Double total coalitions in graphs*

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Abstract

Let G be a graph of minimum degree at least two. A set D of vertices of a graph G with the vertex set V is a double total dominating set of G, if every vertex v has at least two neighbors in D. A double total coalition consists of two disjoint sets of vertices V_1 and V_2, neither of which is a double total dominating set but their union $V_1 \cup V_2$ is a double total dominating set. A double total coalition partition of a graph G is a partition $\Theta = \{V_1, V_2, ..., V_k\}$ of V such that no subset of Θ is a double total dominating set of G, but for every set $V_i \in \Theta$, there exists a set $V_j \in \Theta$ such that V_i and V_j form a double total coalition. In this paper we initiate the study of the double total coalition by setting some basic results, giving exact values and bounds for the double total coalition number.

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1 Introduction

Throughout this article, we only consider finite and simple graphs with minimum degree at least two. The open neighborhood $N_G(v)$ of a vertex v in G is the set of vertices adjacent to v, while the closed neighborhood of v is the set $N_G[v] = \{v\} \cup N_G(v)$. Each vertex of $N(v)$ is called a neighbor of v, and the cardinality of $|N(v)|$ is called the degree of v, denoted by $deg(v)$. The minimum and maximum degree of graph vertices are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Inspired by a solution to the famous Five Queens Problem, Cockayne, Dawes and Hedetniemi introduced the definition of total domination. Various aspects of domination are well studied in the literature, and are surveyed in [12, 13]. A set $S \subseteq V$ is called a dominating set if every vertex of $V \setminus S$ is adjacent to at least one vertex in S. Given a graph G, a set $D \subseteq V(G)$ is said to be a double total dominating set of G if every vertex of G is adjacent to at least two vertices in D. The double total domination number of G, denoted by $\gamma_{\times 2,t}(G)$, is the cardinality of a minimum double total dominating set of G. It is worth mentioning that this parameter is also called as 2-tuple total domination number or total 2-domination number in the literature. The double total domination in graphs has been well studied in [6, 7, 14, 15, 16]. A k-tuple total domatic partition is a partition of vertices of a graph into k-tuple total dominating sets. The maximum cardinality of a k-tuple total domatic partition is called the k-tuple total domatic number, denoted by $d_{\times k,t}(G)$. The k-tuple domatic total number of a graph was introduced by Sheikholeslami and Volkmann in [17]. A k-tuple total domatic number where $k = 2$ is known as the double total domatic number, and denoted by $d_{\times 2,t}(G)$. Fairly recently, the concept of coalition in graphs has triggered a great deal of interest due to its definition, which is based on the dominating sets. A coalition in a graph G is made up of two disjoint sets of vertices A and B of G, neither of which is a dominating set but whose union $A \cup B$ is a dominating set of G. A coalition partition is a vertex partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of V such that for every $i \in \{1, 2, \ldots, k\}$ the set V_i is either a dominating set and $|V_i| = 1$, or there exists another set V_j so that they form a coalition. The maximum cardinality of a coalition partition is called the coalition number of the graph, and denoted by $C(G)$. Coalitions in graphs were introduced and first studied by Haynes et al. in [8], and have been studied further in several works including [4, 9, 10, 11]. Different types of domination coalitions have been studied [1, 2, 3, 5]. These variations are mainly formed by imposing additional conditions on the domination coalition. In order to develop future research, we propose a new perspective on coalitions involving double total dominating sets in graphs.

The main contributions of this work are as follows. In Section 2, we introduce the double total coalitions in graphs and derive some bounds on the double total coalition number which we denote it by $DTC(G)$. In Section 3, we obtain the exact values of the double total coalition number and characterize the graphs G of order n that satisfying $DTC(G) = n - 1$.

2 Existence and bounds

In this section, after stating the definition of the double total coalition, we investigate the existence of the double total coalition and obtain some bounds for the double total coalition number. We begin with the following definitions.

Definition 1 (double total coalition) *Two sets $V_1, V_2 \subseteq V(G)$ form a double total coalition in a graph G if they are not double total dominating sets but their union is a double total dominating set in G.*

Definition 2 (double total coalition partition) *A double total coalition partition, abbreviated dtc-partition, of a graph G is a partition $\Theta = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V such that any $V_i \in \Theta, 1 \leq i \leq k$, is not a double total dominating set but forms a double total coalition with another set $V_j \in \Theta$ that is not a double total dominating set. The maximum cardinality of a double total coalition partition is called the double total coalition number of the graph and denoted by $DTC(G)$. A dtc-partition of G of cardinality $DTC(G)$ is called a $DTC(G)$-double total partition.*

We state the following trivial observation about the double total coalition number of a graph G. The proof follows readily from the definitions and is omitted.

Observation 1 *If G is a graph with $\delta(G) < 2$, then $DTC(G) = 0$.*

The primary objective of this section is to prove that every graph G with $\delta(G) \geq 2$ has a *dtc*-partition.

Theorem 1 *Every graph G with $\delta(G) \geq 2$ has a dtc-partition.*

Proof. Suppose that G has a double total domatic partition $\mathcal{D}_t = \{D_{t,1}, D_{t,2}, \ldots, D_{t,k}\}$ with $d_{\times 2,t}(G) = k$, where each set $D_{t,i}$ has at least three vertices as $\gamma_{\times 2,t}(G) \geq 3$. In what follows we demonstrate the process of constructing a *dtc*-partition Θ of G. For any integer $1 \leq i \leq k-1$, if $D_{t,i}$ is not a minimal double total dominating set, then there is a set $D'_{t,i} \subset D_{t,i}$ which is a minimal double total dominating set. So, we swap $D_{t,i}$ by $D'_{t,i}$ in $\mathcal{D}_t$, and add $D_{t,i} \setminus D'_{t,i}$ to $D_{t,k}$. We therefore infer that all sets $D_{t,i} \in \mathcal{D}_t$ with $1 \leq i \leq k-1$ are minimal double total dominating sets of G. Now, we establish a *dtc*-partition Θ for G. We start by dividing each set $D_{t,i} \in \mathcal{D}_t$ with $1 \leq i \leq k-1$ into two nonempty sets $D^1_{t,i}$ and $D^2_{t,i}$, and then, we add $D^1_{t,i}$ and $D^2_{t,i}$ to Θ. It is clear that neither $D^1_{t,i}$ nor $D^2_{t,i}$ is a double total dominating set but $D^1_{t,i} \cup D^2_{t,i}$ is a double total dominating set. Then, $|\Theta| \geq 2k-2$. We next consider the set $D_{t,k} \in \mathcal{D}_t$. If $D_{t,k}$ is a minimal double total dominating set, then we split $D_{t,k}$

into two nonempty sets $D^1_{t,k}$ and $D^2_{t,k}$, and then, we add $D^1_{t,k}$ and $D^2_{t,k}$ to Θ. Hence, $|\Theta| = 2k$. Further, we suppose that $D_{t,k}$ is not a minimal double total dominating set. Then, there is a set $D'_{t,k} \subset D_{t,k}$ which is a minimal double total dominating set. Now, we split $D'_{t,k}$ into two sets $D'^1_{t,k}$ and $D'^2_{t,k}$, and append them to Θ. Now, consider the set $D''_{t,k} := D_{t,k} \setminus D'_{t,k}$. Since $\mathcal{D}_t$ is a double total domatic partition of G with the maximum cardinality $d_{\times 2,t}(G) = k$, the set $D''_{t,k}$ is not a double total dominating set. If $D''_{t,k}$ forms a double total coalition with one of the sets of Θ, then we add $D''_{t,k}$ to Θ, and therefore $|\Theta| = 2k + 1$. If $D''_{t,k}$ does not form a double total coalition with any set of Θ, then we remove $D'^2_{t,k}$ from Θ and append $D'^2_{t,k} \cup D''_{t,k}$ to Θ, and consequently $|\Theta| = 2k$, which completes the proof.□

From the proof of Theorem, one can derive the following result.

Corollary 1 *If a graph G has $\delta(G) \geq 2$, then $DTC(G) \geq 2d_{\times 2,t}(G)$.*

It is clear that for all graphs G with $\delta(G) \geq 2$, $d_{\times 2,t}(G) \geq 1$. Hence as a consequence of Corollary 1 and Theorem 1, we infer the following result.

Corollary 2 *If G is a graph of order n with $\delta(G) \geq 2$, then $2 \leq DTC(G) \leq n - 1$.*

Now, we recall the following theorem.

Theorem 2 [17] *If G is a graph of order n with minimum degree $\delta(G) \geq k$, then $d_{\times k,t}(G) \geq \left\lfloor \frac{n}{k(n-\delta)+1} \right\rfloor$.*

By Theorem 2, we deduce that $d_{\times 2,t}(G) \geq \left\lfloor \frac{n}{2(n-\delta)+1} \right\rfloor$ where $k = 2$. Combining this result with Corollary 1 yields the following result.

Corollary 3 *If G is a graph of order n with $\delta(G) \geq 2$, then $DTC(G) \geq 2 \left\lfloor \frac{n}{2(n-\delta)+1} \right\rfloor$.*

Here, we present a technical key lemma, which permits us to estimate the number of double total coalitions embracing any set in a $DTC(G)$-partition of G.

Lemma 1 *If G is a graph with $\delta(G) \geq 2$, then for any $DTC(G)$-partition Θ and for any $X \in \Theta$, the number of double total coalitions formed by X is at most $\Delta(G) - 1$.*

Proof. Since $X \in \Theta$, X is not a double total dominating set. So, there is a vertex w such that $|N(w) \cap X| \leq 1$. We now consider two cases.

Case 1. Let $|N(w) \cap X| = 1$. We first assume that $w \in X$. If a set $A \in \Theta$ forms a double total coalition with X, then $A \cup X$ is a double total dominating set of G. Since w has exactly one neighbor in X, so w must have at least one neighbor in A. Thus, there are at most $|N(w)| - 2 \leq \Delta(G) - 2$ other sets that can form a double total coalition with X, and consequently, X is in at most $\Delta(G) - 1$ double total coalitions. Next suppose that $w \notin X$. Assume that there is a set $U \in \Theta$ such that $w \in U$ and X contains exactly one of the members of $N(w)$, such as y. If U and X do not form a double total coalition, any set in Θ that forms a double total coalition with X must contain at least one vertex of $N(w)$. Then, there are at most $|N(w)| - 1$ sets that can form a double total coalition with X, and therefore, X is in at most $\Delta(G) - 1$ double total coalitions. Now may assume that U and X produce a double total coalition. We consider the following subcases.

Subcase 1.1. Assume that y has no neighbor in X, while y has at least two neighbors in U such that $w \in N(y)$. If a set $B \in \Theta$ forms a double total coalition with X, then y must have at least two neighbors in B. So, there are at most $\lceil \frac{|N(y)|-4}{2} \rceil$ other sets that can form a double total coalition with X, and therefore, X is in at most $\lceil \frac{\Delta(G)-4}{2} \rceil + 2 = \lceil \frac{\Delta(G)}{2} \rceil$double total coalitions.

Subcase 1.2. Let y has at least one neighbor in X, while y has at least one neighbor in U such that $w \in N(y)$. Suppose that there exists a set $C \in \Theta$ forms a double total coalition with X. Since $X \cup C$ is a double total dominating set and w has only one neighbor in X, the vertex w must has a neighbor in C. Moreover, since $X \cup U$ is a double total dominating set and w has only one neighbor in X, the vertex w must has a neighbor in U. Therefore, since w has a neighbor in X named y, as well as a neighbor in U and a neighbor in C, there are at most $|N(w)| - 3$ other sets that can form a double total coalition with X, and consequently, X is in at most $|N(y)| - 3 + 2 = |N(y)| - 1 \leq \Delta(G) - 1$ double total coalitions.

Case 2. Let $|N(w) \cap X| = 0$. First suppose that $w \in X$ and $X \cap N(w) = \emptyset$. Assume that there exists a set $D \in \Theta$ forms a double total coalition with X. Then w must have at least two neighbors in D. Hence, there are at most $\lceil \frac{|N(w)|-2}{2} \rceil$ other sets that can form a double total coalition with X, and therefore, X is in at most $\lceil \frac{\Delta(G)-2}{2} \rceil + 1 = \lceil \frac{\Delta(G)}{2} \rceil$ double total coalitions. Now we assume that $w \notin X$ and $X \cap N(w) = \emptyset$. So, each set of Θ which is in a double total coalition with X must contain at least two of the members of $N(w)$. Hence, we can easily see that the total number of sets of Θ forming a double total coalition with X is at most $\frac{\Delta(G)}{2}$.

In light of the above, we conclude that X is in at most $\Delta(G) - 1$ double total coalitions.□

3 Exact values

In this section, we deal with the problem of obtaining the exact value of the double total coalition number of some graphs. We begin with complete graph.

Proposition 1 *For any $n \geq 3$, $DTC(K_n) = n - 1$.*

Proof. By directly applying Theorem from Henning et al. [14], we deduce that $\gamma_{\times 2,t}(K_n) = 3$. Now we show that there is no *dtc*-partition of order n. Suppose, to the contrary, that $DTC(K_n) = n$. It follows that the only possible partition of n sets is n singleton sets. Since $\gamma_{\times 2,t}(K_n) = 3$, no two singleton sets form a double total coalition. Hence, $DTC(K_n) < n$. We next proceed to establish a maximum double total coalition partition of order $n - 1$ for K_n as follows.

$$\Theta(K_n) = \{V_1 = \{v_1, v_2\}, V_2 = \{v_3\}, \ldots, V_{n-1} = \{v_n\}\}.$$

Note that each of V_i for $2 \leq i \leq n-1$ form a double total coalition with V_1. This completes the proof. □

Theorem 3 *For any graph G of order n, $DTC(G) = n - 1$ if and only if $K_2 + \overline{K}_{n-2}$ is a subgraph of G.*

Proof. Let G be a graph of order n. We begin by assuming that $DTC(G) = n - 1$. Consider a $DTC(G)$-partition Θ. By definition, Θ includes one set of cardinality two and $n - 2$ singleton sets. Denote the set of cardinality two as $\{x, y\}$ and the singleton sets as $\{v_1\}, \{v_2\}, \ldots, \{v_{n-2}\}$. It is evident that neither the set $\{x, y\}$ nor any of the singleton sets $\{v_i\}$ for $i = 1, \ldots, n - 2$ can be a double total dominating set. Furthermore, any pair of singleton sets $\{v_i\}$ and $\{v_j\}$ cannot form a double total dominating set either. Consequently, each singleton set $\{v_i\}$ must be involved in a double total coalition with the pair $\{x, y\}$. This implies that the set $\{x, y, v_i\}$ forms a double total dominating set for each i. According to the definition, the induced subgraph of G formed by the vertices x, y, and v_i is isomorphic to K_3. Therefore, we can conclude that $K_2 + \overline{K}_{n-2}$ is a subgraph of G.

The converse of this statement is also straightforward and follows from the definitions involved.□

The following proposition gives us the double total coalition number of complete bipartite graph.

Proposition 2 *Let $G = K_{p,q}$ be a complete bipartite graph such that $q \geq p \geq 4$, then $DTC(K_{p,q}) = p + q - 4$.*

Proof. Let $G = K_{p,q}$ be a complete bipartite graph with two partite sets $X = \{v_1, v_2, \ldots, v_p\}$ and $Y = \{u_1, u_2, \ldots, u_q\}$. One can observe that the vertex partition

$$\Theta = \{\{v_1, v_2, u_1\}, \{v_3\}, \{v_4\}, \ldots \{v_{p-1}\}, \{u_2\}, \{u_3\}, \ldots, \{u_{q-2}\}, \{v_p, u_{q-1}, u_q\}\}$$

is a *dtc*-partition of G of order $p+q-4$. Thus, $DTC(K_{p,q}) \geq p+q-4$. Next, we shall show that $DTC(K_{p,q}) \leq p+q-4$. We first assume that there is a *dtc*-partition of G of order $p+q=n$. By Corollary 2, we deduce that the partition does not exist. If we suppose that there is a *dtc*-partition of G of order $p+q-1=n-1$, then the partition consists of a set of cardinality 2 and $p+q-2$ singleton sets. Since $\gamma_{\times 2,t}(K_{p,q})=4$, no two sets can be in a double total coalition. Thus, the partition does not exist. We proceed further with the following cases.

Case 1. Let Θ_1 is a *dtc*-partition of G of order $p+q-2=n-2$. We consider the following subcases.

Subcase 1.1. Θ_1 consists of a set of cardinality 3, say A, and $p+q-3$ singleton sets. Since $\gamma_{\times 2,t}(K_{p,q})=4$, each singleton set must form a double total coalition with A. To produce a double total coalition, the vertex v either belongs to the set Y and is adjacent to exactly two vertices in X or belongs to the set X and is adjacent to exactly two vertices in Y. Assume that A contains exactly one vertex of X and two vertices of Y. Then there are $q-2$ singleton sets belonging to the set Y which form double total coalitions with A, while the remaining $p-1$ singleton sets belonging to the set X do not form double total coalitions with A. Hence, Θ_1 does not exist. Note that we have the same result when A contains exactly one vertex of Y and two vertices of X.

Subcase 1.2. Θ_1 consists of two sets of cardinality 2, say A and B, and $p+q-4$ singleton sets. Since $\gamma_{\times 2,t}(K_{p,q})=4$, no two singleton sets form a double total coalition. Furthermore, neither A nor B can form a double total coalition with any singleton set of Θ_1. Hence, Θ_1 does not exist.

Case 2. Let Θ_2 is a *dtc*-partition of G of order $p+q-3=n-3$. We consider the following subcases.

Subcase 2.1. Θ_2 consists of a set of cardinality 4 and $p+q-4$ singleton sets. Since no two singleton sets form a double total coalition, each of them must be in a double total coalition with a set of cardinality 4, such as A. On the other side, by Lemma 1, A is in at most $q-1$ (or $p-1$) double total coalitions. Then there are $p-3$ (or $q-3$) singleton sets which cannot form double total coalitions with A. Thus, this partition does not exist.

Subcase 2.2. Θ_2 consists of a set of cardinality 3, a set of cardinality 2 and $p+q-5$ singleton sets. Let $A=\{a,b,c\} \in \Theta_2$ be a set of cardinality 3, and $B=\{z,w\} \in \Theta_2$ be a set of cardinality 2. It is evident that no singleton set in Θ_2 forms a double total coalition with B, as this would imply the existence of an odd cycle in $K_{p,q}$. Consequently, any singleton set in Θ_2 can only form a double total coalition with A. Therefore, B must form a double total coalition exclusively with A. Since A forms a double total coalition with any singleton set, it follows that $A \cap X \neq \emptyset$ and $A \cap Y \neq \emptyset$. Without loss of generality, assume $a,b \in X$ and $c \in Y$. Because A and B form a double total coalition, the set $A \cup B$ constitutes a double total dominating set. By the definition of a double total dominating set, in order to dominate z and w via $A \cup B$, we must have $z,w \in Y$ or $z \in X$ and $w \in Y$ (or $z \in Y$ and

$w \in x$). Suppose, without loss of generality, that $z, w \in Y$, or $z \in X$ and $w \in Y$. If $z, w \in Y$, since a and b are adjacent to both z and w, we can partition B into two singleton sets, $\{z\}$ and $\{w\}$. Subsequently, we remove B from Θ_2 and reassign its elements back to Θ_2. This process results in a new double total coalition partition with cardinality $n-2$. However, this leads to a contradiction because in **Case 1**, it was demonstrated that the scenario where $n-2$ occurs is not possible. Now, if $z \in X$ and $w \in Y$, we can transform Θ_2 into a new double total coalition partition by redefining A as $A = \{a, b, c, z\}$ and B as $B = \{w\}$. It is straightforward to observe that this scenario reduces to **Subcase 2.1**, which has already been shown to be impossible.

Subcase 2.3. Θ_2 consists of three sets of cardinality 2 and $p + q - 6$ singleton sets. The similar argument from **Subcase 1.2** can be used to show that Θ_2 does not exist.

Based on the analysis of all the above cases, we conclude that $DTC(K_{p,q}) \leq p + q - 4$. Therefore, $DTC(K_{p,q}) = p + q - 4$.□

Next we determine the double total coalition number of cycles. Before presenting the next result, we recall the following theorem.

Theorem 4 [14] *If G is a graph of order n with a minimum degree 2 and a maximum degree at most $\frac{n}{2}$, $\gamma_{\times 2,t}(G) = n$.*

Theorem 5 *For a cycle C_n of order $n \geq 3$, $DTC(C_n) = 2$.*

Proof. Assume that Θ is a *dtc*-partition of C_n. By Corollary 2, we have $DTC(C_n) \geq 2$ for any cycle C_n. Now we show that $DTC(C_n) \neq 3$. Suppose, to the contrary, that $DTC(C_n) = 3$. Let $\Theta = \{A, B, C\}$ be a $DTC(C_n)$-partition. By Lemma 1, each set of Θ is in double total coalition with at most one set of Θ. So, without loss of generality, assume that each of B and C forms a double total with A. Since $\gamma_{\times 2,t}(C_n) = n$, it holds that $|A| + |B| = n$ and $|A| + |C| = n$. Therefore, $2|A| + |B| + |C| = 2n$. On the other hand, we know that $|A| + |B| + |C| = n$. Then $|A| = n$, which is a impossible as $|A| < n$. Hence, $DTC(C_n) \neq 3$ and $DTC(C_n) \leq 2$. Now, we establish a maximum double total coalition partition of order 2 for C_n as follows.

$$\Theta(C_n) = \{V_1 = \{v_{2i} | 1 \leq 2i \leq n\}, V_2 = \{v_{2i+1} | 1 \leq 2i + 1 \leq n\}\}.$$

Note that V_1 and V_2 form a double total coalition. This completes the proof. □

4 Conclusion

In this paper, we initiated a study of double total coalition in graphs. We studied the existence of a double total coalition partition and established some bounds for the double total coalition number. In addition, we determined this number for specific classes of graphs and characterized the graphs G with large double total coalition number. Numerous problems for future research remain within this topic. We state some of them here.

(a) Establish the upper and lower bounds for double total coalition number.
(b) Characterization graphs of order n whose double total coalition number is $n-2$.
(c) Study the double total coalitions in regular graphs.
(d) Study the double total coalition of some product of two graphs.

References

[1] S. Alikhani, D. Bakhshesh, H.Golmohammadi, Total coalitions in graphs, *Quaest.Math.* (2024) **47**(11) 2283-2294. https://doi.org/10.2989/16073606.2024.2365365

[2] S. Alikhani, D. Bakhshesh, H. Golmohammadi, S. Klavžar, On independent coalition in graphs and independent coalition graphs, *Discuss. Math. Graph Theory* (2024). https://doi.org/10.7151/dmgt.2543.

[3] S. Alikhani, D. Bakhshesh, H. Golmohammadi, E.V. Konstantinova, Connected coalitions in graphs, *Discuss. Math. Graph Theory* **44** (2024) 1551–1566. https://doi.org/10.7151/dmgt.2509

[4] D. Bakhshesh, M.A. Henning and D. Pradhan, On the coalition number of trees, *Bull. Malays. Math. Sci. Soc.* **46** (2023) 95. https://doi.org/10.1007/s40840-023-01492-4.

[5] J. Barát, Z.L. Blázsik, General sharp upper bounds on the total coalition number, *Discuss. Math. Graph Theory* 44 (2024) 1567–1584. https://doi.org/10.7151/dmgt.2511.

[6] S. Bermudo, J. C. Hernandez-Gomez, J. M. Sigarreta, Total k-dominaiton in strong product graphs, *Discrete Appl. Math.*, **263** (2019), 51–58. https://doi.org/10.1016/j.dam.2018.03.043.

[7] A. Cabrera-Martınez, F. A. Hernandez-Mira, New bounds on the double total domination number of graphs, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 443–453. https://doi.org/10.1007/s40840-021-01200-0.

[8] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, Introduction to coalitions in graphs, *AKCE Int. J. Graphs Combin.* **17** (2) (2020) 653–659. https://doi.org/10.1080/09728600.2020.1832874.

[9] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Coalition graphs of paths, cycles, and trees, *Discuss. Math. Graph Theory* **43** (2023) 931–946. https://doi.org/10.7151/dmgt.2416.

[10] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae and R. Mohan, Upper bounds on the coalition number, *Austral. J. Combin.* **80** (3) (2021), 442–453.

[11] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, R. Mohan, Coalition graphs, *Comm. Combin. Optim.* **8** (2023), no. 2, 423–430. https://doi.org/10.22049/cco.2022.27916.1394.

[12] T.W. Haynes, S. T. Hedetniemi, and M. A. Henning, Domination in Graphs: Core Concepts Series: Springer Monographs in Mathematics, Springer, Cham, 2023. xx + 644 pp.

[13] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, in: Chapman and Hall/CRC Pure and Applied Mathematics Series, Marcel Dekker, Inc. New York, 1998.

[14] M.A. Henning, A.P. Kazemi, k-tuple total domination in graphs, *Discrete Appl. Math.* **158** (2010) 1006–1011. https://doi.org/10.1016/j.dam.2010.01.009.

[15] M.A. Henning, A.Yeo, Strong transversals in hypergraphs and double total domination in graphs, *SIAM J. Discrete Math.* **24** (2010) 1336–1355. https://doi.org/10.1137/090777001.

[16] N.J. Rad, Upper bounds on the k-tuple domination number and k-tuple total domination number of a graph, *Australas. J. Combin.* **73** (2019) 280–290.

[17] S.M. Sheikholeslami, L. Volkmann, The k-tuple total domatic number of a graph, *Util. Math.* *95* (2014) 189-197. https://utilitasmathematica.com/index.php/Index/article/view/1024.