ON SOLVING *HAMILTON'S* GENERAL QUADRATIC EQUATION

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ABSTRACT. Based on *Hamilton's* method of solving the quadratic equation (couple equation), defined over the field of couple numbers, as he calls them, in this paper the method of solving the general *Hamilton* quadratic equation is reduced to solving of the matrix quadratic equation, which is defined over the field of bireal matrices, which correspond to bireal numbers. As the matrix equation is decomposed into a system of nonlinear algebraic equations (SNAE), the solutions of the matrix equation are also the solutions of the SNAE, such that the solutions of the matrix quadratic equation implicitly determine the symmetric matrices S of the null space of the SNAE. In the second part of the paper, the matrix method for solving the SNAE, which is obtained via decomposition of *Hamilton's* quadratic equation (1.11), presented.

1. INTRODUCTION

In the middle of the 19th century, on the basis of the previously defined product of ordered pairs of real numbers (number couples) [4, 5], as follows

$$(a,b)\,(c,d) = (ac - bd, ad + bc)\,, \tag{1.1}$$

the process of solving the quadratic equation (couple equation)

$$(x,y)^2 + (b,0)\,(x,y) + (c,0) = (0,0)\,, \tag{1.2}$$

Hamilton reduced to solving an ordinary quadratic equation $x^2 + bx + c = 0$. More precisely, *Hamilton* reduced the process of solving the equation (1.2) to the process of solving a system of two separate equations

$$x^2 - y^2 + bx + c = 0 \text{ and } 2xy + by = 0, \tag{1.3}$$

which always allows real solutions, regardless of whether the discriminant $b^2/4 - c$ is a positive or negative real number, [5]. In the first case, $y = 0$ in the second equation of the system, which leads us to the first solution

$$(x,y) = (-\frac{b}{2} \pm \sqrt[2]{b^2/4 - c}, 0). \tag{1.4}$$

In the second case, the second factor $2x + b$, in the second equation of system (1.3), should be equalized to zero and thus reach another solution

$$(x,y) = (-\frac{b}{2}, \pm\sqrt[2]{c - b^2/4}). \tag{1.5}$$

On the other hand, systems of nonlinear algebraic equations (SNAE), to which system (1.3) itself belongs, are more ubiquitous in many numerous applications.

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Solving SNAE, in general, requires numerical simulation, and increasingly robust and efficient methods for solving SNAE are continuously sought. At the end of the last century, as well as at the beginning of this century, the so-called method of neural networks was imposed, [3, 7, 8]. In [2] an overview of existing algorithmic approaches for solving SNAE, namely: reduction to *Grebner's* basis, the multidimensional resulting method and the spectral method. In addition, it is shown that the problem of solving SNAE is equivalent to the problem of finding a rank 1 matrix in a given subspace of a space of matrices. All of these methods start from a system of k nonlinear algebraic equations ($k \geq 1$) with n unknowns x_i, which are elements of the $n \times 1$ column matrix $\mathbf{x} = (x_i)^T = [x_1\ x_1\ ...\ x_n]^T$ $(n \geq 2)$, as follows

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})_k\, x_i x_j + 2\sum_{i=1}^{n}(b_i)_k\, x_i + c_k = 0. \tag{1.6}$$

In the matrix notation of system (1.6)

$$\mathbf{x}^T A_k \mathbf{x} + 2\mathbf{b}_k^T \mathbf{x} + c_k = 0, \tag{1.7}$$

$A_k = (a_{ij})_k$ are symmetric $n \times n$ matrices, $\mathbf{b}_k^T = (b_i)_k^T$ are $1 \times n$ row matrices and c_k are constants. Let $\hat{A}_k$ denote the accompanying square symmetric matrices of higher order by 1 of the matrices A_k, obtained by bordering the matrices A_k on the right and from below by the column matrices $\left(\mathbf{b}_k^T, c_k\right)^T$ and the row matrices $\left(\mathbf{b}_k^T, c_k\right)$. The null space of system (1.6) is a set L_0 of all symmetric square matrices S of order $n+1$ satisfying a system of equalities for traces

$$tr(\hat{A}_k S) = 0. \tag{1.8}$$

Theorem 1, which follows, and is taken from [2], allows for simplification in the solving of some concrete systems of type (1.6).

Theorem 1. *For each ordered set of numbers* $s_1, s_2, ..., s_n$*, the matrix* S *may be formed as* $S = [s_1\ s_2\ ...\ s_n\ 1]^T\,[s_1\ s_2\ ...\ s_n\ 1]$ *that provides the one-to-one correspondence between all solutions of system* (1.6) *and the set of all affine matrices of rank* 1 *of zero-space of system* (1.6)*.*

As emphasized in [2], for the practical use of *Theorem* 1, it is enough to determine a basis $E_1, ..., E_q$ of the null space L_0 of the system (1.6). The problem of solving system (1.6) is equivalent to the problem of searching all such numbers $\alpha_1, ..., \alpha_q$, for which the matrices $\alpha_1 E_1 + ... + \alpha_q E_q$ are affine matrices of rank 1.

In [1] and [6] an explicit (qualitative) analysis of solutions of SNAE (1.6), in the case where $n = k = 2$, was presented. This type of system (1.6) is obtained via decomposition of the general *Hamilton* quadratic equation

$$(a_1, a_2)\,(x, y)^2 + (b_1, b_2)\,(x, y) + (c_1, c_2) = (0, 0)\,. \tag{1.9}$$

This paper brings an implicit method for solving this type of SNAE (1.6), which is based on solving a matrix quadratic equation

$$AX^2 + BX + C = O, \tag{1.10}$$

where $A = \begin{bmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}$, $C = \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}$, $X = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ and O is a square zero matrix. The matrix quadratic equation (1.10) corresponds to the quadratic equation (1.9). In the second part of the paper, the matrix method for solving the SNAE is presented, which is obtained via decomposition of the following *Hamilton's* quadratic equation (couple equation)

$$(a_{11}, a_{21})(x, 0)^2 - (a_{12}, a_{22})(0, y)^2 + 2(b_2, -b_1)(x, 0)(0, y) + \tag{1.11}$$
$$+(c_{11}, c_{21})(x, 0) + (c_{22}, -c_{12})(0, y) + (d_1, d_2) = (0, 0).$$

2. Solutions of Hamilton's general quadratic equation

Solutions of quadratic equations $ax^2 + bx + c = 0$, over the field of real numbers $\mathbb{R}$, exist if the discriminant of the system $\Delta = b^2 - 4ac$ is nonnegative. On the other hand, the elements w of the field $\mathbb{R}^2$ (the *Cartesian* square of the set of real numbers $\mathbb{R}$) are ordered pairs (x, y) of real numbers (*Hamilton's* number couples), which we can rename bireal numbers. As $(x, y) = x(1, 0) + y(0, 1)$, where $(1, 0)$ and $(0, 1)$ are the basis elements of the field of bireal numbers $\mathbb{R}^2$ [4], the quadratic equation, over the field of bireal numbers $\mathbb{R}^2$,

$$(a_1, a_2)w^2 + (b_1, b_2)w + (c_1, c_2) = (0, 0), \tag{2.1}$$

where $w = (x, y)$, can be decomposed, such that

$$(a_1, a_2)w^2 + (b_1, b_2)w + (c_1, c_2) = \tag{2.2}$$
$$= a_1 w^2 + b_1 w + (c_1, 0) + a_2 w w_\perp + b_2 w_\perp + (0, c_2) =$$
$$= \left[a_1 w^2 + b_1 w + (c_1, 0)\right](1, 0) + \left[a_2 w^2 + b_2 w + (c_2, 0)\right](0, 1) = (0, 0),$$

where according to relation (1.1) $w_\perp = (0, 1)w$ is the bireal number $(-y, x)$, which is obtained by rotating the bireal number w, by an angle of $\pi/2$ radians, in the positive mathematical direction.

Between the elements $w = (x, y)$ of the field of bireal numbers $\mathbb{R}^2$ and square matrices of the second order $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, which are the elements of the field of bireal matrices $\mathcal{R}^2$, with a basis consisting of the identity matrix $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the antisymmetric matrix $\hat{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, there is a one-to-one correspondence. Accordingly, the product of bireal numbers (1.1) corresponds to the product of the corresponding bireal matrices and vice versa

$$(a, b)(c, d) \rightleftharpoons \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \tag{2.3}$$
$$= \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix} \leftrightharpoons (ac - bd, ad + bc).$$

The inverse bireal number w^{-1} $(w \neq (0, 0))$ corresponds to the inverse matrix $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1}$ $(x \neq 0$ and $y \neq 0)$, so that

$$w^{-1} \rightleftharpoons \frac{1}{x^2 + y^2} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \leftrightharpoons \frac{1}{\|w\|^2} \bar{w}, \tag{2.4}$$

where $\|w\| = \left(x^2+y^2\right)^{1/2}$ is the *Euclidean* norm of the field of bireal numbers $\mathbb{R}^2$ and $\bar{w} = (x, -y)$. Since $X = xE + y\hat{E}$, the general *Hamiltonian* quadratic equation (1.9), in matrix form, obtained from the matrix equation (1.10), is as follows

$$(2.5)\quad AX^2+BX+C = \begin{bmatrix} a_1\left(x^2-y^2\right)-2a_2xy & a_2\left(x^2-y^2\right)+2a_1xy \\ -\left[a_2\left(x^2-y^2\right)+2a_1xy\right] & a_1\left(x^2-y^2\right)-2a_2xy \end{bmatrix}+$$

$$+\begin{bmatrix} b_1x-b_2y & b_1y+b_2x \\ -\left(b_1y+b_2x\right) & b_1x-b_2y \end{bmatrix}+\begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}=$$

$$=a_1\begin{bmatrix} x^2-y^2 & 2xy \\ -2xy & x^2-y^2 \end{bmatrix}+b_1\begin{bmatrix} x & y \\ -y & x \end{bmatrix}+c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+$$

$$+a_2\begin{bmatrix} -2xy & x^2-y^2 \\ -\left(x^2-y^2\right) & -2xy \end{bmatrix}+b_2\begin{bmatrix} -y & x \\ -x & -y \end{bmatrix}+c_2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}=O,$$

so that $AX^2+BX+C = \left(a_1X^2+b_1X+c_1E\right)E+\left(a_2X^2+b_2X+c_2E\right)\hat{E} = O$. According to (2.2), that is (2.5), the field of bireal matrices $\{X : \hat{E}Y_1 = Y_2\} \subset \mathcal{R}^2$, where $Y_i : \mathcal{R}^2 \to \mathcal{R}^2$ $(i = 1, 2)$ are square matrix functions $a_iX^2 + b_iX + c_iE$, is the null field $\mathcal{L}_0$ (the set of solutions) of the matrix quadratic equation (2.5). Obviously, according to *Theorem* 1, the real numbers $s_1 = x$ and $s_2 = y$, where x and y are elements of the bireal matrices $X \in \mathcal{L}_0$, are elements of the matrices $[s_1\ s_2\ 1]$, such that the matrices $S = [s_1\ s_2\ 1]^T\,[s_1\ s_2\ 1]$ are symmetric matrices of the null space of SNAE

$$(2.6)\quad \begin{aligned} \sum_{i=1}^{2}\sum_{j=1}^{2}\left(a_{ij}\right)_1 x_ix_j+2\sum_{i=1}^{2}\left(b_i\right)_1 x_i+c_1 &= 0 \\ \sum_{i=1}^{2}\sum_{j=1}^{2}\left(a_{ij}\right)_2 x_ix_j+2\sum_{i=1}^{2}\left(b_i\right)_2 x_i+c_2 &= 0, \end{aligned}$$

which can be written in a more concise form

$$(2.7)\quad \begin{aligned} \mathbf{x}^TA_1\mathbf{x}+2\mathbf{b}_1^T\mathbf{x}+\frac{c_1}{a_1} &= 0 \\ \mathbf{x}^TA_2\mathbf{x}+2\mathbf{b}_2^T\mathbf{x}+\frac{c_2}{a_2} &= 0, \end{aligned}$$

where $\mathbf{x}^T = [x_1\ x_2] = [x\ y]$,

$$A_1 = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{a_2}{a_1} \\ -\frac{a_2}{a_1} & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_1}{a_2} \\ \frac{a_1}{a_2} & -1 \end{bmatrix},$$

$$\mathbf{b}_1^T = \begin{bmatrix} b_1^{(1)} & b_2^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{b_1}{2a_1} & -\frac{b_2}{2a_1} \end{bmatrix} \text{ and } \mathbf{b}_2^T = \begin{bmatrix} b_1^{(2)} & b_2^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{b_2}{2a_2} & \frac{b_1}{2a_2} \end{bmatrix}.$$

By (1.8), matrices S are affine matrices of rank 1, which satisfy a system of equalities for traces

$$(2.8)\quad tr(\hat{A}_1S) = 0 \text{ and } tr(\hat{A}_2S) = 0,$$

where

$$(2.9)\quad \hat{A}_1 = \begin{bmatrix} 1 & -\frac{a_2}{a_1} & \frac{b_1}{2a_1} \\ -\frac{a_2}{a_1} & -1 & -\frac{b_2}{2a_1} \\ \frac{b_1}{2a_1} & -\frac{b_2}{2a_1} & \frac{c_1}{a_1} \end{bmatrix} \text{ and } \hat{A}_2 = \begin{bmatrix} 1 & \frac{a_1}{a_2} & \frac{b_2}{2a_2} \\ \frac{a_1}{a_2} & -1 & \frac{b_1}{2a_2} \\ \frac{b_2}{2a_2} & \frac{b_1}{2a_2} & \frac{c_2}{a_2} \end{bmatrix}.$$

On the other hand, the solutions of system (2.7) are the solutions of matrix equation (2.5). Since bireal matrices, if they are not zero matrices, are regular and commutative matrices, bireal matrices

$$X_1 = -\frac{A^{-1}B}{2} + \sqrt[2]{\frac{\left(A^{-1}B\right)^2}{4} - A^{-1}C} \text{ and} \tag{2.10}$$

$$X_2 = -\frac{A^{-1}B}{2} - \sqrt[2]{\frac{\left(A^{-1}B\right)^2}{4} - A^{-1}C},$$

are solutions of the matrix equation (2.5). Considering the fact that

$$A^{-1}B = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix} = \tag{2.11}$$

$$= \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1 b_1 + a_2 b_2 & a_1 b_2 - a_2 b_1 \\ a_2 b_1 - a_1 b_2 & a_1 b_1 + a_2 b_2 \end{bmatrix},$$

$$A^{-1}C = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix} = \tag{2.12}$$

$$= \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1 c_1 + a_2 c_2 & a_1 c_2 - a_2 c_1 \\ a_2 c_1 - a_1 c_2 & a_1 c_1 + a_2 c_2 \end{bmatrix} \text{ and}$$

$$\left(A^{-1}B\right)^2 = \frac{1}{\left(a_1^2 + a_2^2\right)^2} \left[\begin{matrix} \left(a_1^2 - a_2^2\right)\left(b_1^2 - b_2^2\right) + 4a_1 a_2 b_1 b_2 \\ 2\left[\left(b_1^2 - b_2^2\right) a_1 a_2 - \left(a_1^2 - a_2^2\right) b_1 b_2\right] \end{matrix} \right. \tag{2.13}$$

$$\left. \begin{matrix} 2\left[\left(a_1^2 - a_2^2\right) b_1 b_2 - \left(b_1^2 - b_2^2\right) a_1 a_2\right] \\ \left(a_1^2 - a_2^2\right)\left(b_1^2 - b_2^2\right) + 4a_1 a_2 b_1 b_2 \end{matrix} \right],$$

the bireal matrix $D^2 = \begin{bmatrix} d_1^2 - d_2^2 & 2d_1 d_2 \\ -2d_1 d_2 & d_1^2 - d_2^2 \end{bmatrix}$, whose elements satisfy SNAE

$$d_1^2 - d_2^2 = \frac{\left(a_1 b_1 + a_2 b_2\right)^2 - \left(a_1 b_2 - a_2 b_1\right)^2 - 4\left(a_1^2 + a_2^2\right)\left(a_1 c_1 + a_2 c_2\right)}{4\left(a_1^2 + a_2^2\right)^2} \tag{2.14}$$

$$2d_1 d_2 = \frac{2\left(a_1 b_1 + a_2 b_2\right)\left(a_1 b_2 - a_2 b_1\right) - 4\left(a_1^2 + a_2^2\right)\left(a_1 c_2 - a_2 c_1\right)}{4\left(a_1^2 + a_2^2\right)^2},$$

is the matrix discriminant of the matrix equation (2.5). SNAE (2.14) can be solved graphically and analytically. The graphic solution is the coordinates of the points of intersection of two second order curves, as shown in Fig.1

The analytical solution can be obtained by converting *Cartesian* coordinates into polar coordinates

$$d_1 = r\cos\varphi \text{ } and \text{ } d_2 = r\sin\varphi. \tag{2.15}$$

Therefore, in polar coordinates

$$r^2 \sin 2\varphi = \frac{2\left(a_1 b_1 + a_2 b_2\right)\left(a_1 b_2 - a_2 b_1\right) - 4\left(a_1^2 + a_2^2\right)\left(a_1 c_2 - a_2 c_1\right)}{4\left(a_1^2 + a_2^2\right)^2} \text{ and} \tag{2.16}$$

$$r^2 \cos 2\varphi = \frac{\left(a_1 b_1 + a_2 b_2\right)^2 - \left(a_1 b_2 - a_2 b_1\right)^2 - 4\left(a_1^2 + a_2^2\right)\left(a_1 c_1 + a_2 c_2\right)}{4\left(a_1^2 + a_2^2\right)^2},$$

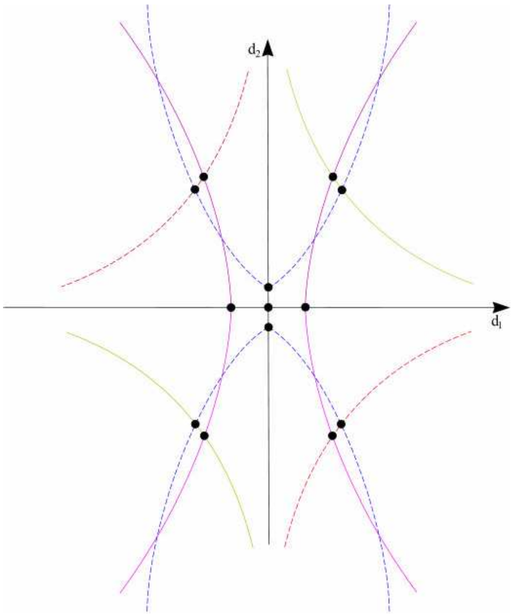

FIGURE 1

tath is,

$$\tan 2\varphi = \frac{2\left(a_1b_1 + a_2b_2\right)\left(a_1b_2 - a_2b_1\right) - 4\left(a_1^2 + a_2^2\right)\left(a_1c_2 - a_2c_1\right)}{\left(a_1b_1 + a_2b_2\right)^2 - \left(a_1b_2 - a_2b_1\right)^2 - 4\left(a_1^2 + a_2^2\right)\left(a_1c_1 + a_2c_2\right)}. \tag{2.17}$$

Finally, after determining the elements d_1 and d_2 of the matrix $D = \begin{bmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{bmatrix}$, the solutions of the matrix equation (2.5), are as follows

$$X_{1,2} = -\frac{A^{-1}B}{2} \pm D. \tag{2.18}$$

On the basis of (2.14), it can be concluded that if

$$\left(a_1b_1 + a_2b_2\right)\left(a_2b_1 - a_1b_2\right) = 2\left(a_1^2 + a_2^2\right)\left(a_1c_2 - a_2c_1\right) \text{ and} \tag{2.19}$$

$$\left(a_1b_1 + a_2b_2\right)^2 - \left(a_2b_1 - a_1b_2\right)^2 \geq 4\left(a_1^2 + a_2^2\right)\left(a_1c_1 + a_2c_2\right),$$

the solutions are the bireal numbers $(x_1, 0)$ and $(x_2, 0)$, that is, the bireal numbers (x_0, y_1) and (x_0, y_2), if the inequality $\geq$, in the previous condition, replaces the inequality $<$. For example, let $a_1c_2 = a_2c_1$ and $a_2b_1 = a_1b_2$. If the condition $b_2^2 \geq 4a_2c_2$ ($b_1^2 \geq 4a_1c_1$) is satisfied, to which the inequality condition (2.19) is reduced, the solutions of the general *Hamiltonian* quadratic equation are the bireal numbers $(x_1, 0)$ and $(x_2, 0)$. If $a_1c_2 = a_2c_1$ and $a_1b_1 = -a_2b_2$, and $b_1^2 \geq -4a_2c_2$ ($b_2^2 \geq -4a_1c_1$), the solutions are bireal numbers $(0, y_1)$ and $(0, y_2)$, [1].

As emphasized above, real solutions of the equation $ax^2 + bx + c = 0$ exist if the discriminant $\Delta = b^2 - 4ac$ is nonnegative. The solutions of the quadratic equation (2.1), over the field of bireal numbers $\mathbb{R}^2$, which also includes *Hamilton's* couple equation (1.2), are in the form of bireal numbers. Accordingly, in addition to the fact that there is a complete analogy between the solution of the quadratic equation (2.1) and the quadratic equation with complex coefficients, the question arises: Whether it was necessary to introduce the imaginary unit i at all? *Hamilton* himself emphasized the same thing in [5], but he did not insist on it, so it was not sufficiently noticed by the scientific public.

2.1. **Examples.** 1. The general *Hamiltonian* quadratic equation

$$(a_1, a_2)\, w^2 + (b_1, b_2)\, w + (c_1, c_2) = (0,0)\,, \tag{2.20}$$

for $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$, can be decomposed, such that

$$(1,1)\, w^2 + (1,1)\, w + (1,1) = \left[w^2 + w + (1,0)\right] [(1,0) + (0,1)] = (0,0)\,, \tag{2.21}$$

that is,

$$w^2 + w + (1,0) = (0,0)\,, \tag{2.22}$$

which is obviously an ordinary *Hamiltonian* quadratic equation (couple equation), the solutions of which, according to (1.5), are as follows

$$w_{1,2} = \frac{1}{2}(-1, \pm \left|\sqrt[2]{3}\right|). \tag{2.23}$$

Therefore, the bireal matrix $D^2 = \begin{bmatrix} d_1^2 & d_2^2 \\ -d_2^2 & d_1^2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{4} \\ -\frac{3}{4} & 0 \end{bmatrix}$, whose elements are solutions of the SNAE

$$\begin{aligned} d_1^2 - d_2^2 &= -\frac{3}{4} \\ 2d_1 d_2 &= 0, \end{aligned} \tag{2.24}$$

is a matrix discriminant of the following matrix equation

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X^2 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = O, \tag{2.25}$$

which corresponds to the quadratic equation (2.22). The solutions of the matrix equation (2.25) are the matrices

$$X_{1,2} = \frac{1}{2} \begin{bmatrix} -1 & \pm\left|\sqrt[2]{3}\right| \\ \mp\left|\sqrt[2]{3}\right| & -1 \end{bmatrix}. \tag{2.26}$$

2. Between the general *Hamiltonian* quadratic equation

$$(1,-1)\, w^2 + (-2,-2)\, w + (1,-1) = (0,0) \tag{2.27}$$

and the matrix equation

$$\begin{aligned} AX^2 + BX + C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^2 - y^2 & 2xy \\ -2xy & x^2 - y^2 \end{bmatrix} + \\ + \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x & y \\ -y & x \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = O, \end{aligned} \tag{2.28}$$

whose solutions are as follows

$$X_{1,2} = -\frac{A^{-1}B}{2} \pm D = (1 \pm \sqrt[2]{2})\hat{E}, \tag{2.29}$$

there is a one-to-one correspondence.

3. Matrix method for solving the SNAE

By decomposing *Hamilton's* quadratic equation (1.11), the following SNAE

$$a_{11}x^2 + a_{12}y^2 + 2b_1xy + c_{11}x + c_{12}y + d_1 = 0 \tag{3.1}$$
$$a_{21}x^2 + a_{22}y^2 + 2b_2xy + c_{21}x + c_{22}y + d_2 = 0.$$

is obtained. However, *Hamilton's* quadratic equation (1.11) can also be written in matrix form

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} (A \begin{bmatrix} x^2\mathbf{1} \\ y^2\hat{\mathbf{1}} \end{bmatrix} + B \begin{bmatrix} xy\mathbf{1} \\ xy\hat{\mathbf{1}} \end{bmatrix} + C \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} + \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix}) = (0,0), \tag{3.2}$$

where

$$A = \begin{bmatrix} a_{11}\mathbf{1} & -a_{12}\hat{\mathbf{1}} \\ a_{21}\hat{\mathbf{1}} & a_{22}\mathbf{1} \end{bmatrix} = \begin{bmatrix} (a_{11},0) & (0,-a_{12}) \\ (0,a_{21}) & (a_{22},0) \end{bmatrix},$$

$$B = \begin{bmatrix} b_1\mathbf{1} & -b_1\hat{\mathbf{1}} \\ b_2\hat{\mathbf{1}} & b_2\mathbf{1} \end{bmatrix} = \begin{bmatrix} (b_1,0) & (0,-b) \\ (0,b_2) & (b_2,0) \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} c_{11}\mathbf{1} & -c_{12}\hat{\mathbf{1}} \\ c_{21}\hat{\mathbf{1}} & c_{22}\mathbf{1} \end{bmatrix} = \begin{bmatrix} (c_{11},0) & (0,-c_{12}) \\ (0,c_{21}) & (c_{22},0) \end{bmatrix}.$$

Since

$$\begin{bmatrix} x^2\mathbf{1} \\ y^2\hat{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} x\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & y\mathbf{1} \end{bmatrix} \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} xy\mathbf{1} \\ xy\hat{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} y\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & x\mathbf{1} \end{bmatrix} \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix},$$

it follows that

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} (A \begin{bmatrix} x^2\mathbf{1} \\ y^2\hat{\mathbf{1}} \end{bmatrix} + B \begin{bmatrix} xy\mathbf{1} \\ xy\hat{\mathbf{1}} \end{bmatrix} + C \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} + \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix}) = \tag{3.3}$$

$$= \begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} [(A \begin{bmatrix} x\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & y\mathbf{1} \end{bmatrix} + B \begin{bmatrix} y\mathbf{1} & 0\hat{\mathbf{1}} \\ 0\hat{\mathbf{1}} & x\mathbf{1} \end{bmatrix} + C) \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} + \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix}] = (0,0),$$

that is,

$$D \begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} + \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} (0,0) \\ (0,0) \end{bmatrix}, \tag{3.4}$$

where $D = \begin{bmatrix} (a_{11}x + b_1y + c_{11}, 0) & -(0, a_{12}y + b_1x + c_{12}) \\ (0, a_{21}x + b_2y + c_{21}) & (a_{22}y + b_2x + c_{22}, 0) \end{bmatrix}$. On the assumption that the matrix $\underset{\sim}{D} = \begin{bmatrix} a_{11}x + b_1y + c_{11} & a_{12}y + b_1x + c_{12} \\ a_{21}x + b_2y + c_{21} & a_{22}y + b_2x + c_{22} \end{bmatrix}$, such that $|D| = |\underset{\sim}{D}|\,\mathbf{1}$, is a regular matrix, one obtains that

$$\begin{bmatrix} x\mathbf{1} \\ y\hat{\mathbf{1}} \end{bmatrix} = -D^{-1} \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix} = -\frac{adjD}{|D|} \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix} = \tag{3.5}$$

$$= -\frac{\begin{bmatrix} (a_{22}y + b_2x + c_{22}, 0) & (0, a_{12}y + b_1x + c_{12}) \\ -(0, a_{21}x + b_2y + c_{21}) & (a_{11}x + b_1y + c_{11}, 0) \end{bmatrix} \begin{bmatrix} d_1\mathbf{1} \\ d_2\hat{\mathbf{1}} \end{bmatrix}}{|\underset{\sim}{D}|} =$$

$$= \frac{\begin{bmatrix} [(b_1d_2 - b_2d_1)\,x + (a_{12}d_2 - a_{22}d_1)\,y + (c_{12}d_2 - c_{22}d_1)]\mathbf{1} \\ [(a_{21}d_1 - a_{11}d_2)\,x - (b_1d_2 - b_2d_1)\,y + (c_{21}d_1 - c_{11}d_2)]\hat{\mathbf{1}} \end{bmatrix}}{(a_{11}x + b_1y + c_{11})\,(a_{22}y + b_2x + c_{22}) - (a_{21}x + b_2y + c_{21})\,(a_{12}y + b_1x + c_{12})}.$$

On the basis of the previous equation, it follows that

$$(3.6)\qquad [|\underline{D}| - (b_1 d_2 - b_2 d_1)]xy = (a_{12} d_2 - a_{22} d_1)\, y^2 + (c_{12} d_2 - c_{22} d_1)\, y = \\ = (a_{21} d_1 - a_{11} d_2)\, x^2 - 2\,(b_1 d_2 - b_2 d_1)\, yx + (c_{21} d_1 - c_{11} d_2)\, x,$$

where

$$(3.7)\quad |\underline{D}| = (a_{11} b_2 - a_{21} b_1)\, x^2 + (a_{22} b_1 - a_{12} b_2)\, y^2 + |\underline{A}|\, xy + (a_{11} c_{22} - a_{21} c_{12})\, x + \\ + (a_{22} c_{11} - a_{12} c_{21})\, y + |\underline{C}|\,,$$

$|\underline{A}| = a_{11} a_{22} - a_{12} a_{21}$ and $|\underline{C}| = c_{11} c_{22} - c_{12} c_{21}$. The solutions x and y of SNAE (3.1) belong to the set $\{x, y, c\}$ of solutions of the following SNAE

$$(3.8)\qquad [|\underline{D}| - (b_1 d_2 - b_2 d_1)]xy = c \\ (a_{12} d_2 - a_{22} d_1)\, y^2 + (c_{12} d_2 - c_{22} d_1)\, y - c = 0 \\ (a_{21} d_1 - a_{11} d_2)\, x^2 - [2\,(b_1 d_2 - b_2 d_1)\, y - (c_{21} d_1 - c_{11} d_2)]x - c = 0,$$

where c is an arbitrary real constant.

3.1. **Examples.** 1. The following SNAE

$$(3.9)\qquad x^2 - y^2 = d \\ 2xy = \frac{d}{b},$$

identical to SNAE (2.14), is obtained from SNAE (3.1) for

$$(3.10)\qquad a_{21} = a_{22} = b_1 = c_{11} = c_{12} = c_{21} = c_{22} = 0, \\ b_2 = b \neq 0,\ a_{11} = -a_{12} = 1 \text{ and} \\ d_1 = d_2 = -d \neq 0.$$

In that case, according to (3.8), solutions x and y of SNAE (3.9) are solutions of SNAE

$$(3.11)\qquad (|\underline{D}| - bd)xy - c = 0 \\ x^2 - 2byx = \frac{c}{d} \\ y^2 = c/d.$$

From the last equation of the previous system, it follows that $y = \pm\sqrt[2]{c/d}$. If this value for y is included in the second equation of SNAE (3.11), the value for x can be obtained, so that $x = (b \pm \sqrt[2]{1+b^2})y$. On the basis of (3.7),

$$(3.12)\qquad |\underline{D}| = b(x^2 + y^2) = \frac{bc}{d}[1 + (b \pm \sqrt[2]{1+b^2})^2] = \\ = \frac{bc}{d}(1 + \frac{b \pm \sqrt[2]{1+b^2}}{-b \pm \sqrt[2]{1+b^2}}) = \frac{2bc}{d}\frac{\pm\sqrt[2]{1+b^2}}{-b \pm \sqrt[2]{1+b^2}}.$$

On the other hand, according to the first equation of SNAE (3.11), one obtains that

$$(3.13)\qquad \frac{|\underline{D}|}{d} = \frac{1}{(b \pm \sqrt[2]{1+b^2})} + b = \pm\sqrt[2]{1+b^2},$$

since $xy = c(b \pm \sqrt[2]{1+b^2})/d$. Accordingly,

$$(3.14)\qquad \frac{c}{d} = \frac{d}{2b}(-b \pm \sqrt[2]{1+b^2}) \geq 0,$$

so that

$$y = \pm \sqrt[2]{\frac{d}{2b}(-b \pm \sqrt[2]{1+b^2})} \text{ and} \tag{3.15}$$

$$x = y(b \pm \sqrt[2]{1+b^2}) = \pm \sqrt[2]{\frac{d}{2b}(b \pm \sqrt[2]{1+b^2})}.$$

2. The following SNAE

$$\begin{aligned} x^2 - y - d &= 0 \\ y^2 + x - d &= 0, \end{aligned} \tag{3.16}$$

is obtained from SNAE (3.1) for

$$\begin{gathered} a_{12} = a_{21} = b_1 = b_2 = c_{11} = c_{22} = 0, \\ a_{11} = a_{22} = -c_{12} = c_{21} = 1 \text{ and} \\ d_1 = d_2 = -d \neq 0. \end{gathered} \tag{3.17}$$

In this case, according to (3.8), solutions x and y of SNAE (3.16) are solutions of SNAE

$$\begin{gathered} |\underset{\sim}{D}|\, xy = c \\ y^2 + y - \frac{c}{d} = 0 \\ x^2 - x - \frac{c}{d} = 0. \end{gathered} \tag{3.18}$$

From the second and third equations of the previous system, for $-1/4 \leq c/d$,

$$y = -\frac{1}{2} \pm \sqrt[2]{\frac{1}{4} + \frac{c}{d}} \text{ and} \tag{3.19}$$

$$x = \frac{1}{2} \pm \sqrt[2]{\frac{1}{4} + \frac{c}{d}}.$$

According to the first equation, it follows that $|\underset{\sim}{D}| = d$, since $xy = c/d$. On the other hand, on the basis of (3.7),

$$|\underset{\sim}{D}| = xy + 1 = \frac{c}{d} + 1. \tag{3.20}$$

Therefore, $c/d = d - 1$, so that, for $3/4 \leq d$,

$$y = -\frac{1}{2} \pm \sqrt[2]{d - \frac{3}{4}} \text{ and} \tag{3.21}$$

$$x = \frac{1}{2} \pm \sqrt[2]{d - \frac{3}{4}}.$$

By analyzing the results above obtained, it can be concluded that for $d < 3/4$, according to (1.4) and (1.5), the following bireal numbers

$$w_1 = (\frac{1}{2}, \pm \sqrt[2]{\frac{3}{4} - d}) \text{ and } w_2 = (-\frac{1}{2}, \pm \sqrt[2]{\frac{3}{4} - d}), \tag{3.22}$$

with the bireal numbers $w_1 = (1/2 \pm \sqrt[2]{d - 3/4}, 0)$ and $w_2 = (-1/2 \pm \sqrt[2]{d - 3/4}, 0)$, for $3/4 \leq d$, are solutions for the following SNAE

$$\begin{aligned} w_1^2 - w_2 - (d, 0) &= (0, 0) \\ w_2^2 + w_1 - (d, 0) &= (0, 0)\,. \end{aligned} \tag{3.23}$$

3. The following SNAE, see [9],

$$x^2 + 4y^2 - 4 = 0 \tag{3.24}$$

$$x^2 + y^2 - 2x = 0,$$

is obtained from SNAE (3.1) for

$$a_{11} = a_{21} = a_{22} = 1 \tag{3.25}$$

$$a_{12} = -2c_{21} = -d_1 = 4 \text{ and}$$

$$b_1 = b_2 = c_{11} = c_{12} = c_{22} = d_2 = 0.$$

According to (3.8), solutions x and y of SNAE (3.24) are solutions of SNAE

$$y^2 = c \tag{3.26}$$

$$|\underset{\sim}{D}|\, xy = 4c$$

$$x^2 - 2x + c = 0.$$

On the basis of (3.7) $|\underset{\sim}{D}| = 8y - 3xy = (8-3x)y$, so that $|\underset{\sim}{D}|\, xy = (8-3x)xy^2 = 4c$, that is, $3x^2 - 8x + 4 = 0$, since $y^2 = c$. The solutions of this quadratic equation: $x_1 = 2$ and $x_2 = 2/3$, are at the same time the solutions of the third equation of SNAE (3.26), only if $c = 0$ or $c = 8/9$. Accordingly, $y_1 = 0$ and $y_2 = \pm 2\sqrt[2]{2}/3$.

Remark 1. *Hamilton's quadratic equation* (*couple equation*)

$$|(s_1, s_2)\, E - A| = |A|\, (s_1, s_2)^2 - tr\,(EadjA)\,(s_1, s_2) + |A| = (0,0)\,, \tag{3.27}$$

where

$$A = \begin{bmatrix} a_{11}\mathbf{1} & -a_{12}\hat{\mathbf{1}} \\ a_{21}\hat{\mathbf{1}} & a_{22}\mathbf{1} \end{bmatrix} = \begin{bmatrix} (a_{11},0) & (0,-a_{12}) \\ (0,a_{21}) & (a_{22},0) \end{bmatrix} \text{ and} \tag{3.28}$$

$$E = \begin{bmatrix} e_{11}\mathbf{1} & -e_{12}\hat{\mathbf{1}} \\ e_{21}\hat{\mathbf{1}} & e_{22}\mathbf{1} \end{bmatrix} = \begin{bmatrix} (e_{11},0) & (0,-e_{12}) \\ (0,e_{21}) & (e_{22},0) \end{bmatrix},$$

is the characteristic polynomial for Hamilton's linear differential equation

$$(e_{11}, e_{21})\,(d_t x, 0) + (e_{22}, -e_{12})\,(0, d_t y) = (a_{11}, a_{21})\,(x,0) + (a_{22}, -a_{12})\,(0,y)\,, \tag{3.29}$$

which also has its own matrix form

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} E d_t \begin{bmatrix} (x,0) \\ (0,y) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} A \begin{bmatrix} (x,0) \\ (0,y) \end{bmatrix}. \tag{3.30}$$

By decomposition, Hamilton's quadratic equation (3.27) *is reduced to SNAE*

$$|\underset{\sim}{E}|\left(s_1^2 - s_2^2\right) - tr(\underset{\sim}{E}adj\underset{\sim}{A})s_1 + |\underset{\sim}{A}| = 0 \tag{3.31}$$

$$2\,|\underset{\sim}{E}|\, s_1 s_2 - tr(\underset{\sim}{E}adj\underset{\sim}{A})s_2 = 0,$$

where $\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *and* $\underset{\sim}{E} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$, *whose solutions are qualitatively comparable to the solutions of SNAE* (1.3).

References

[1] Bardell, N. S., Visualizing the roots of quadratic equations with compex coefficients, *Australian Senior Mathematics Journal*, 2014, **28**(**1**), 7–28

[2] Borisevich V. D., Potemkin V. G. and Strunkov S. P., Global Methods for Solving Systems of Nonlinear Algebraic Equations, *Computers and Mathematics with Applications,* 2000, **40**, 1015-1025

[3] Goulianas K., Margaris A., Adamopoulos M., Finding all real roots of 3×3 nonlinear algebraic systems using neural networks, *Applied Mathematics and Computation*, 2013, **219** (**9**), 4444-4464

[4] Hamilton, R.W., Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time, *Transactions of the Royal Irish Academy*, 1837, **17** (**1**), 293–422

[5] Hamilton, R.W., Lectures on Quaternions. London: Hodges and Smith, 1853

[6] Hardy, G. H., A course of pure mathematics centenary edition, Cambridge: Cambridge University Press, 2008

[7] Li G. and Zhezhao Z. A., Neural network algorithm for solving nonlinear equation systems, 2008, Proceedings of 20^{th} international conference on computational intelligence and security, Los Alamito, USA, 20–23

[8] Margaris A., Goulianas K., Finding all roots of 2×2 nonlinear algebraic systems using back propagation neural networks, *Neural Computing and Applications*, 2012, **21**, 891-904

[9] Sakkalis T. and Patrikalakis M. N., Resolution of multiple roots of nonlinear polynomial systems, *International Journal of Shape Modeling*, 2005, **11** (**1**), 121-147

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