

THE FOKAS' METHOD FOR HEAT TRANSFER EQUATION ON SYMMETRIC GRAPHS

M.R. ESHIMBETOV 

Communicated by P.P. PETROV

Abstract: In the present paper we obtained integral-representation of solutions in terms of given initial and boundary data for the initial-boundary value problems for heat transfer equation on symmetric metric graph via Fokas' unified transformation method.

Keywords: Heat transfer equation, metric graphs, branched structures, Fokas' method, unified transformation, Fourier transformation, initial problem, boundary value problem.

1 Introduction

During the last years the branched thin structures and metric graphs are widely used as a model in theoretical investigation of many complex systems from physics, biology, ecology, sociology, economy and finance[1]-[3]. Despite to high interest for diffusive and dispersive wave transfer problems in such system, here only few papers on the initial-boundary value problems (IBVP) on metric graphs. Firstly IBVP for heat equation on graphs are investigated in [7]. Some numerical methods for solution of IBVP for heat equation can be found in[8]. In the paper [13] it is proved existence and uniqueness of the generalized (weak) solution of initial boundary value problem on general

ESHIMBETOV, M.R., ON SOME PROBLEMS OF COMMUTATIVE ALGEBRA.

© 2025 ESHIMBETOV M.R.

The work is supported by RFFI (grant FL-8223102079).

Received January, 1, 2025, Published December, 31, 2025.

graphs in Sobolev spaces. In [9] and [10] thermal diffusion in branched structures based on linear and nonlinear heat equation and some other physical applications are discussed.

Usually, for investigation physical properties of quantum graphs used static Schrödinger equation [3]–[7]. In the references [16] was investigated nonlinear Schrödinger equation on two dimensional thin tabular branched domain and proved that the problem on metric graphs for one dimensional nonlinear Schrödinger equation on metric graph, with gluing (Kirchhoff) conditions on the vertex point, can be obtained when width of the branches tends to zero. Similar convergence result in the case of linear Schrödinger equation with different approaches can be found in [6]–[7].

Schrödinger equation can be also called to be heat equation with imaginary time. The heat equation on branched structures firstly used in the 50's of the nineteenth century. Thomson (Lord Kelvin) used heat equation (Thomson's cable equation) as mathematical models of signal decay in submarine (under water) telegraphic cables (Ch. IV in [9]). Later this method was widely used in neuroscience for theoretical analyzing data collected from intracellular micro electrode recordings and for analyzing the electrical properties of neuronal dendrites (see [10]). Initial and boundary value problems for some other types of PDE on metric graphs and their possible applications can be found in [11], [16]–[17].

In this paper we construct the integral representation of solution of IBVP for heat equation on general star graph with both finite and semi infinite bonds [11]. For this purpose we generalize so called Fokas' method in the case of metric graph [12]–[15]. It should be noted that this method was used in [16], [17] for the IBVP for heat equation on star graphs with only finite or semi infinite bonds.

2 Formulation of the problem and main result

We consider symmetric metric graph which obtained by connecting n finite B_1, \dots, B_n and m semi infinite $B_{n+1}, B_{n+2}, \dots, B_{n+m}$ bonds at one point, called to be vertex of the graph. We correspond the bonds B_j , ($j = \overline{1, n}$) to the intervals $(0, L_j)$ and the bonds B_r , ($r = \overline{n+1, n+m}$) to intervals $(0, \infty)$ to define coordinates in each bond. Here vertex of the graph corresponds to 0 on each bond (Figure 1).

In each bond of the graph consider the heat transfer equation

$$V_t^{(j)}(x, t) = V_{xx}^{(j)}(x, t) + f^{(j)}(x, t), \quad j = \overline{1, n+m}. \quad (1)$$

Define initial conditions

$$V^{(j)}(x, 0) = V_0^{(j)}(x), \quad x \in \overline{B_j}, \quad j = \overline{1, n+m}. \quad (2)$$

Boundary and the asymptotic conditions

$$\left(\alpha_j V^{(j)}(x, t) + \beta_j V_x^{(j)}(x, t) \right) |_{x=L_j-0} = 0, \quad \alpha_j^2 + \beta_j^2 \neq 0, \quad j = \overline{1, n}, \quad (3)$$

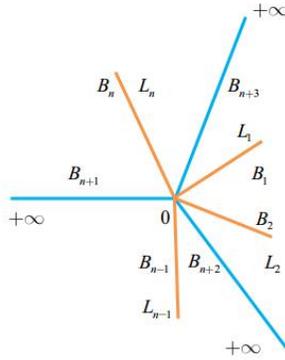


Figure 1. Symmetric metric graph

$$\lim_{x \rightarrow \infty} V^{(r)}(x, t) = 0, \quad t \geq 0, \quad r = \overline{n+1, n+m} \quad (4)$$

on finite and semi infinite bonds, respectively.

Moreover, we need to define the following gluing conditions for connectivity of the graph

$$V^{(1)}(+0, t) = V^{(2)}(+0, t) = \dots = V^{(n+m)}(+0, t), \quad (5)$$

$$\sum_{j=1}^{n+m} \delta_j^2 V_x^{(j)}(+0, t) = 0. \quad (6)$$

The last conditions usually called continuity and Kirchhoff conditions on branching point of the graphs. We suppose, that initial data are smooth enough functions and they satisfies the conditions (3)–(6).

Step 1. Write the PDE as a one-parameter "family" of equations, each of which is in divergence form [14]–[15].

For problem (1), a clear choice of an integrating factor to achieve this is $\tilde{q}(x, t) = e^{-ikx+k^2t}$, leading to the form

$$\begin{aligned} \left[e^{-ikx+k^2t} V^{(j)}(x, t) \right]_t &= \left[e^{-ikx+k^2t} \left(V_x^{(j)}(x, t) + ikV^{(j)}(x, t) \right) \right]_x + \\ &+ e^{-ikx+k^2t} f^{(j)}(x, t), \quad j = \overline{1, n+m}. \end{aligned} \quad (7)$$

Step 2. Use the divergence form to determine a global relation coupling the various boundary values.

In the case of (1), this takes the form

$$\begin{aligned} e^{wt} \widehat{V}^{(j)}(k, t) - \widehat{V}_0^{(j)}(k) &= e^{-ikL_j} \left(h_1^{(j)}(w, t) + ikh_0^{(j)}(w, t) \right) - \\ &- (g_j(w, t) + ikg_0(w, t)) + \widehat{F}^{(j)}(k, t), \quad j = \overline{1, n}; \end{aligned} \quad (8)$$

$$e^{wt} \widehat{V}^{(r)}(k, t) - \widehat{V}_0^{(r)}(k) = -g_r(w, t) - ikg_0(w, t) + \widehat{F}^{(r)}(k, t), \quad r = \overline{n+1, n+m}. \quad (9)$$

where $w = k^2, \{k \in \mathbb{C} : Imk > 0\}$.

However dispersion relation $w = k^2$ invariant with respect to substitution $k \rightarrow -k$ then functions $g_0(w, t)$, $g_j(w, t)$, $g_r(w, t)$, $h_0^{(j)}(w, t)$, $h_1^{(j)}(w, t)$, $j = \overline{1, n}$, $r = \overline{n+1, n+m}$ also be invariant. Hence from (8) and (9) we have:

$$e^{wt}\widehat{V}^{(j)}(-k, t) - \widehat{V}_0^{(j)}(-k) = e^{ikL_j} \left(h_1^{(j)}(w, t) - ikh_0^{(j)}(w, t) \right) - (g_j(w, t) - ikg_0(w, t)) + \widehat{F}^{(j)}(-k, t), \quad j = \overline{1, n}; \quad (10)$$

$$e^{wt}\widehat{V}^{(r)}(-k, t) - \widehat{V}_0^{(r)}(-k) = -g_r(w, t) + ikg_0(w, t) + \widehat{F}^{(r)}(-k, t), \quad (11)$$

where $r = \overline{n+1, n+m}$, $\{k \in \mathbb{C} : \text{Im}k < 0\}$.

We can write solution in following form with using inverse Fourier transformation in global relation (8) and (9) [16]–[17].

$$\begin{aligned} V^{(j)}(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} \left(\widehat{V}_0^{(j)}(k) + \widehat{F}^{(j)}(k, t) \right) dk - \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-ikL_j-wt} \left(h_1^{(j)}(w, t) + ikh_0^{(j)}(w, t) \right) dk - \\ &- \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} (g_j(w, t) + ikg_0(w, t)) dk, \quad j = \overline{1, n}, \end{aligned} \quad (12)$$

$$\begin{aligned} V^{(r)}(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} \left(\widehat{V}_0^{(r)}(k) + \widehat{F}^{(r)}(k, t) \right) dk - \\ &- \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} (g_r(w, t) + ikg_0(w, t)) dk, \quad r = \overline{n+1, n+m}. \end{aligned} \quad (13)$$

The integrand of the second integral in (12) and (13) is entire and decays as $k \rightarrow \infty$ for $k \in \mathbb{C}^- \setminus D^-$. The third integral has an integrand that is entire and decays as $k \rightarrow \infty$ for $k \in \mathbb{C}^+ \setminus D^+$. Using the analyticity of the integrand and applying Jordan's Lemma we can replace the contour of integration of the second integral by $-\int_{\partial D^-}$ and of the third integral by $-\int_{\partial D^+}$ (see [12], [14]–[17]):

$$\begin{aligned} V^{(j)}(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} \left(\widehat{V}_0^{(j)}(k) + \widehat{F}^{(j)}(k, t) \right) dk - \\ &- \frac{1}{2\pi} \int_{\partial D^-} e^{ikx-ikL_j-wt} \left(h_1^{(j)}(w, t) + ikh_0^{(j)}(w, t) \right) dk - \\ &- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-wt} (g_j(w, t) + ikg_0(w, t)) dk, \quad j = \overline{1, n}, \end{aligned} \quad (14)$$

$$\begin{aligned} V^{(r)}(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} \left(\widehat{V}_0^{(r)}(k) + \widehat{F}^{(r)}(k, t) \right) dk - \\ &- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-wt} (g_r(w, t) + ikg_0(w, t)) dk, \quad r = \overline{n+1, n+m}, \end{aligned} \quad (15)$$

where $D^\pm = \{k \in \mathbb{C}^\pm : \text{Re}k^2 < 0\}$, $\mathbb{C}^+ = \{k \in \mathbb{C} : \text{Im}k > 0\}$,

$$\mathbb{C}^- = \{k \in \mathbb{C} : \text{Im}k < 0\}.$$

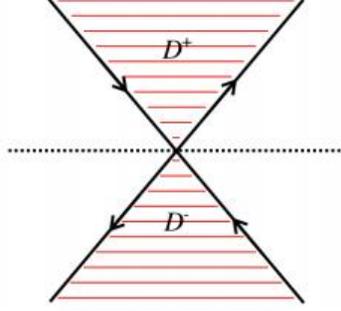


Figure 2. The domains D^+ and D^- for the heat transfer equation

We need to find unknowns $g_0(w, t), g_j(w, t), g_r(w, t), h_0^{(j)}(w, t), h_1^{(j)}(w, t)$, $j = \overline{1, n}, r = \overline{n+1, n+m}$ representation of the solution.

Now, using vertex conditions we get

$$\begin{cases} e^{wt}\widehat{V}^{(j)}(k, t) - \widehat{V}_0^{(j)}(k) = e^{-ikL_j}h_1^{(j)}(w, t) + ike^{-ikL_j}h_0^{(j)}(w, t) - \\ -g_j(w, t) - ikg_0(w, t) + F^{(j)}(k, t); \\ e^{wt}\widehat{V}^{(j)}(-k, t) - \widehat{V}_0^{(j)}(-k) = e^{ikL_j}h_1^{(j)}(w, t) - ike^{ikL_j}h_0^{(j)}(w, t) - \\ -g_j(w, t) + ikg_0(w, t) + \widehat{F}^{(j)}(-k, t); \\ e^{wt}\widehat{V}^{(r)}(k, t) - \widehat{V}_0^{(r)}(k) = -g_r(w, t) - ikg_0(w, t) + \widehat{F}^{(r)}(k, t); \\ \sum_{j=1}^6 \delta_j^2 g_j(w, t) = 0; \\ \alpha_j h_0^{(j)}(w, t) + \beta_j h_1^{(j)}(w, t) = 0. \end{cases} \quad (16)$$

Where $j = \overline{1, n}, r = \overline{n+1, n+m}$.

Solving this equations for $ikg_0(w, t)$, we have

$$\begin{aligned} ikg_0(w, t) = & \frac{1}{\sum_{j=1}^{n+m} \delta_j^2} \left[\sum_{j=1}^n \frac{\delta_j^2}{A_j + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \cdot B_j} \left[e^{ikL_j} \left(1 + i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}_0^{(j)}(k) - \right. \right. \\ & \left. \left. - e^{-ikL_j} \left(1 - i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}_0^{(j)}(-k) \right] + \sum_{r=n+1}^{n+m} \delta_r^2 \widehat{V}_0^{(r)}(k) + \right. \\ & \left. + \sum_{j=1}^n \frac{\delta_j^2}{A_j + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \cdot B_j} \left[e^{ikL_j} \left(1 + i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{F}^{(j)}(k, t) - \right. \right. \\ & \left. \left. - e^{-ikL_j} \left(1 - i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{F}^{(j)}(-k, t) \right] + \sum_{r=n+1}^{n+m} \delta_r^2 \widehat{F}^{(r)}(k, t) - \right. \\ & \left. - \sum_{j=1}^n \frac{\delta_j^2 e^{wt}}{A_j + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \cdot B_j} \left[e^{ikL_j} \left(1 + i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}^{(j)}(k, t) - \right. \right. \end{aligned} \quad (17)$$

$$-e^{-ikL_j} \left(1 - i \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}^{(j)}(-k, t) \Big] - \sum_{r=n+1}^{n+m} \delta_r^2 e^{wt} \widehat{V}^{(r)}(k, t) \Big],$$

where $A_j = e^{ikL_j} - e^{-ikL_j}$, $B_j = e^{ikL_j} + e^{-ikL_j}$, $j = \overline{1, n}$, $\alpha_j \neq 0$.

Now putting $G^{(j)}(k, t) = \widehat{V}_0^{(j)}(k) - ikg_0(w, t) + \widehat{F}^{(j)}(k, t)$, $j = \overline{1, n}$ we can rewrite

$$\begin{cases} e^{wt} V^{(j)}(k, t) = G^{(j)}(k, t) + e^{-ikL_j} \left(1 - i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) h_1^{(j)}(w, t) - g_j(w, t), \\ e^{wt} V^{(j)}(-k, t) = G^{(j)}(-k, t) + e^{ikL_j} \left(1 + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) h_1^{(j)}(w, t) - g_j(w, t). \end{cases} \quad (18)$$

So, we have

$$h_1^{(j)}(w, t) = \frac{1}{A_j + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \cdot B_j} \left[G^{(j)}(k, t) - G^{(j)}(-k, t) - e^{wt} \left(\widehat{V}^{(j)}(k, t) - \widehat{V}^{(j)}(-k, t) \right) \right], \quad (19)$$

where $h_0^{(j)}(w, t) = -\frac{\beta_j}{\alpha_j} h_1^{(j)}(w, t)$, $\alpha_j \neq 0$, $j = \overline{1, n}$.

$$\begin{aligned} g_j(w, t) &= \frac{1}{A_j + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \cdot B_j} \left[e^{ikL_j} \left(1 + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) G^{(j)}(k, t) - \right. \\ &- e^{-ikL_j} \left(1 - i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) G^{(j)}(-k, t) - e^{wt+ikL_j} \left(1 + i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}^{(j)}(k, t) - \\ &\left. - e^{wt-ikL_j} \left(1 - i \cdot \frac{\beta_j}{\alpha_j} \cdot k \right) \widehat{V}^{(j)}(-k, t) \right], \quad (20) \end{aligned}$$

this $\alpha_j \neq 0$, $j = \overline{1, n}$.

Solving (11) for $g_r(w, t)$ we find

$$g_r(w, t) = \widehat{V}_0^{(r)}(-k) + ikg_0(w, t) + \widehat{F}^{(r)}(-k, t) - e^{wt} \widehat{V}^{(r)}(-k, t). \quad (21)$$

Replacing in equation (15) $g_r(w, t)$ with the RHS of (21) we find [16]–[17]

$$V^{(r)}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx-wt} \left(\widehat{V}_0^{(r)}(k) + \widehat{F}^{(r)}(k, t) \right) dk - \quad (22)$$

$$- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-wt} \left(2ikg_0(w, t) + \widehat{V}_0^{(r)}(-k) + \widehat{F}^{(r)}(-k, t) \right) dk, \quad r = \overline{n+1, n+m},$$

The term $e^{wt} \widehat{V}^{(r)}(-k, t)$ gives rise to the term [13]

$$- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx} \widehat{V}^{(r)}(-k, t) dk, \quad 0 < x < \infty, \quad t > 0,$$

which vanishes, since both e^{wt} and $\widehat{V}^{(r)}(-k, t)$ are bounded and analytic in the upper half of the complex k plane, and furthermore $\widehat{V}^{(r)}(-k, t)$ is of $O\left(\frac{1}{k}\right)$ as $k \rightarrow \infty$:

$$\widehat{V}^{(r)}(-k, t) = \int_0^\infty e^{ikx} V^{(r)}(x, t) dx \sim -\frac{V^{(r)}(0, t)}{ik}, \quad k \rightarrow \infty.$$

Thus, Cauchy's theorem supplemented with Jordan Lemma in the domain D^+ [12].

We next substitute $g_1^{(j)}(w, t)$ and $h_1^{(j)}(w, t)$ in (14). We claim that the terms involving $\widehat{V}^{(j)}(\pm k, t)$ yield a zero contribution. Indeed, since this is a well-posed BVP, the relevant terms are bounded as $k \rightarrow \infty$. Let us verify this explicitly: the term in $g_1^{(j)}(w, t)$ involves the following contribution form $\widehat{V}^{(j)}(\pm k, t)$:

$$\frac{e^{ikL_j} \widehat{V}^{(j)}(k, t) - e^{-ikL_j} \widehat{V}^{(j)}(-k, t)}{e^{ikL_j} \left(1 + i\frac{\beta_j}{\alpha_j} k\right) - e^{-ikL_j} \left(1 - i\frac{\beta_j}{\alpha_j} k\right)}.$$

Since $Imk \geq 0$, e^{-ikL_j} grows, and then the above expression, as $k \rightarrow \infty$, becomes

$$\frac{-\widehat{V}^{(j)}(-k, t) + e^{ikL_j} \int_0^{L_j} e^{ik(L_j-x)} V^{(j)}(x, t) dx}{i\frac{\beta_j}{\alpha_j} k - 1},$$

which is clearly bounded as $k \rightarrow \infty$ with $Imk \geq 0$. Similarly the term in $h_1^{(j)}(w, t)$ involves the following contribution from $\widehat{V}^{(j)}(\pm k, t)$:

$$\frac{\widehat{V}^{(j)}(k, t) - \widehat{V}^{(j)}(-k, t)}{e^{ikL_j} \left(1 + i\frac{\beta_j}{\alpha_j} k\right) - e^{-ikL_j} \left(1 - i\frac{\beta_j}{\alpha_j} k\right)},$$

which as $k \rightarrow \infty$, $Imk \leq 0$, simplifies to the expression

$$\frac{\int_0^{L_j} e^{-ik(L_j+x)} V^{(j)}(x, t) dx - e^{-ikL_j} \widehat{V}^{(j)}(-k, t)}{i\frac{\beta_j}{\alpha_j} k - 1},$$

which is clearly bounded as $k \rightarrow \infty$, $Imk \leq 0$ [12].

References

- [1] R. Albert, A. L. Barabasi, *Statistical mechanics of complex networks*, Rev. Mod. Phys. A, 74, 47. 2002.
- [2] R. Cohen, S. Havlin, *Complex Networks: Structure, Robustness and Function*, Cambridge University Press. 2010.
- [3] Tsampikos Kottos and Uzy Smilansky, *Quantum Chaos on Graphs*, Ann. Phys., 76. 274. 1999.
- [4] Sven Gnutzmann and Uzy Smilansky, *Quantum graphs: Applications to quantum chaos and universal spectral statistics*, Adv. Phys. 55 527. 2006.
- [5] S. Gnutzmann, J. P. Keating, F. Piotet, *Eigenfunction statistics on quantum graphs*, Ann.Phys., 325 2595. 2010.

- [6] G. F. Dell'Antonio and E. Costa, *Effective Schrödinger dynamics on o-thin Dirichlet waveguides via quantum graphs: I. Star-shaped graphs*, J. Phys. A, Math. Theor., 43, 474014, 23. 2010.
- [7] P. Exner, O. Post, *A General Approximation of Quantum Graph Vertex Couplings by Scaled Schrödinger Operators on Thin Branched Manifolds*, Commun. Math. Phys., 1322 207-227. 2013.
- [8] H. Uecker, D. Grieser, Z. Sobirov, D. Babajanov, D. Matrasulov, *Soliton transport in tubular networks: Transmission at vertices in the shrinking limit*, Phys. Rev. E 91, 023209, 2015.
- [9] Gustav Doetsch, *Guide to the applications of the Laplace and Z-transforms*, Munchen, Wien : Oldenbourg. 1971.
- [10] Wilfrid Rall, *Branching Dendritic Trees and Motoneuron Membrane Resistivity*, Experimental neurology 1, 491-527. 1959.
- [11] Z.A. Sobirov, M.I. Akhmedov, H. Uecker, *Cauchy problem for the linearized KdV equation on general metric star graphs*, Наносистемы: физика, химия, математика, 6:2 (2015), 198–204.
- [12] A.S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-NSF Regional Conference Series in Applied Mathematics. 2008., p: 352.
- [13] A.S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs*, In Proc. R. Soc. A, volume 453, pages 1411–1443. The Royal Society, 1997.
- [14] N.E. Sheils and D.A. Smith, *Heat equation on a network using the Fokas method*, J. Phys. A: Math. Theor. 48 335001. 2015.
- [15] N.E. Sheils, *Multilayer diffusion in a composite medium with imperfect contact*, Applied Mathematical Modelling Volume 46, June 2017, Pages 450-464.
- [16] G.Khudayberganov, Z.A.Sobirov, M.R.Eshimbetov, *Unified Transform method for the Schrödinger Equation on a Simple Metric Graph*, Journal of Siberian Federal University. Mathematics & Physics 2019, 12(4), 412–420.
- [17] З.А.Собиров, М.Р.Эшимбетов, *Метод Фокаса для уравнения теплопроводности на метрических графах*, Современная математика. Фундаментальные направления, 2021, том 67, выпуск 4, страницы 766–782.

ESHIMBETOV MARDONBEK REYIMBOYEVICH
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER MIRZO ULUGBEK,
STR. UNIVERSITY, 4,
100174, TASHKENT, UZBEKISTAN
E-mail address: mr.eshimbetov@nuu.uz