On the domatic polynomial of graphs

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set V. A subset T of V is called a dominating set if every vertex in $V \setminus T$ has at least one neighbor in T. A domatic partition of a graph G is a partition of vertices into disjoint dominating sets. The domatic number $d(G)$ is the maximum number of disjoint dominating sets. The domatic polynomial $DP(G, y)$ of graph G is defined as $DP(G, y) = \sum_{j=1}^{d(G)} dp(G, j)y^j$, where $dp(G, j)$ is the number of distinct domatic partitions of G with size j. In this article, we investigate the domatic polynomial of a complete graph and a path and its recurrence relation with vertex deletion operation, allowing us to generate coefficients. We investigate domatic roots and establish bounds for roots using Eneström-Kakeya theorem. We analyze the stability of the domatic polynomial of a complete graph, focusing on the conditions that cause the domatic roots to be located in the closed left half-plane. Further, we demonstrate that the domatic polynomial $DP(K_n, x)$ is vertex-reconstructible.

Keywords: Domination, domatic partition, domatic polynomial, complete graph

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$ and $|E(G) = m$. The set $N_G(w) = \{v|vw \in E(G)\}$ is called the open neighborhood of a vertex $w \in V$ and the set $N_G[w] = N_G(w) \cup \{w\}$ is called the closed neighbourhood of $w \in V$. In G, a set $T \subseteq V(G)$ is called a *dominating set* if $N_G[T] = V(G)$, or every vertex in $V \setminus T$ has at least one neighbor in T. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. For more details on domination, we refer [17, 9, 10, 16]. In 2009, Alikhani and Peng [14] defined the domination polynomial $D(G, y)$ of a graph G.

A *domatic partition* of a graph G is a partition of its vertices into disjoint dominating sets The maximum number of disjoint dominating sets in a domatic partition of a graph G is called its *domatic number*. Cockayne and Hedetniemi [1] introduced the domatic number of a graph G. More details on the domatic number can be seen in [1, 2, 3, 4].

Graph polynomials are a well-developed field that can be utilized to analyze graph properties. The polynomials associated with graphs are the chromatic polynomial, the domination polynomial etc. It is worth mentioning that, the concept of dominating set is present in a wide range of problems in communication networks, linear algebra, optimization, online social networks, cloud architecture for video distribution services and wireless sensor networks. For more information and motivation of domination polynomial, we refer to [14].

Definition 1.1. [13] *Let $\mathcal{DP}(G,j)$ be the family of domatic partitions of a graph G with cardinality j and let $dp(G,j) = |\mathcal{DP}(G,j)|$. Then the domatic polynomial $DP(G,y)$ of G is defined as*

$$DP(G,y) = \sum_{j=1}^{d(G)} dp(G,j)y^j$$

where $d(G)$ is called domatic number of G.

For example, the complete graph K_4 on 4 vertices, has one domatic partition of 4 disjoint dominating sets, six domatic partitions of 3 disjoint dominating sets, seven domatic partitions of 2 disjoint dominating sets and one domatic partition of 1 dominating set. So its domatic polynomial is $DP(K_4,y) = y^4 + 6y^3 + 7y^2 + y$.

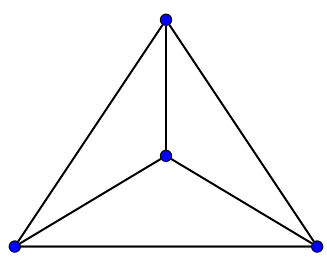

Figure 1: Complete graph K_4

In section 2, we examine the domatic polynomial of a complete graph. Coefficients of the domatic polynomial of the complete graphs are of the form of Stirling numbers of 2nd kind. Accordingly, Stirling numbers of second kind can be manipulated by the Bell polynomial. So the domatic polynomial of a complete graph can be represented by the Bell Polynomial. Also, we examine the domatic roots or the roots of a domatic polynomial. It is evident that the domatic polynomial does not contain a constant term. Therefore, 0 is the root of every domatic polynomial. As the coefficients are positive integers, the interval $(0,\infty)$ is zero-free. If every domatic root lies in $(\infty, 0]$ (that is, closed left half-plane) the complete graph K_p is said to be domatic stable graph. Otherwise, we call it domatic unstable graph. Also, we investigate the stability of domatic polynomials of complete graphs.

Li and Shi [18] more explicitly studied the derivatives of the graph polynomials, which have implications in chemistry, control theory and computer science. Also, we prove that the domatic polynomial of K_n is vertex-reconstructible, as it is related to graph polynomial derivatives. One of the most captivating problems within the field of graph theory is the Reconstruction Conjecture. The conjecture (see [12, 15]) states that any graph with a minimum of three vertices may be reconstructed in a unique manner by removing each vertex of the original graph individually, resulting in a multiset of subgraphs.

In section 3, we explore the domatic polynomial for path and obtain some of its properties and its recurrence relation with the vertex deletion operation. Also, we prove that the domatic roots of a path are real and lie in $(-1, 0]$. In section 4, we summarize our research work.

2 Domatic polynomial of a complete graph

We start with the definition of the domatic polynomial of a complete graph.

Definition 2.1. *The domatic polynomial of the graph K_n of order n is the polynomial*

$$DP(K_n, y) = \sum_{j=1}^{n} dp\begin{Bmatrix} n \\ j \end{Bmatrix} y^j,$$

where $dp\left\{{n \atop j}\right\}$ is the number of j-domatic partitions of K_n.

Note that $dp\left\{{n \atop j}\right\} = dp\left\{{n-1 \atop j-1}\right\} + j\ dp\left\{{n-1 \atop j}\right\}$, where $dp\left\{{n \atop 1}\right\} = dp\left\{{n \atop n}\right\} = 1$, $dp\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, $dp\left\{{n \atop n-1}\right\} = \binom{n}{2}$, $dp\left\{{n \atop 0}\right\} = 0$.

The following result gives the domatic polynomial of a complete graph.

Theorem 2.1. *The domatic polynomial of a complete graph is*

$$DP(K_n, y) = y\{DP(K_{n-1}, y) + \frac{d}{dy}(DP(K_{n-1}, y))\} \qquad for \quad all \quad n \geq 2.$$

Proof. The domatic polynomial of a complete graph is

$$DP(K_n, y) = \sum_{j=1}^{n} dp\begin{Bmatrix} n \\ j \end{Bmatrix} y^j \tag{1}$$

and

$$dp\begin{Bmatrix} n \\ j \end{Bmatrix} = dp\begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix} + j\ dp\begin{Bmatrix} n-1 \\ j \end{Bmatrix}. \tag{2}$$

Now, from Eq. (1) and (2), we get

$$\begin{aligned} DP(K_n, y) &= \sum_{j=1}^{n} \left(dp\begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix} + j\ dp\begin{Bmatrix} n-1 \\ j \end{Bmatrix} \right) y^j \\ &= \sum_{j=1}^{n} dp\begin{Bmatrix} n-1 \\ j-1 \end{Bmatrix} y^j + \sum_{j=1}^{n} j\ dp\begin{Bmatrix} n-1 \\ j \end{Bmatrix} y^j \\ &= y\{DP(K_{n-1}, y) + \frac{d}{dy}(DP(K_{n-1}, y))\} \end{aligned}$$

□

We will use the following vertex deletion operation for graphs.

Vertex deletion: Let $G = (V, E)$ be a graph and $u \in V$. Then $G - u$ is the graph obtained by removing u and all edges incident to u.

Now, we obtain a recurrence relation for $DP(K_n, y)$ with vertex deletion operation.

Theorem 2.2. *Let u be a vertex of the complete graph K_n. Then*

$$DP(K_n, y) = y\{DP(K_n - u, y) + \sum_{i=1}^{n-1} \binom{n-1}{i} DP(K_{n-1} - u_i, y)\}.$$

Proof. We separate the domatic sets of K_n into two sets. In the first set, the polynomial $DP(K_n - u, y)$ counts the domatic sets without u. Then $yDP(K_n - u, y)$ counts the domatic sets which include u.
In second set, $\sum_{i=1}^{n-1} \binom{n-1}{i} DP(K_{n-1} - u_i, y)$ counts the total domatic sets of $\binom{n-1}{i}$ sets without u_i in $(K_n - u_i, y)$. If we add u back, we get all the domatic sets which involve u. These sets are counted by the polynomial $y \sum_{i=1}^{n-1} \binom{n-1}{i} DP(K_{n-1} - u_i, y)$. Therefore,

$$DP(K_n, y) = y\{DP(K_n - u, y) + \sum_{i=1}^{n-1} \binom{n-1}{i} DP(K_{n-1} - u_i, y)\}$$

□

We find some properties of the coefficients of the domatic polynomial of a complete graph. The recurrence in *Theorem 2.2* allows us to generate quickly the domatic polynomial for complete graphs.

$n \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	1	3	1									
4	1	7	6	1								
5	1	15	25	10	1							
6	1	31	90	65	15	1						
7	1	63	301	350	140	21	1					
8	1	127	966	1701	1050	266	28	1				
9	1	255	3025	7770	6951	2646	462	36	1			
10	1	511	9330	34105	42525	22827	5880	750	45	1		
11	1	1023	28501	145750	246730	179487	63987	11880	1155	55	1	
12	1	2047	86526	611501	1379400	1323652	627396	159027	22275	1705	66	1

Table 1: $dp(\left\{ {n \atop j} \right\})$ is the number of domatic partition of K_n with j-disjoint dominating sets.

The next theorem follows easily from the definition of the domatic polynomial of a complete graph.

Theorem 2.3. *If K_n is a complete graph, then* *i.* $dp\left\{{n \atop j}\right\} = 0$ *if* $n < 0$ *or* $j \leq 0$ *or* $n < j$,

ii. the leading coefficient of $DP(K_n, y)$ *is always equal to 1,*

iii. $DP(K_n, y)$ *has constant term as zero,*

iv. zero is always a root of $DP(K_n, y)$,

v. the coefficient of the y^{n-1} *term in* $DP(K_n, y)$ *is the number of edges.*

Now, we discuss the roots of the domatic polynomials and bounds for the absolute value of all the domatic roots. As domatic polynomials have positive coefficients and no constant term, so the roots must be negative.

Let K_n be a complete graph with domatic polynomial $DP(K_n, y)$. A root of $DP(K_n, y)$ is referred to as a *domatic root* of K_n and the collection of all domatic roots of K_n is denoted by $\mathbb{Z}(DP(K_n, y))$. *D-number* of K_n is the total number of distinct real domatic roots of the graph K_n and is denoted by $D_R(K_n)$.

Example 1. *The domatic polynomial of the complete graph* K_3 *shown in Figure 2 is*

$$DP(K_3, y) = y^3 + 3y^2 + y.$$

The domatic roots of K_3 *are* $0, -2.61803, -0.381966$. *Therefore,* $D_R(K_3) = 3$.

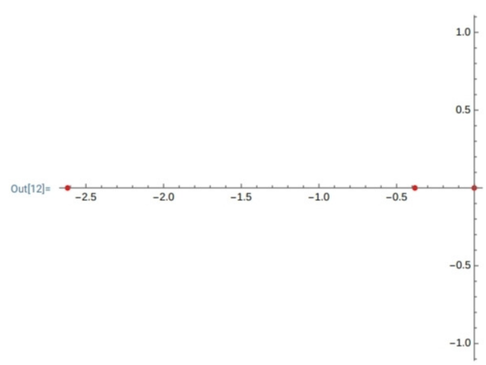

Figure 2: Domatic roots of K_3

Theorem 2.4. *(Descarte's Rule Of Signs* [8]*) Let* $f(y) = b_n y_n + b_{n-1} y_{n-1} + \cdots + b_1 y + b_0$ *be a polynomial with real coefficients. Then the number of positive real roots of* f *is less than or equal to the number of sign changes in the sequence* $(b_n, b_{n-1}, \ldots, b_1, b_0)$.

Using Descarte's rule of signs, the following is evident.

Theorem 2.5. *For a complete graph* K_n, $D_R(K_n) = n, \quad for \quad all \quad n \geq 1.$

Now, we present general root bounds for domatic complete graph polynomials. Depending on the value of n, we provide an annulus containing the domatic roots of a complete graph by using Eneström-Kakeya theorem.

Theorem 2.6. *(Eneström-Kakeya [6]) If the ppolynomial $f(y) = b_0 + b_1 y \cdots + b_n y^n$ has positive real coefficients, then all the roots of f lie in the annulus $r \leq |z| \leq R$, where $r = \min\left\{\frac{b_i}{b_{i+1}} : 0 \leq i \leq n-1\right\}$ and $R = \max\left\{\frac{b_i}{b_{i+1}} : 0 \leq i \leq n-1\right\}$.*

Theorem 2.7. *All the domatic roots of the complete graph K_n lie in the annulus $0 \leq |z| \leq \frac{n(n-1)}{2}$.*

Proof. The domatic polynomial of K_n is

$$DP(K_n, y) = \sum_{j=1}^{n} dp\begin{Bmatrix} n \\ j \end{Bmatrix} y^j$$

where $dp\{{n \atop j}\}$ is the number of j-domatic partition of K_n.

Clearly, $y = 0$ is the root of $DP(K_n, y)$, so in this case, $r = min\left\{dp\{{n \atop n-1}\}, dp\{{n \atop n-2}\}, \ldots, dp\{{n \atop n-(n-2)}\}\right\}$ and

$R = max\left\{dp\{{n \atop n}\},\ dp\{{n \atop n-1}\}, \ldots, dp\{{n \atop n-(n-1)}\}\right\}$. □

The domatic roots of the complete graph K_n for $1 \leq n \leq 100$ are shown in Figure 3 below.

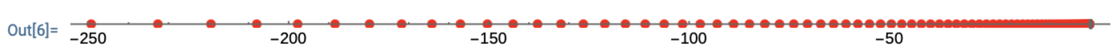

Figure 3: Domatic roots of K_n for $1 \leq n \leq 100$

Further, we observe that all the domatic roots lie in the closed left half-plane. Also, we prove that domatic polynomial of a complete graph is stable, by using Hermite-Biehler theorem. In order to accomplish this, we describe terminology and results related to polynomials.

If every nonzero domatic root lies on the left half plane, the graph K_n is said to be *domatic stable* graph. A polynomial $f(y) \in \mathbb{R}$ is *standard* if its leading coefficient is positive or it is equal to zero. Let $f_1(y)$ and $g_1(y)$ be polynomials with all real roots, say $s_1 \leq s_2 \leq \cdots \leq s_n$ and $t_1 \leq t_2 \leq \cdots \leq t_m$ being the roots of f_1 and g_1 respectively. We say that

- f_1 interlaces g_1 if $m = n + 1$ and $t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq s_n \leq t_{n+1}$ and
- f_1 alternates left of g_1 if $m = n$ and $s_1 \leq t_1 \leq s_2 \leq \cdots \leq s_n \leq t_n$.

We write $f_1 \prec g_1$ if either f_1 interlaces g_1 or f_1 alternates left of g_1.

Given a polynomial

$$P(y) = \sum_{j=0}^{n} b_j y^j, \text{ let } P_{\text{even}}(y) = \sum_{j=0}^{n/2} b_{2j} y^j \quad \text{and} \quad P_{\text{odd}}(y) = \sum_{j=0}^{(n-1)/2} b_{2j+1} y^j,$$

where $P_{\text{even}}(y)$ and $P_{\text{odd}}(y)$ are the "even" and "odd" parts of the polynomial $P(y)$ with $P(y) = P_{\text{even}}(y^2) + yP_{\text{odd}}(y^2)$.

The *Hermite-Biehler theorem* gives the necessary and sufficient conditions for the stability of a real polynomial.

Theorem 2.8. *(Hermite-Biehler theorem for stability* [5]*) Let $P(y) = P_{even}(y^2) + yP_{odd}(y^2)$ be standard. Then $P(y)$ is stable iff both P_{even} and P_{odd} are standard, have only non-positive roots and $P_{odd} \prec P_{even}$.*

Sturm's theorem provides a helpful method for determining the case of a nonreal root of a polynomial. Sturm sequence is highly effective method for identifying the characteristic roots of polynomials (see [11]).

Theorem 2.9. *(Sturm's theorem* [8]*) Let $f(y) \in \mathbf{R}[y]$ have positive degree, and suppose $(f_0, f_1, \cdots, f_k)$ is its Sturm sequence. Let $c, d \in \mathbb{R}$ such that $c < d$ and c and d are not roots of f. Then $V(c) - V(d)$ represents the number of distinct roots of f in (c, d), where $V(e) = Var(f_0(e), f_1(e), \cdots, f_k(e))$.*

Corollary 2.10. [7] *Let $f(y)$ be a positive degree real polynomial with a positive leading coefficient. Then $f(y)$ has only real roots if its Sturm sequence has no degree gaps and no negative leading coefficient.*

Theorem 2.11. *The complete graph K_n is domatic stable graph.*

Proof. Expanding the domatic polynomial of K_n into its even $P_e^n(y)$ and odd $P_o^n(y)$ parts, we have

$$K_n(y) = P_e^n(y^2) + y\, P_o^n(y^2) \qquad \textit{for all} \;\; n \geq 3.$$

For $n = 3$, $DP(K_3, y) = y^3 + 3y^2 + y$, where $P_e(y) = 3y$ and $P_o(y) = y + 1$. Sturm sequence of $DP(K_3, y)$ is as follows.

$$\begin{aligned} f_0(y) &= y^3 + 3y^2 + y \\ f_1(y) &= 3y^2 + 6y + 1 \\ f_2(y) &= \frac{4}{3}y + \frac{1}{3} \\ f_3(y) &= \frac{5}{16} \end{aligned}$$

Again, for $n = 7$, $DP(K_7, y) = y^7 + 21y^6 + 140y^5 + 350y^4 + 301y^3 + 63y^2 + y$, where $P_e(y) = 21y^3 + 350y^2 + 63y$ and $P_o(y) = y^3 + 140y^2 + 301y + 1$. Also, the Sturm sequence of $DP(K_7, y)$ is

$$f_0(y) = y^7 + 21y^6 + 140y^5 + 350y^4 + 301y^3 + 63y^2 + y$$
$$f_1(y) = 7y^6 + 126y^5 + 700y^4 + 1400y^3 + 903y^2 + 126y + 1$$
$$f_2(y) = 14y^5 + 150y^4 + 428y^3 + 342y^2 + \frac{372}{7}y + \frac{3}{7}$$
$$f_3(y) = \frac{423}{7}y^4 + \frac{2311}{7}y^3 + \frac{2586}{7}y^2 + \frac{6645}{98}y + \frac{55}{98}$$

$$f_4(y) = \frac{10595536}{178929}y^3 + \frac{7343791}{59643}y^2 + \frac{1757987}{59643}y + \frac{45479}{178929}$$

$$f_5(y) = \frac{196703398722951}{2291130267904}y^2 + \frac{3844015038350667}{112265383127296}y + \frac{35534000554389}{112265383127296}$$

$$f_6(y) = \frac{773662042435955256211093120}{741715087253713075940477 67}y + \frac{835895426065693938317708 8}{741715087253713075940477 67}$$

$$f_7(y) = \frac{4536018656150121999761389580746254212 3709}{104499157344510038035873806035264973389440 0}$$

By calculating Sturm sequence, the even and odd polynomials for $n = 3$ and $n = 7$ have a real root. Likewise, we have a real root for any $n \geq 3$. Also, their Sturm sequence contains no degree gaps and no negative leading coefficients. Therefore, $DP(K_n, y)$, for all $n \geq 3$, has only real roots using Corollary 2.10 and the roots are non-positive as its coefficients are positive. Further, we can easily check that P_{even} and P_{odd} are standard and $P_{\text{odd}} \prec P_{\text{even}}$. Therefore, the domatic polynomial of a complete graph is stable by Hermite-Biehler theorem. □

Theorem 2.12. *The domatic roots of K_n are all non-positive and lie in $(-\infty, 0]$.*

Proof. Since all the coefficients of the polynomial are positive, so by Descarte's rule of sign, the only possible roots are negative and complex. But by Sturm's theorem, we have only negative real roots and zero is one of the roots of this polynomial. Therefore, all the real roots are non-positive. Hence the domatic roots of K_n are all non-positive and lie in $(-\infty, 0]$. □

The problem of polynomial reconstruction is related to the graph polynomial derivative. Kelly [12] and Ulam [15] stated in their well-known (and unsolved) reconstruction conjecture that every graph G with at least 3 vertices is reconstructible.

For a graph $G = (V, E)$ with a vertex $v \in V$, the *deck* of G is the multiset of the vertex-deleted subgraphs $G - v$ for $v \in V$.

A graph polynomial $P(G)$ is *polynomial reconstructible*, if $P(G)$ can be determined from the multiset of vertex-deleted subgraphs with respect to $P(G)$.

Theorem 2.13. *The domatic polynomial of a complete graph is vertex-reconstructible. That is,*

$$\frac{d}{dy}(DP(K_n, y)) = \sum_{i=1}^{n} \binom{n}{i} DP(K_n - u_i, y).$$

Proof. We will prove that the two polynomials LHS $= \frac{d}{dy}(DP(K_n, y))$ and RHS $= \sum_{i=1}^{n} \binom{n}{i} DP(K_n - u_i, y)$ have precisely the same terms with equal coefficients.

Let $G = K_n$ and y^j be a term of LHS. Then $DP(G, y)$ has a term in y^{j+1}. It follows that G has a domatic set C with $j+1$ vertices. Let $u_i \in V(G)$. Then $G - u_i$ will contain the domatic set $C - u_i$. Hence RHS also contains a term in y^j.

Conversely, if RHS contains a term in y^j, then there exists a vertex u_i such that $G - u_i$ has a domatic set with j vertices. Therefore, G has a domatic set with $j+1$ vertices. It follows that $DP(G, y)$ has a term in y^{j+1}. Hence LHS has a term in y^j. We conclude that LHS and RHS have the same kinds of terms.

Again, we will show that the coefficients of like terms are equal. Let $dp\left\{ {n \atop j} \right\} y^j$ be a term in $DP(G, y)$. Then the corresponding term in LHS will be $dp\left\{ {n \atop j} \right\} j y^{j-1}$. Thus the coefficients of y^j in LHS will be $j.dp\left\{ {n \atop j} \right\}$. Now, for each domatic set in G with j vertices, there will be exactly corresponding domatic sets in the graphs $G - u_i$ with $j-1$ vertices. Since G contains $dp\left\{ {n \atop j} \right\}$ domatic partitions, RHS will contain the term y^{j-1} with coefficient $j.dp\left\{ {n \atop j} \right\}$. Therefore the coefficients of like terms are equal. □

3 Domatic polynomial of a path

In this section, we explore the domatic polynomial of a path and its properties. The following are immediate.

Proposition 3.1. *If P_n is a path of order n, then*

$$DP(P_n, y) = y + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} y^2, \qquad \forall\, n \geq 2$$

where the coefficients of y^2 and y are the total number of domatic partitions of cardinality 2 and cardinality 1 respectively.

Proposition 3.2. *For the path P_n, $D_R(P_n) = n$, for all $n \geq 1$.*

As a direct consequence of Proposition 3.1, we obtain the following result.

Proposition 3.3. *Let P_n be a path of order n. Then*

i. $y^2 + \sum_{k=2}^{n} DP(P_k, y) - (n-2)y = y + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} y^2$, *for all* $n \geq 2$.

ii. $y^2 + \sum_{k=2}^{n} DP(P_{2k}, x) - (n-2)y = y + \sum_{k=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-k-1}{k} y^2$, *for all* $n \geq 2$.

iii. $y^2 + \sum_{k=2}^{n} DP(P_{2k-1}, y) - (n-2)y = y + \sum_{k=0}^{\lfloor \frac{2n-2}{2} \rfloor} \binom{2n-k-2}{k} y^2$, *for all* $n \geq 2$.

Remark 1. *Part (i) of Proposition 3.3 gives the domatic polynomial of a path with any number of vertices.*
Part (ii) of Proposition 3.3 gives the domatic polynomial of a path with odd number of vertices.
Part (iii) of Proposition 3.3 gives the domatic polynomial of a path with even number of vertices. vertices.

Theorem 3.1. *The recurrence relation for* $DP(P_n, y)$ *is*

$$DP(P_n, y) = DP(P_{n-1}, y) + DP(P_{n-2}, y) - y, \qquad \textit{for all } n \geq 4$$

with $DP(P_2, y) = y^2 + y = DP(P_3, y)$.

Proof. In P_n, the possible domatic numbers are 1 and 2 only, that is, $d(P_n) = 1$ *or* 2. We consider two cases.
Case 1. For each P_n, we will always have only one domatic partition of one disjoint dominating set or the graph itself, that is, $d(P_n) = 1$. Hence $DP(P_n, y) = y$.
Case 2. We find possible domatic partition of each P_p into 2 disjoint dominating sets only. that is, $d(P_n) = 2$. As there are $(n-2-k)$ of k ways to get the total number of domatic partitions of 2 disjoint dominating sets, that is, $dp\left\{ {n \atop k} \right\} = \binom{n-2-k}{k}$, so

$$DP(P_n, y) = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} y^2$$

Combining Case 1 and Case 2, we get the result. □

We obtain a recurrence relation for $DP(P_n, y)$ with vertex deletion operation.

Theorem 3.2. *For each vertex* u *in* P_n*, we have*

$$DP(P_n, y) = \sum_{i=1}^{n-(n-2)} DP(P_n - u_i, y) - DP(P_{n-(n-2)} - u, y).$$

Proof. Let $u \in V(P_n)$. We separate the domatic sets of P_n into two sets.
In the first set, $\sum_{i=1}^{n-(n-2)} DP(P_n - u_i, y)$ counts the total domatic sets of $n-(n-2)$ sets without u_i in $(P_n - u_i, y)$.
In the second set, the polynomial $DP(P_{n-(n-2)} - u, y)$ counts the domatic sets without u in $(P_{n-(n-2)} - u, y)$.
Therefore,

$$DP(P_n, y) = \sum_{i=1}^{n-(n-2)} DP(P_n - u_i, y) - DP(P_{n-(n-2)} - u, y)$$

□

Table 2: The number of domatic partition of P_n with k-disjoint dominating sets.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12
1	1											
2	1	1										
3	1	1										
4	1	2										
5	1	3										
6	1	5										
7	1	8										
8	1	13										
9	1	21										
10	1	34										
11	1	55										
12	1	89										

The recurrence in Theorem 3.2 allows us to generate the domatic polynomial for paths. The number of domatic partitions of P_n with k-disjoint dominating sets is shown in Table 2.

Theorem 3.3. *For a path P_n, we have*

i. $DP(P_n, y)$ *is quadratic polynomial,*

ii. $DP(P_n, y)$ *has zero constant term,*

iii. zero is a always root of $DP(P_n, y)$,

iv. $DP(P_n, y)$ *is a strictly increasing function in* $(-\infty, \infty)$.

Theorem 3.4. *The domatic roots of P_n are all real and lie in* $(-1, 0]$.

Proof. Let P_n be a path with n vertices and k edges. By Proposition 3.1, the domatic polynomial of the path of order n is

$$DP(P_n, y) = y + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} y^2 \qquad \forall\, n \geq 2$$

The roots of $DP(P_n, y)$ are

$$\frac{(-1 \pm 1)}{2. \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k}},$$

which lie in $\left((-1)/ \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k}\right]$ and thus all the domatic roots of path are real. Hence, the domatic roots of P_n are all real and lie in $(-1, 0]$. $\square$

4 Conclusion

In this study, we have explored the domatic polynomial $DP(G, y)$ of complete graphs and paths. We derived recurrence relations for the domatic polynomial under vertex deletion operations, which allowed us to systematically generate the coefficients of the polynomial. Additionally, we examined the roots of the domatic polynomial using the Eneström-Kakeya theorem and analyzed their stability, specifically in the context of complete graphs, demonstrating that the roots lie within the closed left half-plane. We proved the vertex-reconstructibility of the domatic polynomial for complete graphs. In general, these findings contribute to the understanding of domatic partitions and open avenues for future research in the study of polynomial invariants of graphs.

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