. G is a non-cyclic graph of order $2p$, then G has only elements of order $1, 2, p$. Thus, the vertex set can be partitioned into $\{e\}$,

$$A_1 = \{g \in G : |g| = 2\}$$
$$A_2 = \{g \in G : |g| = p\}$$

clearly $\{e\}$ and every element of A_1 is adjacent to A_2, also $\{e\}$ is adjacent to every element of A_1. Therefore, none of the pair of sets in $\{e\}, A_1, A_2$ form an independent set. Also if

$\mathcal{G}_{\phi(G)}$, then $< A_i >, i = 1, 2$ does not form a complete graph since $\phi(r^2) \neq \phi(r)^2, r = 2, p$. Therefore, if $|A_1| > 1$, then $A_1 \cup \{e\}$ does not form a complete graph. Therefore, $\mathcal{G}_{\phi(G)}$ is the complete split graph $CS(2p, 2p-2)$ if $|A_1| = 1$ and $\phi(r^2) \neq \phi(r)^2, r = 2, p$. □

Theorem 25. *Let G be a non cyclic group of order p_1p_2, where $p_1 > 2, p_2 > 2$ are distinct primes and ϕ be a multiplicative function. Then*

(a.) $\mathcal{G}_{\phi(G)}$ is a complete tripartite graph if and only if $\phi(p^2) \neq \phi(p)^2$ for primes $p = p_1, p_2$.

(b.) $\mathcal{G}_{\phi(G)}$ is a complete graph if and only if $\phi(p^2) = \phi(p)^2$ for primes $p = p_1, p_2$.

Proof. Since the $|G| = p_1p_2$ and G is non-cyclic, the vertex set of G can be partitioned into $\{e\}$ and

$$A_1 = \{g \in G : |g| = p_1\}$$
$$A_2 = \{g \in G : |g| = p_2\}$$

Now, since ϕ is multiplicative, for every $a \in A_1$ and $b \in A_2$, a, b are adjacent in $\mathcal{G}_{\phi(G)}$, also, every vertex is adjacent to e as well. Since $p_1, p_2 > 2$, we have $|A_1| \geq 2, |A_2| \geq 2$.

Now, $\mathcal{G}_{\phi(G)}$ is a tripartite graph if and only if $gg' \notin \mathcal{G}_{\phi(G)}$ for any $g, g' \in A_i, i = 1, 2$, which is if and only if $\phi(p^2) \neq \phi(p)^2$ for $p = p_1, p_2$. Therefore, the part $(a.)$ follows.

Now, for part $(b.)$ $\mathcal{G}_{\phi(G)}$ is complete if and only if $gg' \in \mathcal{G}_{\phi(G)}$ for any $g, g' \in A_i, i = 1, 2$. Which is if and only if $\phi(p^2) = \phi(p)^2$ for $p = p_1, p_2$. Therefore, the result follows. □

Theorem 26. *Let G be a non-abelian group of order $p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are distinct primes and ϕ be a multiplicative function such that $\phi(r^2) \neq \phi(r)^2$, for any prime r. Then, $\mathcal{G}_{\phi(G)}$ contains a complete $n+1$-partite graph.*

Proof. Since G is non-abelian with $|G| = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are distinct primes. Order of every element in G divides $p_1p_2 \ldots p_n$. We partition the elements of G as $\{e\}$, $A_1 = \{g_1 \in G : |g_1| = p_1\}$, $A_2 = \{g_2 \in G : |g_2| = p_2\}$, $\ldots$, $A_n = \{g_n \in G : |g_n| = p_n\}$. If we take any two elements $g_i \in A_i$ and $g_j \in A_j$ where $i \neq j$. Since ϕ is multiplicative and $(p_i, p_j) = 1$, we can say that all elements in A_i are adjacent to all elements in A_j. This is true for all i and j. Thus in $\mathcal{G}_{\phi(G)}$ all vertices in $\{e\}, A_1, A_2, \ldots, A_n$ are mutually adjacent. Now, any two vertices $g, g' \in A_i, i = 1, 2, \ldots, n$ are not adjacent, since $\phi(r^2) \neq \phi(r)^2$, for any prime r. Thus, $\{e\}, A_1, A_2, \ldots, A_n$ are independent sets in $\mathcal{G}_{\phi(G)}$. Therefore, $\mathcal{G}_{\phi(G)}$ contains a complete $n+1$-partite graph. □

Theorem 27. *For any group G of order $p_1p_2\dots p_n$, with $n \geq 3$, where $p_1, p_2, \dots, p_n$ are distinct odd primes, $\mathcal{G}_{\phi(G)}$ is non-planar.*

Proof. We divide the possibilities into two cases.

Case I: $|G| = p_1p_2\dots p_n, n \geq 4$ and p_i are odd primes. Now, for each prime p_i, there exist at least one element of order p_i. Let A_i denote the collection of all elements of order $p_i, i = 1, 2, \dots n$. Now, each element of A_i is adjacent to each element of A_j whenever $i \neq j$. Since $i \geq 4$, the set $\{e, a_1, a_2, a_3, a_4\}, a_i \in A_i, i = 1, 2, 3, 4$ forms an induced K_5 in $\mathcal{G}_{\phi(G)}$. Therefore, $\mathcal{G}_{\phi(G)}$ is non-planar.

Case II: $|G| = p_1p_2p_3$ and p_i are odd primes. Now, since each p_i is an odd prime, there exist at least two distinct elements of order $p_i, i = 1, 2, 3$ for each prime p_i. Let A_1, A_2, A_3 denote the collection of all elements of order p_1, p_2, p_3 respectively. Now, each element of A_i is adjacent to each element of A_j whenever $i \neq j$. Therefore, $\{a_1^1, a_1^2, a_2^1, a_2^2, a_3^1, a_3^2\}, a_i^j \in A_i, i = 1, 2, 3$ forms an induced $K_{3,3}$ in $\mathcal{G}_{\phi(G)}$.

□

Now, we provide some basic results regarding the $\mathcal{G}_\phi$ of a vectors space V over a finite field F_p. Now, in the case of vector spaces, for every $a \neq 0 \in V$ we consider $i(a)$ as the minimum number of elements in the basis which spans a and $i(0) = 1$.

Theorem 28. *Let V be a vector space over a finite field F_p with basis $\{x_1, x_2, \dots, x_k\}$. Then*

(a.) $\mathcal{G}_{\phi(V)}$ *is connected.*

(b.) $\mathcal{G}_{\phi(V)}$ *is of diameter 2.*

(c.) $\gamma(\mathcal{G}_{\phi(V)}) = 1$.

Proof. The proof follows directly from the fact that every element of $\{x_1, x_2, \dots, x_k\}$ is adjacent to every other element x of V. Since

$$\phi(i(x_j)i(x)) = \phi(i(x)) = \phi(i(x_j))\phi(i(x))$$

Thus, $\mathcal{G}_{\phi(V)}$ must be connected, for every two vertex $x, y \in V$, $x - x_i - y$ is a path connecting x, y and every x_i is a dominating set of $\mathcal{G}_{\phi(V)}$, therefore the result. □

3 Reduced Form of Graphs

Now, consider the special case of a group and order of an element as the integer associated with the element of the group. In this case, the identity element e in the group has order 1, therefore it must be adjacent to every other vertex of $\mathcal{G}_{\phi(G)}$ and it is not adjacent to any of the element in $\mathcal{G}^c_{\phi(G)}$ for every multiplicative function ϕ. Also, the identity element makes graph $\mathcal{G}_{\phi(G)}$ as a diameter 2 graph. Such a restriction does not allow to study the structural properties of algebraic structures from the associated generalized arithmetic function graph or its complementary generalized arithmetic function graph. Thus, we propose a new class of reduced version of the graph called Reduced Generalized Arithmetic Function Graph and the corresponding Complementary Reduced Generalized Arithmetic Function Graph.

Definition 29. *Let G be a finite algebraic structure and associated with each element of $a \in G$ assume that there exist a positive integer $i(a)$. Let ϕ be an arithmetic function. Then, the reduced generalized arithmetic function graph of a finite structure G is a graph whose vertices are elements of $G - \{e\}$ where e is the identity element and any two elements $a, b \in G - \{e\}$ are adjacent if and only if $\phi(i(a)i(b)) = \phi(i(a))\phi(i(b))$. The graph is denoted by $\mathcal{RG}_{\phi(G)}$.*

Analogous to reduced version of complementary generalized arithmetic function graph is defined as

Definition 30. *Let G be a finite algebraic structure and associated with each element of $a \in G$ assume that there exist a positive integer $i(a)$. Let ϕ be an arithmetic function. Then, the reduced generalized arithmetic function graph of a finite structure G is a graph whose vertices are elements of $G - \{e\}$ where e is the identity element and any two elements $a, b \in G - \{e\}$ are adjacent if and only if $\phi(i(a)i(b)) \neq \phi(i(a))\phi(i(b))$. The graph is denoted by $\mathcal{RG}^c_{\phi(G)}$.*

Now, we discuss some basic properties of reduced graphs and complementary reduced graph of a group. In this section we consider those multiplicative function which are actually positive values. In particular we are not considering Mobius function.

Theorem 31. *Let ϕ be a weakly multiplicative function, then if G is cyclic then $\mathcal{RG}^c_{\phi(G)}$ is connected . As a consequence, if G is cyclic then $\mathcal{RG}_{\phi(G)}$ is not connected.*

Proof. Let $|G| = n$, since G is cyclic, there exist an element $g \in G$ such that $|g| = n$. Also, note that every other element $g' \in G$ must be of order p, where p divides n. Thus, for every other element $g' \neq e$, $\phi(|g||g'|) \neq \phi(|g|)\phi(|g'|)$, therefore the vertex g is adjacent to any other vertex g' in $\mathcal{RG}^c_{\phi(G)}$. Therefore, $\mathcal{RG}^c_{\phi(G)}$ is connected. □

Theorem 32. *Let ϕ be a weakly multiplicative function and G be a group, if $|G|$ is prime, then $\mathcal{RG}^c_{\phi(G)}$ is complete.*

Proof. Assume that $|G| = p$ is prime, then every non-identity element of G must be of order p. Since order of every element is same, then $\phi(i(a)i(b)) \neq \phi(i(a))\phi(i(b))$ for all $a, b \in G - \{e\}$. Thus, every pair of vertices of $\mathcal{RG}_{\phi(G)}$ must be disconnected. □

As a direct consequence, we have

Corollary 33. *Let G be a group, if $|G|$ is prime then $\mathcal{RG}_{\phi(G)}$ is completely disconnected.*

Remark 34. *The converse of the Theorem 32 is not true in general. For example, consider the dihedral group D_8, every element of D_8 is of either order 2 or 4. Thus, for every $a, b \in D_8 - \{e\}$, $gcd(i(a), i(b)) \neq 1$. Hence a, b are not adjacent in $\mathcal{RG}_{\phi(D_8)}$. Therefore, $\mathcal{RG}_{\phi(D_8)}$ is completely disconnected, but $|D_8| = 8$ is not prime.*

Using the arguments of Theorem 32, we can generalize the result for prime power orders,

Theorem 35. *Let ϕ be a weakly multiplicative function and G be a group, then $\mathcal{RG}^c_{\phi(G)}$ is complete if and only if order $|G| = p^k$, where p is prime.*

Proof. Let $|G| = p^k$, then every element g of G must be of order p^m where $0 \leq m \leq k$. Thus for any $g, g' \in G - \{e\}$, $\phi(|g|, |g'|) \neq \phi(|g|)\phi(|g'|)$, therefore any pair of vertices g and g' are adjacent. Therefore, $\mathcal{RG}^c_{\phi(G)}$ is complete.

Conversely, assume that $|G| = n \neq p^k$, for some prime p. Then there exist two distinct primes p_1 and p_2 such that p_1, p_2 divides $|G|$. Now, assume that k_1, k_2 be the largest positive integers such that $p_1^{k_1}, p_2^{k_2}$ divides $|G|$. Then there exist elements g, g' of orders $p_1^{k_1}, p_2^{k_2}$ in $|G|$. Clearly $\phi(|g||g'|) = \phi(|g|)\phi(|g'|)$, therefore, g and g' are not adjacent in $\mathcal{RG}^c_{\phi(G)}$. Thus, $\mathcal{RG}^c_{\phi(G)}$ is not complete, now by contrapositive argument, the result follows. □

Theorem 36. *Let $G \neq \{e\}$ be a group of order $|G| = p^k$ and ϕ is a multiplicative function with $\phi(p^r) = \phi(p)^r$ for every $r \geq 1$. Then $\mathcal{RG}^c_{\phi(G)}$ is complete.*

Proof. Let $|G| = p^k$, then every element g of G must be of order p^m where $0 \leq m \leq k$. Then for every element $g, g' \in G$, $\phi(|g||g'|) = \phi(p^r p^s) = \phi(p)^{r+s} = \phi(p)^r \phi(p)^s = \phi(|g|)\phi(|g'|)$. Thus, every pair of element is adjacent in Using Theorem 35, we can establish that if $|G| = p^k$ for some prime p then $\mathcal{RG}_{\phi(G)}$ is completely disconnected. Then $\mathcal{RG}^c_{\phi(G)}$ is complete.

□

Analogous to the above theorem, we can say about the complementary reduced graph and is discussed in the coming theorem.

Corollary 37. *Let $G \neq \{e\}$ be a group of order $|G| = p^k$ and ϕ is a multiplicative function with $\phi(p^r) = \phi(p)^r$ for every $r \geq 1$. Then $\mathcal{RG}_{\phi(G)}$ is completely disconnected.*

Now, let's discuss the bipartite nature of the reduced graph of a cyclic group

Theorem 38. *Let G be a cyclic group of order n. If at least three distinct primes p_i divides n then $\mathcal{RG}_{\phi(G)}$ is not bipartite.*

Proof. Let p_1, p_2, p_3 be the three prime numbers which divide n. Since G is cyclic, there must exist at least one element each of order p_1, p_2, p_3 respectively. Let g_1, g_2, g_3 be those elements, now clearly $\phi(|g_i||g_j|) = \phi(|g_i|)\phi(|g_j|)$ for $1 \leq i, j \leq 3, i \neq j$. Therefore, $\mathcal{RG}_{\phi(G)}$ has a component which contains an odd cycle C_3. Therefore, $\mathcal{RG}_{\phi(G)}$ is not bipartite. □

Now, we discuss the reduced graph of some classes of graphs. First we consider the dihedral group D_{2n}.

Theorem 39. *Let ϕ be a weakly multiplicative function. Then for the dihedral group D_{2n}, $\mathcal{RG}_{\phi(G)} = K_{n,n-1}$ if and only if n is an odd.*

Proof. First, assume that n is an odd, since $G = D_{2n}$ there exist exactly n elements g of order $|g| = 2$ and $n - 1$ elements g' such that $(g')^n = e$. Thus, we can partition the elements of $D_{2n} - \{e\}$ as A, the collection of elements of order 2 and B, the collection of elements g such that $g^n = e$. Now, for every $g \in A$ and $g' \in B$, gg' is an edge in $\mathcal{RG}_{\phi(G)}$ as $\phi(|g||g'|) = \phi(|g|)\phi(|g'|)$. Also, not that no pair of elements of A as well as B are adjacent in $\mathcal{RG}_{\phi(G)}$. Therefore, $\mathcal{RG}_{\phi(G)} = K_{n,n-1}$.

Conversely, assume that $G = D_{2n}$ and $\mathcal{RG}_{\phi(G)} = K_{n,n-1}$, we have to prove that n is an odd. We prove this by contradiction. Suppose that n is not an odd number, then 2 divides n. Since, $G = D_{2n}$ consist of a cyclic group of order n, for each proper prime divisor p of n, there exist exactly $p-1$ elements of order p. Now, since $p = 2$ divides n, D_{2n} has at least $n+1$ elements of order 2. Therefore, at least $n+1$ elements must be pairwise disjoint in $\mathcal{RG}_{\phi(G)}$. Therefore, if $\mathcal{RG}_{\phi(G)} = K_{n,n-1}$, then n must be odd. □

In the Remark 34, We gave D_8 as an example of a group which is does not have prime order, but its reduced graph is completely disconnected. Now, in the next result we generalize this and establish it for more general order dihedral groups.

Theorem 40. *Let ϕ be a weakly multiplicative function. Then for the dihedral group D_{2n}, $\mathcal{RG}_{\phi(G)}$ is completely disconnected if and only if $n = 2^k$ for some $k \in \mathbb{N}$.*

Proof. First, assume that $|G| = 2^k$ where $k \in \mathbb{N}$. We have to prove that $\mathcal{RG}_{\phi(G)}$ is completely disconnected. Let g, g' be any two vertices in $\mathcal{RG}_{\phi(G)}$. Since, $n = 2^k$, the orders of g and g' must be even. Therefore, $\phi(|g||g'|) \neq \phi(|g|)\phi(|g'|)$. Thus, g, g' are not adjacent in $\mathcal{RG}_{\phi(G)}$. Since the elements chosen here are arbitrary, any two vertices of $\mathcal{RG}_{\phi(G)}$ are not adjacent. Therefore, $\mathcal{RG}_{\phi(G)}$ is completely disconnected.

Conversely, assume that $G = D_{2n}$ and $\mathcal{RG}_{\phi(G)}$ is completely disconnected. We have to prove that $n = 2^k$ for some $k \in \mathbb{N}$. We prove this by contradiction, suppose that $n \neq 2^k$ for some $k \in \mathbb{N}$. Then there exist an odd prime p, such that p divides n. Since, D_{2n} contains a cyclic group of rotations of order n and p divides n there must exist an element g of order p. Now, take an arbitrary element g' of order 2 in D_{2n}, clearly, $\phi(|g||g'|) = \phi(|g|)\phi(|g'|)$. Thus g, g' are adjacent in $\mathcal{RG}_{\phi(G)}$. Therefore, the graph is not disconnected. Which is a contradiction. Thus, $n = 2^k$ for some $k \in \mathbb{N}$. □

Now, let us discuss some basic properties of the reduced graph of symmetric group.

Theorem 41. *Let ϕ be a multiplicative function with $\phi(p^2) \neq \phi(p)^2, p = 2, 3$. Then for the symmetric group $S_n, n \geq 3$, $\mathcal{RG}_{\phi(G)}$ is planar if and only if $G = S_3$*

Proof. First assume that $G = S_3$, among the five non-identity elements of S_3, three is of order 2 and two is of order 3, thus, $\mathcal{RG}_{\phi(S_3)} \cong K_{2,3}$, clearly it is planar. Therefore, when $G = S_3$, $\mathcal{RG}_{\phi(G)}$ is planar.

Conversely, assume that $G = S_n$ and $\mathcal{RG}_{\phi(G)}$ is planar. We have to prove that $G \neq S_3$, we have this by contradiction. Let $G = S_n, n \geq 4$, then G has at least three elements of order 2, say $g_1 = (1,2), g_2 = (1,3), g_3 = (1,4)$ and at least three elements of order 3, say $h_1 = (1,2,3), h_2 = (1,3,2), h_3 = (1,2,4)$. Clearly, $\phi(|g_i||h_j|) = \phi(|g_i|)\phi(|h_j|)$, $1 \leq g_i, h_j \leq 3$. Thus, the induced subgraph $< g_1, g_2, g_3, h_1 h_2, h_3 >$ of $\mathcal{RG}_{\phi(S_n)}$ is $K_{3,3}$, thus $\mathcal{RG}_{\phi(S_n)}$ is not planar, which is a contradiction. Thus $G \neq S_n, n \geq 4$. Therefore, the result follows. □

Conclusion

In this paper we have presented a unified form for graphs originating from algebraic structures using arithmetic functions. This new constructions is a more generalized version of the already existing graphs such as Power graphs, Co-prime graphs, Mobius function graphs and much more. We have also studied the properties various algebraic structures such as group, vector space, using the graphs we proposed. We also obtained some basic properties of these graphs for different algebraic structures. There are several problems in this area which are yet to be explored. We end our study by proposing some class of problems for further studies.

Problem 42. *Characterize the groups, rings, vector space for which the reduced graph is connected, hamiltonian and planar.*

Problem 43. *Study the spectral properties of generalized arithmetic function graph and reduced generalized arithmetic function graph of different algebraic structures.*

Problem 44. *Study the topological properties of generalized arithmetic function graph and reduced generalized arithmetic function graph of different algebraic structures.*

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