

ON NULL COMPLETELY REGULAR CODES IN MANHATTAN METRIC

I.YU. MOGILNYKH  AND A.YU. VASIL'EVA 

Abstract: We investigate the class of completely regular codes in graphs with a distance partition $C_0, \dots, C_\rho$, where each set C_i , for $0 \leq i \leq \rho - 1$, is an independent set. This work focuses on the existence problem for such codes in the n -dimensional infinite grid. We demonstrate that several parameter families of such codes necessarily arise from binary or ternary Hamming graphs or do not exist. Furthermore, employing binary linear programming techniques, we explore completely regular codes in infinite grids of dimensions 3 and 4 for the cases $s = 1$ and $s = 2$.

Keywords: completely regular code, perfect coloring, infinite rectangular grid, Manhattan metric, Hamming metric.

1 Introduction

One-error-correcting perfect codes [12] and diameter perfect codes [10] in the n -dimensional infinite grid graph are well-studied objects; both can be defined as completely regular codes with covering radii 1 and 2. Later variations and generalizations can be viewed somewhat separately and include perfect colorings with multiple colors and completely regular codes (the latter is referred to as CRC) with various covering radii. Perfect k -colorings in the

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infinite grid graph G_n were studied by Axenovich for $k = n = 2$ [1], by Puzynina for $k = 3, n = 2$ [15], by Dobrec et al. [9], Golomb and Welsh [12] for $k = 2$ and arbitrary $n \geq 2$. For $k \leq 9$ and $n = 2$, they were studied by Krotov. Completely regular codes in G_n were considered in [16] and [4].

As the infinite grid graph provides a graph covering for the binary even-length Hamming and modular grid graphs, the characterization problem for CRCs with putative parameters in the infinite rectangular grid graph G_n is at least as hard as the respective long-standing open problems [6], [14] for CRCs in classical binary and ternary Hamming graphs.

We study completely regular codes whose first r diagonal elements of parameter matrix are zeros; we refer to such codes as r -null CRCs. Several important classes of error-correcting codes in Hamming and grid graphs such as perfect codes, diameter perfect codes, Golay and Preparata codes and their derived codes are r -null completely regular with maximal or premaximal possible r [6]. In a similar fashion, one can define all-ones CRCs.

The next three sections are introductory: basic definitions are provided in Section 2; constructions of completely regular codes in the n -dimensional grid graph are reviewed in Section 3 and the structure of the first four layers of the graph is discussed in Section 4. In Section 5, for arbitrary n and any covering radius, we show that any 2-null CRC in G_n with $c_1 = 1$ and $c_2 = 2$ necessarily arises from the binary Hamming graph and that 2-null CRCs with $c_1 = 1$ or 2 and $c_3 = 3$ exist only in the case $c_1 = 1, c_2 = 3$ and $n = 2$. In Section 6, we prove that 1-null completely regular codes with $c_2 = 1$ necessarily have period three. Codes with the all-ones diagonal in the parameter matrix are considered in Section 7, where several subclasses of these codes are also shown either to be derived from the binary Hamming graph or not to exist.

In Section 8, we propose a binary linear programming approach to the existence problem for completely regular codes in the n -dimensional infinite grid for $n = 3, 4$. We characterize all 1-null CRCs in G_3 up to their parameters and rule out several 2-null CRCs in G_4 . A distinctive feature of our approach is a relaxed form of the linear programming problem: the infeasibility of a single instance of linear programming problem implies the nonexistence of multiple codes with different covering radii.

2 Definitions and notations

A vertex partition $(C_0, C_1, \dots, C_{k-1})$ of a graph is called a perfect k -coloring (or equitable k -partition) if for every $i, j \in 0, 1, \dots, k-1$ there exists an integer α_{ij} such that every vertex in C_i has exactly α_{ij} neighbors in C_j . The matrix $A = (\alpha_{ij})$ is called the parameter matrix of the perfect coloring. In the case when the parameter matrix is tridiagonal, the set C_0 is called a completely regular code. The maximum distance between the code and the remaining vertices is called the covering radius of the code and is denoted by ρ . We will use both the classical representation of the parameter

matrix and the following compact notation as the sequence of its tridiagonal elements:

$$[a_0, b_0 | c_1, a_1, b_1 | c_2, a_2, b_2 | \dots | c_\rho, a_\rho].$$

Given a code, the set of vertices at distance equal to the covering radius from the code is called the opposite code. Obviously, if a code is a CRC, then its opposite code is also a CRC and vice versa.

A CRC with parameter matrix A is called an r -null CRC if $a_0 = \dots = a_{r-1} = 0$. A CRC is called an all-nulls (respectively, all-ones) CRC if $a_0 = \dots = a_\rho = 0$ (respectively, $a_0 = \dots = a_\rho = 1$).

Note that, by definition, 1-null CRCs are CRCs with minimum distance at least two. Moreover, 2-null CRCs in Hamming graphs include all CRCs with minimum distance at least four and, in particular, many well-known families of optimal codes, such as Preparata, BCH, Golay codes, perfect codes and shortened perfect codes. All these codes give rise to CRCs in the n -dimensional grid (see Section 3 below).

The n -dimensional infinite rectangular grid is the graph G_n with vertex set $\mathbf{Z}^n$, where $\mathbf{Z}$ denotes the set of integers and edge set

$$\{(x, y) : \sum_{i=1}^n |x_i - y_i| = 1\}.$$

The graph metric in G_n is often called Manhattan metric [10].

For simplicity, we use non-comma notation for elements in $\mathbf{Z}^n$, for example (101-21) designates an element of $\mathbf{Z}^5$.

By $G_{n,q}$ we denote the graph with the vertex set $\{0, \dots, q-1\}^n$ and the edge set

$$\{(x, y) : \sum_{i=1}^n \min\{|x_i - y_i|, |q - x_i + y_i|\} = 1\}.$$

The vertices of the Hamming graph $H(n, q)$ are $\{0, \dots, q-1\}^n$, the edge set is

$$\{(x, y) : |\{i \in \{1, \dots, n\} : x_i \neq y_i\}| = 1\}.$$

We use a word to refer to a vertex of these graphs. The all-zero word is denoted by $\mathbf{0}$. The weight of the word x is its distance between x and $\mathbf{0}$ in G_n . Throughout the paper we denote by e_i the word of weight one with i th symbol equal to 1. The vertex set of infinite triangular grid is $\mathbf{Z}^2$ and the edge set is

$$\{(x, y) : \sum_{i=1}^2 |x_i - y_i| = 1\} \cup \{(x, y) : x_1 - y_1 = x_2 - y_2 \in \{-1, 1\}\}.$$

Both G_n and the infinite triangular grid are Cayley graphs: G_n is the Cayley graph of the n th power of the additive group of integers and the infinite triangular grid is the Cayley graph of its second power. The respective Cayley graph generator sets are $\{\pm e_i : i \in \{1, \dots, n\}\}$ and $\{\pm(10), \pm(01), \pm(11)\}$.

The following restriction on the parameters of completely regular codes in G_n was established in [4].

Theorem 1. [4, Theorem 1] *The following holds for a CRC in G_n with parameter matrix $[a_0, b_0 | c_1, a_1, b_1 | c_2, a_2, b_2 | \dots | c_\rho, a_\rho]$ and covering radius ρ :*

$$\begin{aligned} c_1 &\leq c_2 \leq \dots \leq c_\rho, \\ b_0 &\geq b_1 \geq \dots \geq b_{\rho-1}. \end{aligned}$$

3 Constructions of CRCs in G_n

Below, we provide constructions of completely regular codes in the n -dimensional grid for the current study. The concepts behind these methods are not new. Many of them were presented earlier for CRCs in distance-regular or Cayley graphs and are well-known or can be easily generalized to the n -dimensional grid. Several recent surveys on CRCs [6], [5], [14] provide a solid background for the constructions described below.

3.1. General constructions. Given two graphs Γ and Γ' with vertex sets V and V' , respectively, a surjection $\phi : V \rightarrow V'$ is called a covering (and Γ is called a covering graph) if ϕ bijectively maps the neighborhood of any vertex in Γ to the neighborhood of its image. The role of coverings in the theory of perfect colorings is reflected by the following fact.

Proposition 1. *Let ϕ be a covering with a graph Γ . If $(C_0, \dots, C_{k-1})$ is a perfect coloring of Γ' with parameter matrix A , then $(\phi^{-1}(C_0), \dots, \phi^{-1}(C_{k-1}))$ is a perfect coloring of Γ with parameter matrix A .*

Let $\phi : G \rightarrow G'$ be a homomorphism from the group G to G' , which maps a generator set S of G to S' of G' . Then ϕ is a covering of the Cayley graph of G' with the generator set S' with a covering graph being the Cayley graph of G with the generator set S [5, Lemma 4.1].

The component-wise modulo q mapping is a homomorphism that implies that the graph G_n is a covering graph for $G_{n,q}$, $q \geq 3$. We note that the graphs $G_{n,3}$ and $G_{n,4}$ are isomorphic to the Hamming graphs $H(n, 3)$ and $H(2n, 2)$ respectively. Moreover, G_3 covers an infinite triangular grid. Due to Proposition 1, the CRCs in these graphs are sources for CRCs in the n -dimensional grid.

We say that a code C has a period q in i th direction if $x \pm qe_i \in C$ for $x \in C$.

Construction 1. *Let C be a CRC in the graph $G_{n,q}$, $q \geq 3$ then the code $\{x \in \mathbb{Z}^n : (x_1 \bmod q, \dots, x_n \bmod q) \in C\}$ is a CRC in G_n with the same parameter matrix as C .*

Remark 1. *One can show that a CRC C in G_n is obtained from Construction 1 by showing that q is a period of C in any direction i.e. $x \pm qe_i \in C$, for all $i \in \{1, \dots, n\}$ and x in C . For any fixed $n \geq 2$, there are many perfect (diameter perfect) codes in G_n with different periods [10]. In Sections 5-7 we*

show that some classes of null-CRCs have only specific period q , e.g. $q = 3$ or 4 in each direction.

We specially distinguish two cases of the above construction.

Construction 2. Let C be a CRC in the Hamming graph $H(n, 3) = G_{n,3}$, then the code $\{x \in \mathbf{Z}^n : (x_1 \bmod 3, \dots, x_n \bmod 3) \in C\}$ is completely regular in G_n with the same parameter matrix as C .

As the graph $G_{n,4}$ is isomorphic to $H(2n, 2)$ via the Gray mapping $\tau(0) = (00)$, $\tau(1) = (01)$, $\tau(2) = (11)$, $\tau(3) = (10)$ we obtain the following.

Construction 3. If C is a CRC in the Hamming graph $H(2n, 2)$, then the code $\{x \in \mathbf{Z}^n : (\tau(x_1 \bmod 4), \dots, \tau(x_n \bmod 4)) \in C\}$ is a CRC in G_n with the same parameter matrix as C .

Consider the following mapping:

$$\phi(x_1, x_2, x_3) = (x_1 - x_2, x_3 - x_2)$$

for $(x_1, x_2, x_3) \in \mathbf{Z}^3$. This function is a group homomorphism from the additive group of $\mathbf{Z}^3$ to that of $\mathbf{Z}^2$. Moreover, it maps the Cayley generators of G_3 to those of the infinite triangular grid: $\phi(e_1) = (1, 0)$, $\phi(e_3) = (0, 1)$, $\phi(e_2) = -(1, 1)$. Therefore, ϕ is a covering of the infinite triangular grid with covering graph G_3 and from Proposition 1 we obtain the following.

Construction 4. If C is a CRC in the infinite triangular grid then

$$\{(x_1, x_2, x_3) \in \mathbf{Z}^3 : (x_1 - x_2, x_3 - x_2) \in C\}$$

is a CRC in G_3 with the same parameter matrix as C .

The construction below was firstly described by Fon-der-Flaas in [11] for Hamming graphs. It can be extended to Cayley graphs of abelian groups such as infinite grid.

Construction 5. Let C be a CRC in the graph G_n (or $H(n, q)$). For integer $k \geq 1$, $x \in \mathbf{Z}^{kn}$ we define $\phi(x) = (\sum_{i=1, \dots, k} x_i, \dots, \sum_{i=1, \dots, k} x_{(n-1)k+i})$. The code $\{x : \phi(x) \in C\}$ is a CRC in G_{kn} (or $H(kn, q)$) with parameter matrix kA .

We finish this subsection with the following important result.

Theorem 2. [4, Theorem 2] Let D be a CRC with parameter matrix A in G_n such that $c_i = c_j$, $a_i = a_j$ and $b_i = b_j$ for some $1 \leq i < j \leq \rho - 1$. Then $c_l = c_m$, $a_l = a_m$ and $b_l = a_m$ for any $1 \leq l \neq m \leq \rho - 1$ and CRC D is obtained by Construction 5 from a CRC in G_1 .

3.2. Null-CRC codes. The important classes of optimal error-correcting codes in G_n include perfect and diameter perfect codes. A perfect code (also called a 1-perfect code) in a k -regular graph is a set of vertices that intersects every ball of radius one in the graph in exactly one vertex. Equivalently, a perfect code is a CRC with parameter matrix $\begin{pmatrix} 0 & k \\ 1 & k-1 \end{pmatrix}$. Such codes

were shown to exist in G_n for any $n \geq 1$ in the work of Golomb and Welsh [12].

Construction 6. *The collection of even weight words of a perfect code in G_n is a completely regular code with parameter matrix*

$$\begin{pmatrix} 0 & 2n & 0 & 0 \\ 1 & 0 & 2n-1 & 0 \\ 0 & 2n-1 & 0 & 1 \\ 0 & 0 & 2n & 0 \end{pmatrix}.$$

A code C is called a one-error-correcting diameter perfect code (shortly, diameter perfect code) in G_n if the union of any two balls of radius 1 with adjacent centers contains exactly one vertex of C [10]. In [10, Section II E] the existence of such subgroup codes in G_n for any $n \geq 1$ was settled. In particular, because the codes are subgroups of $\mathbf{Z}^n$, there is a partition of G_n into disjoint diameter perfect codes. The union of t disjoint such codes with words of the same overall parity check forms a completely regular, see [14, Proposition 7.23]. Such completely regular codes exist for all $t, n \geq 1$.

Construction 7. *The union of t disjoint diameter perfect codes in G_n with words of even weight is completely regular with parameter matrix*

$$\begin{pmatrix} 0 & 2n & 0 \\ t & 0 & 2n-t \\ 0 & 2n & 0 \end{pmatrix}.$$

As the Hamming graph is distance-regular, its distance partition with respect to a singleton vertex is a perfect coloring. From Construction 3 we obtain the following.

Construction 8. *The code $\{x : x_i \bmod 4 = 0, i \in \{1, \dots, n\}\}$ is an all-nulls CRC in G_n with covering radius $\rho = 2n$ and parameter matrix*

$$\begin{pmatrix} 0 & 2n & 0 & 0 & \dots & 0 \\ 1 & 0 & 2n-1 & 0 & \dots & 0 \\ 0 & 2 & 0 & 2n-2 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & 2n-1 & 0 & 1 \\ 0 & 0 & \dots & 0 & 2n & 0 \end{pmatrix}$$

By merging the code and its opposite in the above construction we obtain.

Construction 9. *The code $\bigcup_{a=0,2}\{x : x_i \bmod 4 = a, i \in \{1, \dots, n\}\}$ is an all-nulls CRC in G_n with covering radius $\rho = n$ and parameter matrix*

$$\begin{pmatrix} 0 & 2n & 0 & 0 & \dots & 0 \\ 1 & 0 & 2n-1 & 0 & \dots & 0 \\ 0 & 2 & 0 & 2n-2 & \dots & 0 \\ & & & \dots & & \\ 0 & & \dots & n-1 & 0 & n+1 \\ 0 & & \dots & 0 & 2n & 0 \end{pmatrix}$$

We finish this section with an example of an all-ones CRC. If an all-nulls CRC in $H(2n-1, 2)$ with parameter matrix A exists, then the code

$$\{(x_1, \dots, x_{2n-1}, a) : x \in C, a \in \{0, 1\}\}$$

is a CRC in $H(2n, 2)$ with the parameter matrix $A + E$ [11]. For C being a singleton of $H(2n-1, 2)$ we obtain the following from Construction 3.

Construction 10. *For any n there is all-ones CRC in G_n with $\rho = 2n-1$ and parameter matrix*

$$\begin{pmatrix} 1 & 2n-1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 2n-2 & 0 & \dots & 0 \\ 0 & 2 & 1 & 2n-3 & \dots & 0 \\ \dots & & & & & \\ \dots & & & & & \\ 0 & 0 & \dots & 0 & 2n-1 & 1 \end{pmatrix}.$$

4 Local structure of n -dimensional infinite grid

Let $u \in \mathbf{Z}^n$ be a word of weight not more than 4. We say u is of type $abcd$ if the sequence of the absolute values of symbols of u in descending order starts with $abcd$. For example, the word (1000-21) is of type 2110. The type of a word u is called Lee composition vector of u in [17]. Clearly, the sets of the words having fixed type and weight are orbits of the stabilizer of all-zero word in the automorphism group of the graph G_n . The weight two words are only of types 2000 and 1100. Weight three and four words are split into types 3000, 2100, 1110 and types 4000, 3000, 2110, 2200, 1111 respectively.

Let a cap be the set of all weight four words with fixed value 3 or -3 in a fixed position, i.e. caps are the following sets:

$$\{3e_i \pm e_j : j \in \{1, \dots, n\} \setminus \{i\}\},$$

$$\{-3e_i \pm e_j : j \in \{1, \dots, n\} \setminus \{i\}\}, \quad i = 1, \dots, n.$$

We see that the set of all type 3100 words is the disjoint union of all caps.

Define the interval graph $I(x)$ of a vertex x as the subgraph of G_n induced by the following vertex set $\{y : d(y, x) + d(y, \mathbf{0}) = d(x, y)\}$. Up to types, the interval subgraphs for all weight four vertices are depicted on Fig.1.

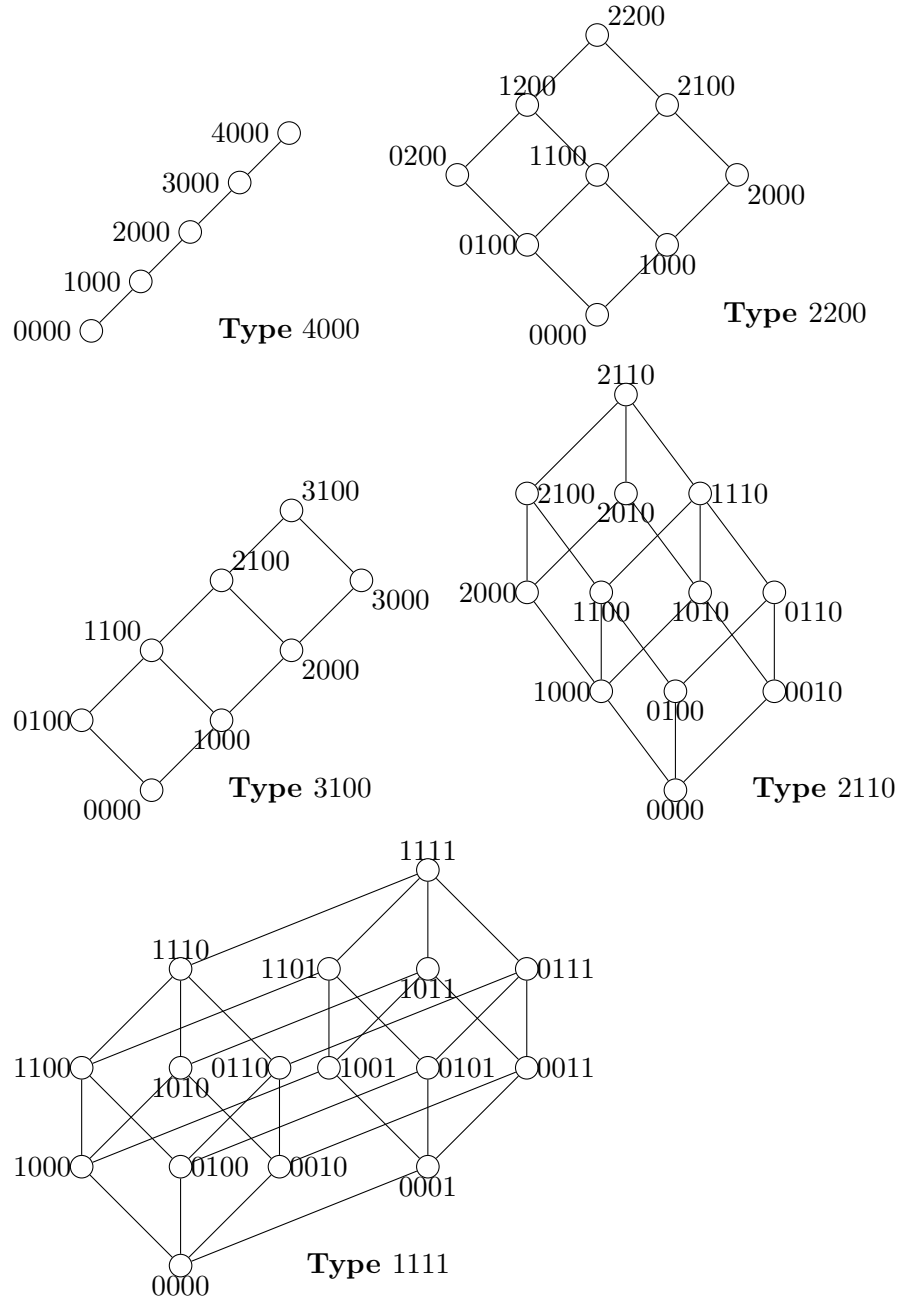


Fig. 1. The interval graphs for weight four words. The last $n - 4$ all-zero symbols are omitted in the figures throughout the paper for simplicity.

5 Null CRCs with $c_2 \leq 3$

In the current section classes of 2-null and 3-null CRCs are considered. We start with 2-null CRCs such that $c_1 < c_2 \leq 3$ (note that c_1 is necessarily not greater than c_2 due to Theorem 1). Throughout the current and next two sections we assume that the all-zero word $\mathbf{0}$ is in C .

Remark 2. 1. Since $\mathbf{0}$ is in C , when $a_0 = 0$, the weight one words are all in C_1 . If, moreover, $a_1 = 0$, then weight two words are a subset of $C \cup C_2$. The condition $c_1 = 1$, $a_0 = a_1 = 0$ is equivalent to the minimum distance of C being at least four. Since the minimum distance of any CRC in G_n is not greater than four [4, Theorem 4] it is exactly four.

2. Note that if a weight four word x is in a 2-null CRC C , then, given that $a_0 = a_1 = 0$, all weight one and three words in the interval graph of x are in C_1 and weight two words are in C or C_2 .

In case of a 2-null CRC C , any type 1100 word is adjacent to two words of weight one, which are in C_1 , we obtain the following.

Lemma 1. *If C is a 2-null CRC in G_n , $n \geq 2$ with $\rho \geq 2$ then any type 1100 word in C_2 is adjacent to exactly $c_2 - 2$ weight three words in C_1 .*

The following statement was announced in [3].

Corollary 1. *Any 2-null CRC in G_n , $n \geq 1$ with $\rho \geq 2$, $c_1 = 1$ and $c_2 = 2$ is obtained by Construction 3 from a CRC in $H(2n, 2)$.*

Proof. Note that since $c_1 = 1$, all weight two and three words in the interval graph of any word of weight four from C are in C_2 and C_1 respectively.

In G_n , any word x of type 2000 e.g. $(20 \dots 0)$ (which is in C_2) is adjacent to only one word of weight one, $(10 \dots 0)$ (which is in C_1). Since c_2 is 2 by Lemma 1, x is at distance two from exactly one nonzero codeword y and by Remark 2.1 y cannot be of weight two, so it is of weight four. The word y cannot be of types 3100 or 2110. Indeed, from Fig.1 the interval graphs of these words contain a word of type 1100 which by Remark 2.1 is in C_2 . In the interval graphs of types 3100 or 2110, this word is adjacent to at least three words of weights one and three. Since, by Remark 2.2, all these words are in C_1 , this contradicts the parameter $c_1 = 2$. We conclude that the word y is $(40 \dots 0)$ of type 4000. Extending this argument to all type 2000 words we conclude that the weight four words in C are exactly all type 4000 words. Thus, we see that the code has period four and is obtained by Construction 3. □

Lemma 2. *Let C be a 2-null CRC in G_n , $n \geq 2$ with $\rho \geq 2$, $c_1 \leq 2$ and $c_2 \geq 3$. Then any word of type 1100 in C_2 is in the interval graph of a weight four word in C .*

Proof. By Lemma 1, any type 1100 word in C_2 is adjacent to a word z of weight three in C_1 . If z is adjacent to a word of weight four in C , we are

done, so we suppose that z is adjacent only to weight two words in C . By Remark 2.1, the case $c_1 = 1$ implies that the minimum distance of C is four, a contradiction. So, c_1 is 2 and z is adjacent to exactly two words of weight two in C . Up to automorphism of G_n , the word z is $(3000 \dots 0)$, $(2100 \dots 0)$ or $(1110 \dots 0)$. The word $(3000 \dots 0)$ is adjacent to only one weight two word in G_n , so it is not z . The word $(2100 \dots 0)$ is adjacent to only two words of weight two: $(1100 \dots 0)$ and $(2000 \dots 0)$ and we see that $(1000 \dots 0) \in C_1$ is adjacent to three words in C , a contradiction with $c_1 = 2$. Similarly, in the case when z is $(1110 \dots 0)$, up to permutation of three first positions, the weight two neighbors of z are $(1100 \dots 0)$ and $(1010 \dots 0)$, giving the same contradiction with $c_1 = 2$ for the word $(10 \dots 0)$ in C_1 . $\square$

Lemma 3. *Let C be any 2-null CRC in G_n , $n \geq 2$, with $\rho \geq 2$, $c_1 \leq 2$ and $c_2 = 3$. Then the following holds:*

1. *Any cap contains not more than one word in C . In particular, there are not more than $2n$ type 3100 words in C .*
2. *Any type 1100 word in C_2 is at distance two from exactly one word in C of weight four, which is necessarily of type 3100.*

Proof. 1. The statement is obvious for the case $c_1 = 1$ because by Remark 2.1 the minimum distance of C is four and the distance between any distinct words in any cap is two.

Let c_1 be 2. Suppose the opposite, up to an automorphism of G_n , we have two cases.

Case A. Let the words $(3100 \dots 0)$ and $(3-10 \dots 0)$ be in C . Therefore, the words $(2100 \dots 0)$, $(3000 \dots 0)$ and $(2-100 \dots 0)$ are in C_1 because C is a 1-null code.

We see that the word $(200 \dots 0)$, which is in C or in C_2 due to Remark 2.2, has four neighbors from C_1 , namely $(100 \dots 0)$, $(2100 \dots 0)$, $(3000 \dots 0)$ and $(2-100 \dots 0)$. Therefore, since $c_2 = 3$, the word $(200 \dots 0)$ having four neighbors in C_1 cannot be in C_2 and it is in C . However, the word $(3000 \dots 0) \in C_1$ is adjacent to three words of C , $(310 \dots 0)$, $(3-10 \dots 0)$ and $(200 \dots 0)$, which contradicts $c_1 = 2$.

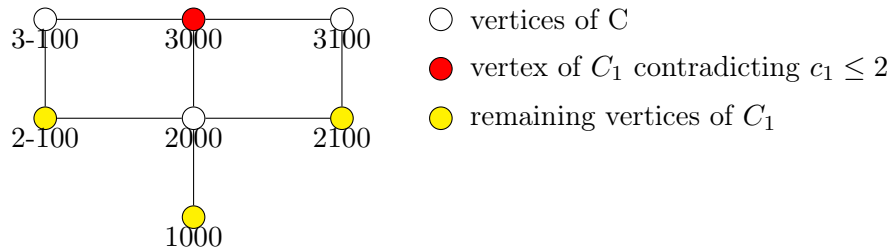


Fig. 2. The contradiction for Lemma 2.1, Case A.

Case B. Let the words $(3100 \dots 0)$ and $(301 \dots 0)$ be in C . Similarly to case A, we see that the word $(200 \dots 0)$ is adjacent to $(100 \dots 0)$, $(2100 \dots 0)$,

$(3000 \dots 0)$ and $(2010 \dots 0)$ and therefore it is in C , which implies the same contradiction for $(3000 \dots 0) \in C_1$ as in Case A.

2. Consider a word x of type 1100 in C_2 . Since $c_2 = 3$, by Lemma 2 the word x is at distance two from exactly one weight four word y in C . By examining the interval graphs on Fig. 1, we see that y is of type 3100 or 2110. Indeed, any type 1100 word in the other interval graphs is adjacent to two words of weight three, which are in C_1 by Remark 2.2, providing contradiction to Lemma 1. Let y be of type 2110, without restricting the generality, it is $(2110 \dots 0)$ and we seek a contradiction. Consider the interval graphs of type 2110 word y in Fig. 3. We see that $x \in C_2$ is already adjacent to three words from C_1 in the interval graph of y and $(1100 \dots 0)$ and $(1010 \dots 0)$ are adjacent to four words from C_1 in $I(y)$. We obtain that x can only be $(0110 \dots 0)$.

Case $c_1 = 1$. Then the minimum distance of C is four. We immediately obtain a contradiction since two more type 1100 words: $(110 \dots 0)$, $(101 \dots 0)$ are in the interval graph of y and they belong to C_2 , contradicting $c_2 = 3$, see Fig. 3.

When c_1 is 2 by Remark 2.2 there are weight two words from C and we see that both $(1100 \dots 0)$ and $(1010 \dots 0)$ must be in C (otherwise we meet the same contradiction as in case $c_1 = 1$). However, in this case, $(1000 \dots 0) \in C_1$ is adjacent to three words in C and we obtain a contradiction with $c_1 = 2$ (see Fig. 4).

□

Theorem 3. *A 2-null CRC in G_n , $n \geq 1$ with $\rho \geq 2$, $c_1 = 1$ and $c_2 = 3$ exists if and only if it is obtained by Construction 6 for $n = 2$. There are no 2-null CRCs in G_n with $c_1 = 2$, $c_2 = 3$ for any n .*

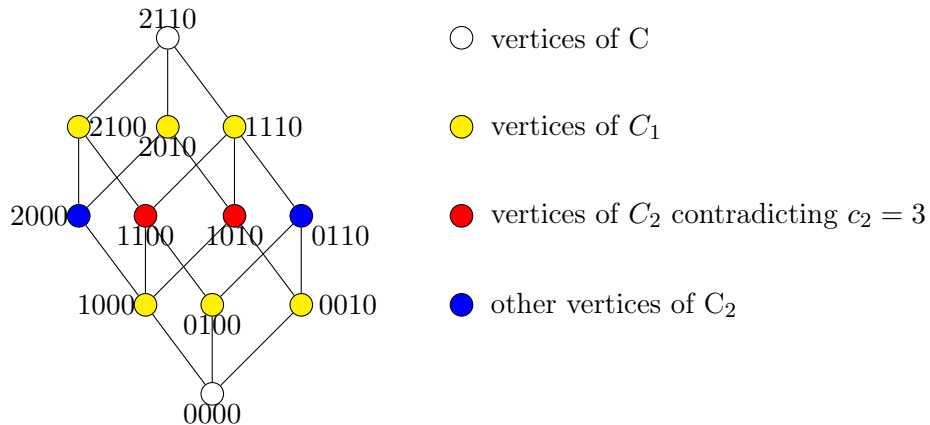


Fig 3. The final contradiction in Lemma 4.2 for $c_1 = 1$.

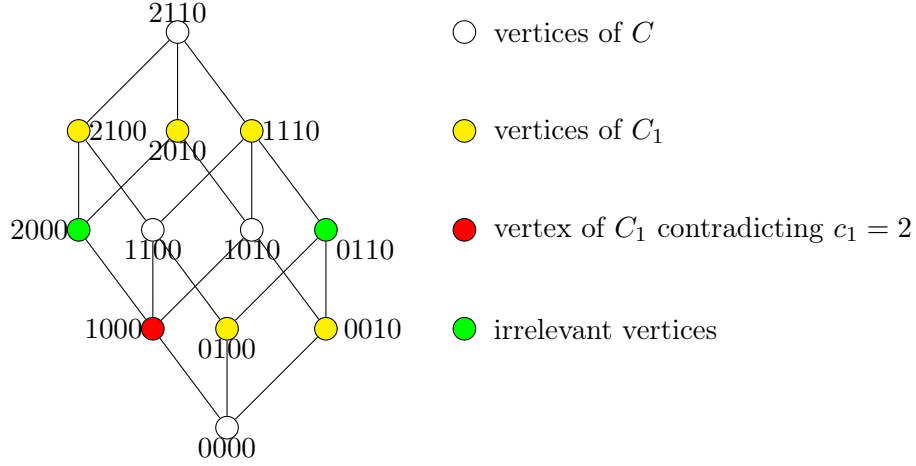


Fig 4. The final contradiction in Lemma 4.2 for $c_2 = 2$.

Proof. We show that there are at least $4\binom{n}{2} - n$ type 1100 words in C_2 .

We have $4\binom{n}{2}$ words of type 1100 in G_n and for $c_1 = 1$ all of them are in C_2 . For $c_1 = 2$ let β of them be in C . By double counting of the edges between these words and the weight one words, we obtain that $\beta \leq n$. We see that there are at least $4\binom{n}{2} - n$ type 1100 words in C_2 . By Lemma 3.2 the number of these words is not more than the number of type 3100 words in C . As by Lemma 3.1, there are not more than $2n$ type 3100 words in C , we obtain that $4\binom{n}{2} - n \leq 2n$, which holds only for $n \leq 2$.

The case $n = 1$ for $c_2 = 3$ is not possible. All parameter matrices of CRCs in G_2 were found in [2] and there are no codes with $a_0 = a_1 = 0$ and $c_1 = 2$, $c_2 = 3$ and only one parameter matrix with $a_0 = a_1 = 0$ and $c_1 = 1$, $c_2 = 3$, namely

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$

and the respective CRCs are obtained by Construction 6.

□

Theorem 4. *The only 3-null CRCs in G_n $n \geq 1$ with $\rho \geq 2$ and $c_2 \leq 3$ to exist are:*

1. CRCs in G_1 with any $\rho \geq 2$, $a_0 = a_1 = 0$, $c_i = b_i = 1$, $i = 1 \dots, \rho - 1$, $a_\rho = 0$ and $\rho \geq 3$, $a_0 = a_1 = 0$, $c_i = b_i = 1$, $i = 1 \dots, \rho - 1$, $a_\rho = 1$.
2. The CRCs obtained from these codes in G_2 with $c_1 = c_2 = 2$ and in G_3 with $c_1 = c_2 = 3$ by Construction 5.
3. CRCs with $c_1 = 1$, $c_2 = 2$ obtained by Construction 3.
4. CRCs with $c_1 = 1$, $c_2 = 3$ obtained by Construction 6 for $n = 2$.

Proof. The parameters of CRCs in G_1 were obtained in [2] and the only 3-null codes in G_1 are series with parameters as in the first and second statement of Theorem.

By Theorem 1, the sequence of elements below the main diagonal of parameter matrix is not decreasing. If $c_1 = c_2$ and $\rho \geq 3$ then because $a_1 = a_2 = 0$, we have $b_1 = b_2$ and then due to Theorem 2 this implies that C in G_n is obtained from Construction 5 applied to a CRC in G_1 , where n is a divisor of all entries of parameter matrix. For $c_1 = c_2 = 2$ or 3 this implies that n is 2 and 3 respectively. If $c_1 = c_2$ and $\rho = 2$ then, since $a_1 = a_2 = 0$ we obtain that $c_1 = c_2$ is equal to graph valency $2n$, which cannot occur.

The case $c_1 < c_2$ follows from Corollary 1 and Theorem 3. This case corresponds to the third and fourth statements of Theorem. $\square$

6 1-null CRCs with $a_1 = 1$

Theorem 5. *Any CRC in G_n , $n \geq 1$ with $\rho \geq 1$, $a_0 = 0$ and $a_1 = 1$ has $c_1 = 1$ and is obtained by Construction 2.*

Proof. Since $a_0 = 0$, all words of weight one are in C_1 , each of which has exactly one weight two neighbor in C_1 because $a_1 = 1$. Moreover, if there is a word of type 1100 as such a neighbor, then $a_1 > 1$, a contradiction. We conclude that all words of type 2000 are also in C_1 . By the same arguments, all words of type 3000 are in C and C has period three in any direction and therefore C arises from a CRC in $H(n, 3)$ by Construction 2.

Consider the respective code in $H(n, 3)$. The graph induced by the ball of radius 1 in $H(n, 3)$ is the union of n cliques of size 3 any two of which have only one common vertex, the center of the ball. In case when the center of the ball is in C_1 , because $a_1 = 1$, only one of the cliques, say K , contains an extra vertex of C_1 . The remaining cliques contain words of C and C_2 , but because a_0 is 0, actually only words of C_2 . We are left with the case where there is only one vertex in the ball from C in the clique K , thus $c_1 = 1$. $\square$

A clear example of a ternary CRC meeting Theorem 5 is a singleton vertex code. The database [14, Tables 7.9–7.11] mentions CRCs in $H(n, 3)$ that satisfy Theorem 5 are the ternary Golay code and its punctured, shortened, and extended versions [6]. Handbooks such as [7], [8] reference linear completely regular codes in Hamming metric via their associated distance-regular graphs; notably, the ternary [56, 50, 4] Hill-cap code with parameters triples [0, 112|1, 1, 110|20, 92]. [8, Section 10.75] is completely regular and also fulfills Theorem 5.

7 All-ones CRC

In this section we study all-ones CRCs. In this case, each connected component of C_i , $i = 0, \dots, \rho$, consists of two adjacent words. Throughout this section we assume that the all-zero word $\mathbf{0}$ and the word $-e_n$ are in C .

We say that the following set of words in the graph G_n

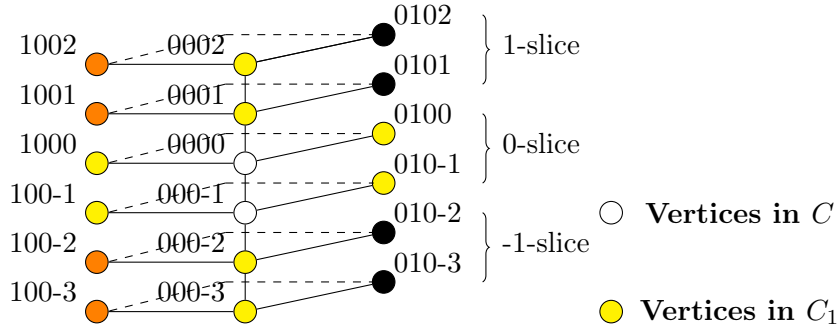
$$\{(x_1, \dots, x_n) : x_1, \dots, x_{n-1} \in \mathbf{Z}, x_n \in \{2s, 2s-1\}\}$$

is the s -slice. We call a collection of words x and $x - e_n$ in s -slice a slice pair. An n -line is a set of all words in G_n that differ only in the n -th symbol. We now show that any n -line in G_n is a collection of slice pairs, whose words are at alternating distances from an all-ones CRC C , forming domino-like structure, see Fig. 5.

Lemma 4. *Let C be an all-ones CRC in G_n , $n \geq 1$. The following holds.*

1. *Two words in any slice pair are in C_i , for some $i \in \{0, \dots, \rho\}$.*
2. *If an n -line contains words of C_i , then the words from C_i in the direction n have period four and are split into slice pairs. In particular, any n -line consists of slice pairs from C_i, C_{i+1} , for some $i \in \{0, \dots, \rho-1\}$.*

Proof. Firstly, we note that the existence of a slice pair in C_i for some $i \in \{0, \dots, \rho\}$ implies that the words in the slice are split into slice pairs at the same distance from C . Indeed, let x and $x - e_n$ be a slice pair in the s -slice for some s in $\mathbf{Z}$, both from C_i , for some $i \in \{0, \dots, \rho\}$. Consider the cycles of length four containing the slice pair: $x, x - e_n, x + ae_j, x + ae_j - e_n$, $a \in \{-1, 1\}$, $j \in \{1, \dots, n-1\}$, where all words are in the s -slice. The words $x + ae_j, x + ae_j - e_n$ are not in C_i because x and $x - e_n$ are in C_i and C is an all-ones CRC. Then they are both in C_{i-1} or in C_{i+1} . Extending this argument for $x + ae_j, x + ae_j - e_n$ instead of x and $x - e_n$ we obtain the required.



● All orange vertices are either in C or in C_2 simultaneously

● All black vertices are either in C or in C_2 simultaneously

Fig 5. Domino property for all-ones CRC, Lemma 4.

1. By convention, $\mathbf{0}$ and $-e_n$ are in C , so by the above the 0-slice is split in to slice pairs with both words at the same distance from C .

Because CRC C has the parameter $a_0 = 1$ the following neighbors of $\mathbf{0}$ and $-e_n$: $ae_i, ae_i - e_n, i \in \{1, \dots, n-1\}, a \in \{-1, 1\}$ are in C_1 . Furthermore, because C has $a_1 = 1$, e_n and $-2e_n$ from C_1 have unique neighbors in C_1 . We show that they are $2e_n$ and $-3e_n$ respectively. Indeed, suppose the opposite and there is a neighbor of e_n , $e_n + ae_i, a \in \{-1, 1\}, i \in \{1, \dots, n-1\}$ which is in C_1 . Then $e_n + ae_i$ in C_1 is adjacent to two words in C_1 , namely e_n and ae_i , a contradiction with the parameter $a_1 = 1$ of CRC C . Therefore, the neighbor of e_n in C_1 can only be $2e_n$. Similar argument for $-2e_n$ shows that $-3e_n$ is in C_1 . By the remark at the beginning of the proof we see that the words in any slice pairs in the 1-slice and in -1 -slice are at the same distance from C .

We extend this argument in s -slices for all $s \in \mathbf{Z}$ to obtain the required.

2. Obviously, any word in a slice pair x and $x - e_n$ from C_i is adjacent to the same number of words from C_{i-1} (C_i and C_{i+1} respectively) in the s -slice. By the first statement of Lemma, $x + e_n, x + 2e_n, x - 2e_n$ and $x - 3e_n$, are in C_{i-1} or C_{i+1} simultaneously. The same argument applied to $x + e_n, x + 2e_n$ implies the period of four for C_i in any n -line. $\square$

From Lemma 4.2 there are two code edges at distance 3 in any n -line with a code word. We now compute the minimum distance between code slice pairs in any slice.

Lemma 5. *Let C be all-ones CRC in G_n , $n \geq 2$ with $\rho \geq 1$ and $c_1 = 1$. Then the minimum distance between slice pairs from C in the same slice is at least 4.*

Proof. Let d be the minimum of the distances between x and y , for all $x \in A, y \in B$ and all distinct slice pairs A, B from C in the same slice. By Lemma 4.1 we see that $d \neq 1$. Considering the shortest path between the closest x and y in C , we see that if $d = 2$, then parameter $c_1 \geq 2$ and if $d = 3$ then $a_1 \geq 2$, a contradiction. $\square$

Remark 3. *Let C be an all-ones CRC in G_n with $c_1 = 1$.*

1. *Note that no weight two word is in C . Otherwise, there is a weight one word adjacent to such a word and all-zero word. This weight one word cannot be in C or C_1 because $a_0 = c_1 = 1$.*

2. *Consider the interval graph of weight four word y from C . From the above, taking into account that $\mathbf{0}$ is in C if type 1100 word is in C_2 , all its neighbors in the interval graph of y are from C_1 .*

The discussion below follows the narrative of Section 5.

Theorem 6. *Any all-ones CRC in G_n , $n \geq 1$ with $\rho \geq 2$, $c_1 = 1$ and $c_2 = 2$ is obtained by Construction 3.*

Proof. Firstly, we note that all type 2000 words in the 0-slice are in C_2 . Indeed, by our convention ae_j is in C_1 for all $a \in \{-1, 1\}, j \in \{1, \dots, n-1\}$ and by Lemma 4.1 the slice pair $ae_j, ae_j - e_n$ is in C_1 . The type 2000 word

$2ae_j$ cannot be in C_1 because otherwise $ae_j \in C_1$ is adjacent to two words in C_1 , contradicting $a_1 = 1$. Moreover, by Remark 3.1, $2ae_j$ is not in C , so all type 2000 words in the 0-slice are in C_2 .

Thus, any word from the 0-slice of type 2000 is in C_2 , which given that $c_2 = 2$, implies it must be at distance two from a codeword y different from $\mathbf{0}$. By Remark 3.1, y cannot be of weight two, so it is of weight four. We observe that y is of type 4000 (in the 0-slice) or of types 2200, 3100, 2110 (in the same slice or not). We will show that the last three cases are impossible.

Consider the interval graph $I(y)$, $y \in C$. Since $\mathbf{0}$ and y are in C , all weight one and three words in the graph $I(y)$ are from $C \cup C_1$. Due to Remark 3.1, the words of weight two in the graph $I(y)$ are in $C_1 \cup C_2$. Similarly to Corollary 1, in case when y is of types 2200, 3100, 2110, we see from Fig. 1. that there is always a type 1100 word z in $I(y)$ having at least three neighbors in $I(y)$. If the word z is in C_2 , by Remark 3.2, moreover, all neighbors of z in $I(y)$ are in C_1 , providing contradiction to $c_2 = 2$. If z is in C_1 , by pigeon hole principle at least two its neighbors are either in C or in C_1 , providing contradiction to $a_1 = c_1 = 1$.

Finally, we conclude that any word of type 4000 in the 0-slice is in C . Since the period of C is four in the n th direction by Lemma 4.2, this implies the period four for C in all n directions. $\square$

Remark 4. *In view of Construction 3 the code in G_n from Theorem 6 corresponds to a CRC in the Hamming graph $H(2n, 2)$. Lemma 4 states that the code is split into pairs with only difference in the last symbol taking values 0 or 3 modulo 4. This implies that the code in $H(2n, 2)$ is obtained from a code D as $\{(x, a) : x \in D, a \in \{0, 1\}\}$, i.e. by adding nonessential symbol using the approach described prior to Construction 10. In turn, D is an all-nulls CRC in $H(2n - 1, 2)$.*

Lemma 6. *Let C be an all-ones CRC in G_n , $n \geq 2$ with $\rho \geq 2$, $c_1 = 1$ and $c_2 = 3$. Then the following holds:*

1. *The intersection of any cap with the 0-slice contains at most one word in C . In particular, there are not more than $2(n - 1)$ type 3100 words of 0-slice in C .*
2. *Any type 1100 word x with $x_n = 0$ is in C_2 and is at distance two from exactly one type 3100 word y in C such that $y_n = 0$.*

Proof. 1. The statement holds because by Lemma 5 the minimum distance between slice pairs in any slice is at least four and the distance between any distinct words in any cap is two.

2. Let x be a type 1100 word with $x_n = 0$. Both of its weight one neighbors are in C_1 , because the only weight one codeword is $-e_n$. We conclude that x cannot be in C_1 because a_1 is 1. It also cannot be in C by Remark 3.1 and so it is in C_2 . Since c_2 is 3, the word x is at distance two from exactly one nonzero word y in C . By Remark 3.1 there are no codewords of weight two, so y in C has weight four.

Case A. Let the word y be of types 2200, 2110 or 1111. As in the proof of Theorem 6, the weight one and three words from $I(y)$ are in $C \cup C_1$ and weight two words from $I(y)$ are in $C_1 \cup C_2$.

From Fig. 1, the interval graph $I(y)$ always contains a word z of type 1100 having four neighbors in the graph $I(y)$. If z is in C_2 , by Remark 3.2, all four weight one and three neighbors of $z \in C_2$ in $I(y)$ are in C_1 , contradicting $c_2 = 3$. If z is in C_1 at least two its neighbors in $I(y)$ are in C or C_1 , contradicting $a_1 = c_1 = 1$.

Case B. Let y be of type 3100 at distance two from type 1100 word x , $x_n = 0$. Then obviously $y_n = 0$, y is in the 0-slice and we obtain the required. $\square$

Theorem 7. *An all-ones CRC in G_n , $n \geq 1$ with $\rho \geq 2$, $c_1 = 1$ and $c_2 = 3$ exists if and only if n is 2.*

Proof. We have $4\binom{n-1}{2}$ words of type 1100 with the zero last symbol and by Lemma 6.2 all of them are in C_2 . By Lemma 6.2 there is a bijection between these words and type 3100 codewords with the zero last symbol. Since by Lemma 6.1 there are at most $2(n-1)$ such words, we conclude that $4\binom{n-1}{2} \leq 2(n-1)$, i.e., $n \leq 2$. The case $n = 1$ and $c_2 = 3$ is impossible and there is only one all-ones parameter matrix of the CRCs in G_2 with $c_1 = 1$

and $c_2 = 3$ [2], namely $\begin{pmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$. $\square$

8 Binary linear programming for null CRC in G_3 and G_4

All parameter matrices for completely regular codes in G_2 were enumerated in [2], so the first open cases occur for $n = 3$ and $n = 4$. These results are consistent with the characterization of parameters of completely regular codes with covering radius 2 in [16]. The discussion below focuses on CRCs in the infinite rectangular grid, but we also state the key points for arbitrary vertex-transitive graphs with an infinite number of vertices.

The "perfect coloring" equation for a completely regular code C with parameter matrix A in a finite graph of order v with adjacency matrix M can be formulated as a classical 0-1 linear programming problem. We associate the elements of the distance partition $C_0 = C, \dots, C_\rho$ with their characteristic vectors $\chi_0, \dots, \chi_\rho$. The existence problem for a completely regular code thus becomes the following 0-1 linear programming problem.

$$\chi_0, \dots, \chi_\rho \in \{0, 1\}^v, \chi_0 + \dots + \chi_\rho = \mathbf{1}, M(\chi_0 \dots \chi_\rho) = (\chi_0 \dots \chi_\rho)A.$$

We now outline two main differences from the original integer linear programming approach discussed above.

Firstly, since G_n (or, in general, vertex-transitive graph Γ) is a graph with infinitely many vertices, the search space must be restricted to a certain finite subgraph. In previous work, a computer-based enumeration of perfect colorings with 9 colors [13] was conducted using a finite subgrid as the search domain. During our testing, we found that the results for our codes using

linear programming are more accurate (i.e., more parameter matrices are excluded) when the subgraph is induced by a ball.

Let M denote the adjacency matrix of the subgraph Γ' of G_n of order v , induced by the vertices in G_3 or G_4 of weight at most six. We denote by J the set of all vertices of Γ' that have degree 6 or 8 in Γ' (i.e., the valencies of G_3 and G_4 , respectively). Let P_J be the binary diagonal matrix of size v such that $(P_J)_{x,x} = 1$ if and only if $x \in J$. Let E denote the identity matrix of size v .

We now express the above considerations in matrix form as follows:

$$LP': \begin{cases} \chi_0, \dots, \chi_\rho \in \{0, 1\}^v, \chi_0 + \dots + \chi_\rho = \mathbf{1}, \\ P_J M(\chi_0 \dots \chi_\rho) = P_J(\chi_0 \dots \chi_\rho) A, \\ (E - P_J) M(\chi_0 \dots \chi_\rho) \leq (E - P_J)(\chi_0 \dots \chi_\rho) A, \\ \chi_0[0] = 1. \end{cases}$$

Here, we index all vertices of Γ' with indices $0, \dots, v-1$ and due to the vertex transitivity of the graph G_n , we assume the zeroth variable (e.g., the all-zero word) to be in $C = C_0$, or equivalently, $\chi_0[0] = 1$. In general, the respective linear programming problem LP' for a CRC with a given parameter matrix can be considered for any subgraph Γ' of any vertex-transitive graph. If the linear programming problem LP' is infeasible, then a CRC with parameter matrix A does not exist in G_n (vertex transitive graph Γ). On the other hand, note that the feasibility of the problem LP' does not imply the existence of a completely regular code in Γ .

The second feature is added to eliminate several parameter matrices (possibly infinitely many, depending on the graph) for CRCs in infinite graphs. The covering radius of CRCs in G_n can be relatively large (up to $2n+1$ for irreducible CRCs [4]). To eliminate several CRCs that share the same submatrix of parameter matrix, we propose the following relaxation. For $r \leq \rho$ let B denote the upper-left $(r+1) \times r$ submatrix of parameter matrix A of the CRC. We introduce additional variables by defining the sum of all χ_i for $i \geq r+1$ as F in the following binary integer linear programming problem.

$$LP_{\geq r}: \begin{cases} \chi_0, \dots, \chi_r, F \in \{0, 1\}^v, \chi_0 + \dots + \chi_r + F = \mathbf{1}, \\ P_J M(\chi_0 \dots \chi_{r-1}) = P_J(\chi_0 \dots \chi_r) B, \\ (E - P_J) M(\chi_0 \dots \chi_{r-1}) \leq (E - P_J)(\chi_0 \dots \chi_r) B, \\ \chi_0[0] = 1, \\ \sum_{x=0}^{v-1} F[x] \geq 0. \end{cases}$$

Variations of the problem $LP_{\geq r}$ are obtained by replacing the last equation with either $\sum_{x=0}^{v-1} F[x] = 0$ or $\sum_{x=0}^{v-1} F[x] \geq 1$. We denote these modified problems by $LP_{=r}$ and $LP_{>r}$, respectively.

Note that the problem $LP_{=r}$ for a matrix B is equivalent to a coloring of the ball using exactly $r + 1$ colors such that the coloring is consistent with parameter matrix obtained from B by appending a column of row sums to the right. Below, we refer to this augmented matrix as the extension of B .

If a matrix B' with at least $r + 2$ rows has a $(r + 1) \times r$ upper-left submatrix B , we call B' a descendant of B . If the problem $LP_{\geq r}$ is infeasible for the matrix B , then it is also infeasible for all descendants of B and no CRCs in G_n exist with covering radius at least r whose parameter matrix has B as its upper-left submatrix.

We summarize the above discussion as follows.

Proposition 2. *Let B be a $(r + 1) \times r$ matrix and let M be the adjacency matrix of a subgraph Γ' of a vertex-transitive graph Γ with v vertices and valency t . Let J be the set of vertices in the subgraph Γ' that have valency t and let P_J be the $v \times v$ diagonal matrix defined by $(P_J)_{x,x} = 1$ if $x \in J$ and 0 otherwise. Then the following holds:*

1. *If the integer problem $LP_{\geq r}$ (respectively, $LP_{>r}$) is infeasible, then there are no completely regular codes (CRCs) in Γ with covering radius at least (respectively, greater than) r whose parameter matrix contains B as its upper-left $(r + 1) \times r$ submatrix.*

2. *If the integer problem $LP_{=r}$ is infeasible, then there is no CRC in Γ with covering radius exactly r whose full parameter matrix is the extension of B .*

In the subsequent considerations, we restrict descendants to those satisfying the ascending and descending conditions for the entries above and below the diagonal, as described in Theorem 1.

Example 1. Let B be $\begin{pmatrix} 0 & 6 \\ 1 & 0 \\ 0 & 5 \end{pmatrix}$, n is 3. The extended matrix of B is $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 1 \end{pmatrix}$. Taking into account restrictions imposed by Theorem 1,

the 4×3 descendants of B are $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$,

$\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. We see that the last two matrices cannot be submatrices of

parameter matrices of CRCs, because of the graph degree is six and third rows, the set C_3 is disjoint to C_2 .

Theorem 8. *Up to opposite code, any 1-null CRC in G_3 has one of the parameters from Table 1.*

TABLE 1. The parameters of 1-null CRCs in G_3

Parameters	Construction
$[0, 6 1, 5]$	A perfect code in G_3
$[0, 6 2, 4]$	A CRC in $H(6, 2)$ from a perfect code in $H(3, 2)$, Construction 5
$[0, 6 3, 3]$	A CRC in G_3 from a perfect code in G_1 , Construction 5
$[0, 6 6, 0]$	Even weight code in G_3
$[0, 6 1, 2, 3 2, 4]$	A CRC from triangular grid [18], Constr. 4
$[0, 6 1, 2, 3 3, 3]$	A CRC from triangular grid [18], Constr. 4
$[0, 6 1, 3, 2 6, 0]$	Constr. 2 for repetition code in $H(3, 3)$
$[0, 6 1, 4, 1 6, 0]$	Constr. 3 for shortened perfect code in $H(6, 2)$
$[0, 6 1, 0, 5 6, 0]$	A diameter perfect codes in $G_{3,q}$, Constr. 7 with $t = 1$
$[0, 6 2, 0, 4 6, 0]$	Two cosets of diameter perfect codes in G_3 , Constr. 7 with $t = 2$
$[0, 6 3, 0, 3 3, 3]$	Construction 5 for a perfect code in G_1
$[0, 6 2, 0, 4 4, 0, 2 6, 0]$	Constr. 3 + Constr. 8 in $H(3, 2)$
$[0, 6 1, 0, 5 2, 0, 4 6, 0]$	$\{(0 \dots 0), (1 \dots 1)\}$ in $H(6, 2)$, Constr. 9
$[0, 6 1, 0, 5 5, 0, 1 6, 0]$	Even weight subcode of perfect code in G_3 , Constr. 6
$[0, 6 1, 1, 4 2, 2, 2 3, 3]$	Constr. 2 for a singleton vertex in $H(3, 3)$
$[0, 6 1, 0, 5 2, 0, 4 3, 0, 3 4, 0, 2 5, 0, 1 6, 0]$	A singleton vertex in $H(6, 2)$, Constr. 8
$[0, 6 3, 0, 3 \dots 3, 0, 3 6, 0]$ for any $\rho \geq 2$	Constr. 5 for CRCs in G_1 [2]

Proof. We split the codes into two main cases: $\rho = 1$ and $\rho \geq 2$.

I. CRCs with $\rho = 1$. It is not hard to see that up to opposite code the only CRC with covering radius 1 and $a_0 = a_1 = 0$ is the even weight code in G_3 . The CRCs with $\rho = 1$ and $a_0 = 0$, $a_1 \geq 1$ are studied by problem $LP_{=1}$.

The problem is feasible only for the following 2×1 matrices $B = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $a = 1, 2, 3$ with extended 2×2 matrices $[0, 6|1, 5]$, $[0, 6|2, 4]$, $[0, 6|3, 3]$, all of which are parameter matrices for CRCs (see Table 1). Alternatively, the parameters $c_1 = 4, 5$ could be excluded based on a necessary condition for CRCs with $\rho = 1$ from [9, Theorem 2].

II. 1-null CRCs with $\rho \geq 2$. First, we make use of our results from previous sections: by Corollary 1 and Corollary 5 all 2-null CRCs with $c_1 = 1$, $c_2 = 2$ are obtained as 2-null CRCs from the Hamming graph $H(6, 2)$ and all 1-null CRCs with $a_1 = 1$ are obtained from CRCs in the Hamming graph $H(3, 3)$. The classification of the latter codes is known, see [14] and the only such codes are singleton vertices in $H(6, 2)$, $H(3, 3)$ and a pair of words at Hamming distance 6 in $H(6, 2)$ with the parameters:

$$[0, 6|1, 0, 5|2, 0, 4|3, 0, 3|4, 0, 2|5, 0, 1|6, 0], [0, 6|1, 1, 4|2, 2, 2|3, 3], \\ [0, 6|1, 0, 5|2, 0, 4|6, 0].$$

The vast majority of 3×2 matrices are excluded because $LP_{\geq 2}$ is infeasible.

A. $LP_{=2}$ is feasible for 3×2 matrix. It turns out that all corresponding 3×3 extended matrices are indeed the parameter matrices for CRCs. We split between several subcases.

A1. $LP_{=2}$ is feasible, $LP_{>2}$ is infeasible. The corresponding extended matrices are:

$$[0, 6|1, 0, 5|6, 0], [0, 6|1, 3, 2|6, 0], [0, 6|1, 4, 1|6, 0][0, 6|2, 0, 4|6, 0], [0, 6|3, 0, 3|6, 0].$$

A2. $LP_{=2}$ is feasible, $LP_{>2}$ is feasible but the matrix has no feasible descendants w.r.t $LP_{\geq 3}$. The respective extended matrices from this case are

$$[0, 6|1, 2, 3|2, 4], [0, 6|1, 2, 3|3, 3].$$

Both are parameter matrices of CRCs from triangular grid (though other constructions can exist).

A3. $LP_{=2}$ is feasible, $LP_{>2}$ is feasible and has feasible descendants w.r.t $LP_{\geq 3}$. There is only one such matrix, which is $\begin{pmatrix} 0 & 6 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}$. The corresponding extended matrix is

$$[0, 6|3, 0, 3|3, 3],$$

which is parameter matrix of a CRC.

For all 4×3 descendants of the matrix $\begin{pmatrix} 0 & 6 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}$, $LP_{\geq 3}$ is feasible only for the following matrices: $\begin{pmatrix} 0 & 6 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 0 & 6 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. Since both matrices have zeros on the main diagonal, the corresponding extended matrices fulfill

the conditions of Theorem 2 and we have a description of such codes. These codes exist for all $\rho \geq 3$.

B. $LP_{\geq 3}$ is feasible, but $LP_{=2}$ is not. In this case, all 4×3 descendants that are feasible with respect to $LP_{\geq 3}$ cannot have descendants due to the rightmost bottom element being six. Indeed, since in this case $c_3 = 6$, by Theorem 3, c_4 is also 6, which implies that C_3 and C_4 are disjoint, leading to a contradiction. Moreover, the corresponding extended matrices are the parameter matrices of existing CRCs. We have three such matrices:

$\begin{pmatrix} 0 & 6 \\ 1 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 6 \\ 4 & 0 \\ 0 & 5 \end{pmatrix}$, feasible descendants are $\begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 5 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix},$
 $\begin{pmatrix} 0 & 6 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 \\ 4 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$. The corresponding extended matrices in the compact form are

$$[0, 6|1, 0, 5|5, 0, 1|6, 0], [0, 6|2, 0, 4|4, 0, 2|6, 0], [0, 6|4, 0, 2|5, 0, 1|6, 0].$$

All these matrices are the parameter matrices of some CRCs but the last matrix is not new as it the parameter matrix of the opposite code obtained by the above mentioned Construction 9.

□

Theorem 9. 1. The only 2-null CRCs in G_4 that exist with $c_1 = 1$, $c_2 = 2$ and any $\rho \geq 2$ are obtained by Constructions 8 and 9.

2. There are no 2-null CRCs in G_4 for $c_1 = 1$, $c_2 = 3, 4, 5, 6$ and any $\rho \geq 2$.

3. The only 2-null CRCs in G_4 that exist with $c_1 = 1$, $c_2 = 7$, $c_2 = 8$ and any $\rho \geq 2$ are obtained from Construction 6 and Construction 7 with $t = 1$.

Proof. By Corollary 1, the CRCs with $c_1 = 1$ and $c_2 = 2$ correspond to CRCs in $H(8, 2)$, which are characterized in [14]. The respective CRCs in G_4 are obtained by Constructions 8 and 9. The CRCs with $c_2 = 3$ were excluded by Theorem 3. The remaining cases $c_2 = 4, 5, 6$ and 7 were analyzed similarly to Theorem 8, considering problems $LP_{\geq r}$, $LP_{=r}$ for $r = 2$ and 3 and descendants if necessary. The only non-excluded parameter matrix was the one from Construction 6. The case $c_2 = 8$ corresponds to the matrix from Construction 7 with $t = 1$. □

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