

**LOCAL AND 2-LOCAL  $\frac{1}{2}$ -DERIVATIONS ON  
QUASI-FILIFORM LIE ALGEBRAS OF MAXIMUM  
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**Abstract:** The present paper is devoted to the study of local and 2-local  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length. We describe local  $\frac{1}{2}$ -derivation of quasi-filiform Lie algebras of maximum length. Therefore, these quasi-filiform Lie algebras of maximum length have local  $\frac{1}{2}$ -derivations are not  $\frac{1}{2}$ -derivations. Moreover, similar problem concerning 2-local  $\frac{1}{2}$ -derivations of such algebras is investigated.

**Keywords:** Quasi-filiform Lie algebras, nilpotent algebras,  $\frac{1}{2}$ -derivation, local  $\frac{1}{2}$ -derivation, 2-local  $\frac{1}{2}$ -derivation.

**1 Introduction**

The notion of  $\delta$ -derivations was introduced by V. Filippov for Lie algebras in [18, 19]. The space of  $\delta$ -derivations includes usual derivations ( $\delta = 1$ ), anti-derivations ( $\delta = -1$ ) and elements from the centroid. In [19] it was proved that prime Lie algebras, as a rule, do not have nonzero  $\delta$ -derivations (provided  $\delta \neq 1, -1, 0, \frac{1}{2}$ ), and all  $\frac{1}{2}$ -derivations of an arbitrary prime Lie algebra  $A$  over the field  $\mathbb{F}$  of characteristic  $p \neq 2, 3$  with a non-degenerate symmetric invariant bilinear form were described. It was proved that if  $A$

is a central simple Lie algebra over a field of characteristic  $p \neq 2, 3$  with a non-degenerate symmetric invariant bilinear form, then any  $\frac{1}{2}$ -derivation  $\varphi$  has the form  $\varphi(x) = \lambda x$  for some  $\lambda \in \mathbb{F}$ .

In [20],  $\delta$ -derivations were investigated for prime alternative and non-Lie Malcev algebras over the ring of operators  $\mathbb{F}$ , and it was proved that alternative and non-Lie Malcev algebras with certain restrictions of  $\mathbb{F}$  have no non-trivial  $\delta$ -derivation. The description of  $\delta$ -derivations of classical Lie superalgebras was given in [26].  $\delta$ -derivations of finite-dimensional semisimple Jordan algebras over the field of characteristic  $p \neq 2$  and  $\delta$ -superderivations of finite-dimensional simple Lie and Jordan superalgebras were investigated in [27]. Zusmanovich in [39] described  $\delta$ -(super)derivations of prime Lie superalgebras. Specifically, he proved that a prime Lie superalgebra has no non-trivial  $\delta$ -(super)derivations for  $\delta \neq 1, -1, 0, \frac{1}{2}$ . Moreover, in [38] it is proved that any  $\delta$ -derivation of a finite-dimensional simple Jordan algebra with values in a finite-dimensional unital irreducible bimodule is, in a sense, trivial.

Nowadays, local and 2-local operators have become popular for some non-associative algebras such as the Lie, Jordan, and Leibniz algebras. The notions of local derivations were introduced in 1990 by Kadison [24] and Larson, Sourour [30]. Later in 1997, Šemrl introduced the notions of 2-local derivations and 2-local automorphisms of algebras [32].

Investigation of local derivations on Lie algebras was initiated in [9] by Ayupov and Kudaybergenov. They proved that every local derivation on semisimple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations that are not derivations. In [6], local derivations of solvable Lie algebras are investigated, and it is shown that in the class of solvable Lie algebras there exist algebras that admit local derivations that are not derivations and also algebras for which every local derivation is a derivation. Several authors investigated local derivations for the finite or infinite dimensional Lie and Leibniz algebras [3, 4, 5, 7, 10, 12, 17, 25, 23, 28, 34, 36]. It was proved that all local derivations of the following algebras are derivations: Borel subalgebras of finite-dimensional simple Lie algebras; Witt algebras; solvable Lie algebras of maximal rank; Cayley algebras; locally finite split simple Lie algebras; the Schrödinger algebras; conformal Galilei algebras.

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of Lie algebras [2, 8, 11, 12, 13, 14, 15, 16, 23, 33, 35]. Specifically, in [8] it is proved that every 2-local derivation on the semisimple Lie algebras is a derivation and that each finite-dimensional nilpotent Lie algebra, with dimension larger than two, admits 2-local derivation which is not a derivation. Let us present the list of Lie algebras for which all 2-local derivations are derivations: finite-dimensional semisimple Lie algebras; Witt algebras; locally finite split simple Lie algebras; Virasoro algebras; Virasoro-like algebra; the Schrodinger-Virasoro algebra; Jacobson-Witt algebras; planar Galilean conformal algebras.

Investigation of local and 2-local  $\delta$ -derivations on Lie algebras was initiated in [29] by A.Khudoyberdiyev and B.Yusupov. Namely, in [24] it is proved we introduce the notion of local and 2-local  $\delta$ -derivations and describe local and 2-local  $\frac{1}{2}$ -derivation of finite-dimensional solvable Lie algebras with filiform, Heisenberg, abelian nilradicals. Moreover, we give the description of local  $\frac{1}{2}$ -derivation of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. B.Yusupov, V.Vaisova and T.Madrakhimov proved similar results concerning local  $\frac{1}{2}$ -derivations of naturally graded quasi-filiform Leibniz algebras of type I in their recent paper [37]. They proved that quasi-filiform Leibniz algebras of type I, as a rule, admit local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations. Similar problem [31] U.Mamadaliyev, A.Sattarov and B.Yusupov investigated local and 2-local  $\frac{1}{2}$ -derivations on Leibniz algebras. They proved that any local  $\frac{1}{2}$ -derivation on the solvable Leibniz algebras with model or abelian nilradicals, whose complementary space with maximal dimension is a  $\frac{1}{2}$ -derivation. They proved that solvable Leibniz algebras with abelian nilradicals, which have 1-dimensional complementary space is a  $\frac{1}{2}$ -derivation. Moreover, similar problem concerning 2-local  $\frac{1}{2}$ -derivations of such algebras are investigated and an example of solvable Leibniz algebra given such that any 2-local  $\frac{1}{2}$ -derivation on it is a  $\frac{1}{2}$ -derivation, but which admit 2-local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations.

In this work, we investigate local and 2-local  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length. We describe local  $\frac{1}{2}$ -derivation of quasi-filiform Lie algebras of maximum length. Therefore, these quasi-filiform Lie algebras of maximum length have local  $\frac{1}{2}$ -derivations are not  $\frac{1}{2}$ -derivations. Moreover, similar problem concerning 2-local  $\frac{1}{2}$ -derivations of such algebras is investigated.

## 2 Preliminaries

This section will discuss the concepts and known results on algebras over the field  $\mathbb{C}$  unless otherwise stated.

**Definition 1.** Let  $(\mathfrak{L}, [-, -])$  be an algebra with a multiplication  $[-, -]$ . A linear map  $\varphi$  is called a  $\delta$ -derivation if it satisfies

$$\varphi[x, y] = \delta([\varphi(x), y] + [x, \varphi(y)]),$$

where  $\delta$  from the ground field  $\mathbb{F}$ .

Note that 1-derivation is a usual derivation and  $(-1)$ -derivation is called an anti-derivation. If  $\varphi_1$  and  $\varphi_2$  are  $\delta_1$  and  $\delta_2$ -derivations, respectively, then their commutator  $[\varphi_1, \varphi_2] = \varphi_1\varphi_2 - \varphi_2\varphi_1$  is a  $\delta_1\delta_2$ -derivation. The set of all  $\delta$ -derivations, for the fixed  $\delta$ , we denote by  $\text{Der}_\delta(\mathfrak{L})$ . For the Lie algebras, the notion of anti-derivations coincides with the notion of reverse derivations, which was studied by Herstein in [22]. Note that the main example of  $\frac{1}{2}$ -derivations is the multiplication by an element from the ground field, i.e.,

$\varphi(x) = \lambda x$  for all  $x \in \mathfrak{L}$ . Such kind of  $\frac{1}{2}$ -derivations are called trivial  $\frac{1}{2}$ -derivations.

Since the notions of local operators can be defined for any type of operator, the notions of local and 2-local  $\delta$ -derivations are defined as follows:

**Definition 2.** A linear map  $\Delta$  is called a local  $\delta$ -derivation, if for any  $x \in \mathfrak{L}$ , there exists a  $\delta$ -derivation  $\varphi_x : \mathfrak{L} \rightarrow \mathfrak{L}$  (depending on  $x$ ) such that  $\Delta(x) = \varphi_x(x)$ . The set of all local  $\delta$ -derivations on  $\mathfrak{L}$  we denote by  $\text{LocDer}_\delta(\mathfrak{L})$ .

**Definition 3.** A map  $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$  (not necessary linear) is called a 2-local  $\delta$ -derivation, if for any  $x, y \in \mathfrak{L}$ , there exists a  $\delta$ -derivation  $\varphi_{x,y} \in \text{Der}_\delta(\mathfrak{L})$  such that

$$\nabla(x) = \varphi_{x,y}(x), \quad \nabla(y) = \varphi_{x,y}(y).$$

It should be noted that 2-local  $\delta$ -derivation is not necessarily linear, but for any  $x \in \mathfrak{L}$  and for any scalar  $\lambda$ , we have

$$\nabla(\lambda x) = \varphi_{x,\lambda x}(\lambda x) = \lambda \varphi_{x,\lambda x}(x) = \lambda \nabla(x).$$

In this work, we focus on investigating local and 2-local  $\frac{1}{2}$ -derivations. Since any  $\frac{1}{2}$ -derivation is a local and 2-local  $\frac{1}{2}$ -derivations, we are interesting on local and 2-local  $\frac{1}{2}$ -derivation, which is not a  $\frac{1}{2}$ -derivation. Such local (resp. 2-local)  $\frac{1}{2}$ -derivations we call non-trivial local (resp. 2-local)  $\frac{1}{2}$ -derivations.

Lower central series for a given Lie algebra  $\mathfrak{L}$  is defined as follows:

$$\mathfrak{L}^1 = \mathfrak{L}, \quad \mathfrak{L}^{k+1} = [\mathfrak{L}^k, \mathfrak{L}], \quad k \geq 1.$$

**Definition 4.** A Lie algebra  $\mathfrak{L}$  is said to be **nilpotent**, if there exists  $k \in \mathbb{N}$  such that  $\mathfrak{L}^k = \{0\}$ . The minimal number  $k$  with such property is said to be the **index of nilpotency** of the algebra  $\mathfrak{L}$ .

**Definition 5.** An  $n$ -dimensional Lie algebra is called **quasi-filiform** if its index of nilpotency is equal to  $n - 1$ .

Now, let us define a maximum length for the nilpotent Lie algebras.

A Lie algebra  $\mathfrak{L}$  is  $\mathbb{Z}$ -graded, if  $\mathfrak{L} = \bigoplus_{i \in \mathbb{Z}} \mathbf{V}_i$ , where  $[V_i, V_j] \subseteq V_{i+j}$  for any  $i, j \in \mathbb{Z}$  with a finite number of non-null spaces  $V_i$ . We say that a nilpotent Lie algebra  $\mathfrak{L}$  admits **the connected gradation**  $\mathfrak{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$ , if  $V_{k_i} \neq \{0\}$  for any  $i$  ( $1 \leq i \leq t$ ).

**Definition 6.** The number  $l(\bigoplus \mathfrak{L}) = l(V_{k_1} \oplus \cdots \oplus V_{k_t}) = k_t - k_1 + 1$  is called **the length of gradation**. A gradation is called **of maximum length**, if  $l(\bigoplus \mathfrak{L}) = \dim(\mathfrak{L})$ .

We denote  $l(\mathfrak{L}) = \max\{l(\bigoplus L) \text{ such that } L = V_{k_1} \oplus \cdots \oplus V_{k_t} \text{ is a connected gradation}\}$  of **the length of an algebra**  $\mathfrak{L}$ .

**Definition 7.** A Lie algebra  $\mathfrak{L}$  is called **of maximum length** if  $l(\mathfrak{L}) = \dim(\mathfrak{L})$ .

In the following theorem we present the classification of quasi-filiform Lie algebras of maximum length given in [21].

**Theorem 1.** *Let  $L$  be an  $n$ -dimensional quasi-filiform Lie algebra of maximum length. Then the algebra  $L$  is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$g_{(n,1)}^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 2 \leq i \leq \frac{n-1}{2}, n \geq 5 \text{ and } n \text{ is odd}; \end{cases}$$

$$g_{(n,1)}^2 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3, n \geq 5; \end{cases}$$

$$g_{(n,1)}^3 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3 \\ [e_2, e_i] = e_{i+3}, & 3 \leq i \leq n-4, n \geq 7; \end{cases}$$

$$g_7^1 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 5, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_i, e_{7-i}] = (-1)^i e_7, & 2 \leq i \leq 3; \end{cases} \quad g_9^2 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 7, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_5] = 3e_7, \\ [e_2, e_6] = 5e_8, \\ [e_3, e_i] = -2e_{i+3}, & 4 \leq i \leq 5, \\ [e_i, e_{9-i}] = (-1)^i e_9, & 2 \leq i \leq 4; \end{cases}$$

$$g_{11}^3 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 9, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_i] = -e_{i+2}, & 6 \leq i \leq 7 \\ [e_3, e_7] = -e_{10}, \\ [e_3, e_i] = e_{i+3}, & 4 \leq i \leq 5, \\ [e_4, e_i] = e_{i+4}, & 5 \leq i \leq 6, \\ [e_i, e_{11-i}] = (-1)^i e_{11}, & 2 \leq i \leq 5, \end{cases}$$

where  $\{e_1, e_2, \dots, e_n\}$  is a basis of the algebra.

In the following theorem we present the  $\frac{1}{2}$ -derivation of quasi-filiform Lie algebras of maximum length given in [1]

**Theorem 2.** *Any  $\frac{1}{2}$ -derivation  $\varphi$  of quasi-filiform Lie algebras of maximum length has the following form:*

- for algebra  $g_{(n,1)}^1$  ( $n = 5$ ) :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, \\ \varphi(e_2) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \frac{1}{2}(\alpha_1 + \beta_2)e_3 + \frac{1}{2}\beta_3 e_4 - \frac{1}{2}\alpha_3 e_5, \\ \varphi(e_4) = \frac{1}{4}(3\alpha_1 + \beta_2)e_4 + \frac{1}{2}\alpha_2 e_5, \\ \varphi(e_5) = \frac{1}{2}\beta_1 e_4 + \frac{1}{4}(\alpha_1 + 3\beta_2)e_5; \end{cases}$$

- for algebra  $g_{(n,1)}^1$  ( $n \geq 7$ ) :

$$\begin{cases} \varphi(e_1) = \sum_{i=1}^n \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2} e_{n-1} - \frac{1}{2}\alpha_{n-2} e_n, \\ \varphi(e_i) = \alpha_1 e_i + \frac{(-1)^i}{2} \alpha_{n-i+1} e_n, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = \alpha_1 e_n; \end{cases}$$

- for algebra  $g_{(n,1)}^2$  ( $n = 5$ ) :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, \\ \varphi(e_2) = \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \frac{1}{2}(\alpha_1 + \beta_2)e_3 + \frac{1}{2}(\beta_3 - \alpha_5)e_4, \\ \varphi(e_4) = \frac{1}{4}(3\alpha_1 + \beta_2)e_4, \\ \varphi(e_5) = -\alpha_2 e_3 + \gamma_4 e_4 + \frac{1}{2}(3\alpha_1 - \beta_2)e_5; \end{cases}$$

- for algebra  $g_{(n,1)}^2$  ( $n = 6$ ) :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6, \\ \varphi(e_2) = \alpha_1 e_2 - 3\alpha_6 e_3 + \beta_4 e_4 + \beta_5 e_5, \\ \varphi(e_3) = \alpha_1 e_3 - 2\alpha_6 e_4 + \frac{1}{2}\beta_4 e_5, \\ \varphi(e_4) = \alpha_1 e_4 - \frac{3}{2}\alpha_6 e_5, \\ \varphi(e_5) = \alpha_1 e_5, \\ \varphi(e_6) = -\alpha_2 e_3 - \alpha_3 e_4 + \gamma_5 e_5 + \alpha_1 e_6; \end{cases}$$

- for algebra  $g_{(n,1)}^2$  ( $n \geq 7$ ) :

$$\begin{cases} \varphi(e_1) = \sum_{i=1}^{n-1} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2} e_{n-1}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = -\sum_{i=3}^{n-2} \alpha_{i-1} e_i + \gamma_{n-1} e_{n-1} + \alpha_1 e_n; \end{cases}$$

- for algebra  $g_{(n,1)}^3$  ( $n = 7$ ) :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6, \\ \varphi(e_2) = \alpha_1 e_2 + 4\alpha_2 e_4 + \beta_5 e_5 + \beta_6 e_6, \\ \varphi(e_3) = \alpha_1 e_3 + 2\alpha_2 e_5 + \frac{1}{2}(\beta_5 - \alpha_3) e_6, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{3}{2}\alpha_2 e_6, \\ \varphi(e_5) = \alpha_1 e_5, \quad \varphi(e_6) = \alpha_1 e_6, \\ \varphi(e_7) = -\alpha_2 e_3 - \alpha_3 e_4 - \alpha_4 e_5 + \gamma_6 e_6 + \alpha_1 e_7; \end{cases}$$

- for algebra  $g_{(n,1)}^3$  ( $n \geq 8$ ) :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \sum_{i=3}^{n-1} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \sum_{i=5}^{n-3} \alpha_{i-2} e_i + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}(\beta_{n-2} - \alpha_{n-4}) e_{n-1}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 4 \leq i \leq n-1, \\ \varphi(e_n) = -\sum_{i=4}^{n-2} \alpha_{i-1} e_i + \gamma_{n-1} e_{n-1} + \alpha_1 e_n; \end{cases}$$

- for algebra  $g_7^1$  :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7, \\ \varphi(e_2) = \alpha_1 e_2 - \frac{1}{3}\alpha_3 e_4 + \beta_5 e_5 + \beta_6 e_6 + \beta_7 e_7, \\ \varphi(e_3) = \alpha_1 e_3 - \frac{2}{3}\alpha_3 e_5 + \frac{1}{2}(\beta_5 - \alpha_4) e_6 - \frac{1}{2}\alpha_5 e_7, \\ \varphi(e_4) = \alpha_1 e_4 - \frac{1}{3}\alpha_3 e_6 + \frac{1}{2}\alpha_4 e_7, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2}\alpha_3 e_7, \\ \varphi(e_6) = \alpha_1 e_6, \\ \varphi(e_7) = \alpha_1 e_7; \end{cases}$$

- for algebra  $g_9^2$  :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9, \\ \varphi(e_2) = \alpha_1 e_2 + \frac{1}{3}\alpha_5 e_6 + \beta_7 e_7 + \beta_8 e_8 + \beta_9 e_9, \\ \varphi(e_3) = \alpha_1 e_3 - \frac{4}{3}\alpha_5 e_7 + \frac{1}{2}(\beta_7 - 5\alpha_6) e_8 - \frac{1}{2}\alpha_7 e_9, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{1}{3}\alpha_5 e_8 + \frac{1}{2}\alpha_6 e_9, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2}\alpha_5 e_9, \\ \varphi(e_i) = \alpha_1 e_i, \quad 6 \leq i \leq 9; \end{cases}$$

- for algebra  $g_{11}^3$  :

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9 + \alpha_{10} e_{10} + \alpha_{11} e_{11}, \\ \varphi(e_2) = \alpha_1 e_2 - \alpha_6 e_7 - \alpha_7 e_8 + \beta_9 e_9 + \beta_{10} e_{10} + \beta_{11} e_{11}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2} \beta_9 e_{10} - \frac{1}{2} \alpha_9 e_{11}, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{1}{2} \alpha_7 e_{10} + \frac{1}{2} \alpha_8 e_{11}, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2} \alpha_6 e_{10} - \frac{1}{2} \alpha_7 e_{11}, \\ \varphi(e_6) = \alpha_1 e_6 + \frac{1}{2} \alpha_6 e_{11}, \\ \varphi(e_i) = \alpha_1 e_i, \quad 7 \leq i \leq 11. \end{cases}$$

### 3 Local and 2-local $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length

**3.1. Local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length..** In this subsection we investigate local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.

**Theorem 3.** *Any local  $\frac{1}{2}$ -derivation  $\Delta$  of quasi-filiform Lie algebras of maximum length has the following form:*

- for algebra  $g_{(n,1)}^1 (n = 5)$  :

$$\begin{cases} \Delta(e_1) = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \\ \Delta(e_2) = a_6 e_1 + a_7 e_2 + a_8 e_3 + a_9 e_4 + a_{10} e_5, \\ \Delta(e_3) = a_{11} e_3 + a_{12} e_4 + a_{13} e_5, \\ \Delta(e_4) = a_{14} e_4 + a_{15} e_5, \\ \Delta(e_5) = a_{16} e_4 + a_{17} e_5; \end{cases}$$

- for algebra  $g_{(n,1)}^1 (n \geq 7)$  :

$$\begin{cases} \Delta(e_1) = \sum_{i=1}^n a_i e_i, \\ \Delta(e_2) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n, \\ \Delta(e_3) = c_1 e_3 + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_i) = c_1 e_i + c_i e_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = d e_n; \end{cases} \quad (1)$$

- for algebra  $g_{(n,1)}^2 (n = 5)$  :

$$\begin{cases} \Delta(e_1) = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \\ \Delta(e_2) = a_6 e_2 + a_7 e_3 + a_8 e_4 + a_9 e_5, \\ \Delta(e_3) = a_{10} e_3 + a_{11} e_4, \\ \Delta(e_4) = a_{12} e_4, \\ \Delta(e_5) = a_{13} e_3 + a_{14} e_4 + a_{15} e_5; \end{cases}$$

- for algebra  $g_{(n,1)}^2$  ( $n = 6$ ) :

$$\begin{cases} \Delta(e_1) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ \Delta(e_2) = a_7e_2 + a_8e_3 + a_9e_4 + a_{10}e_5, \\ \Delta(e_3) = a_{11}e_3 + a_{12}e_4 + a_{13}e_5, \\ \Delta(e_4) = a_{14}e_4 + a_{15}e_5, \\ \Delta(e_5) = a_{16}e_5, \\ \Delta(e_6) = a_{17}e_3 + a_{18}e_4 + a_{19}e_5 + a_{20}e_6; \end{cases}$$

- for algebra  $g_{(n,1)}^2$  ( $n \geq 7$ ) :

$$\begin{cases} \Delta(e_1) = \sum_{i=1}^{n-1} a_i e_i, \\ \Delta(e_2) = a_1 e_2 + c_1 e_{n-2} + c_2 e_{n-1}, \\ \Delta(e_3) = a_1 e_3 + c_n e_{n-1}, \\ \Delta(e_i) = a_1 e_i, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = - \sum_{i=3}^{n-1} c_i e_i + a_1 e_n; \end{cases}$$

- for algebra  $g_{(n,1)}^3$  ( $n = 7$ ) :

$$\begin{cases} \Delta(e_1) = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ \Delta(e_2) = a_1e_2 + a_7e_4 + a_8e_5 + a_9e_6, \\ \Delta(e_3) = a_1e_3 + a_{10}e_5 + a_{11}e_6, \\ \Delta(e_4) = a_1e_4 + a_{12}e_6, \\ \Delta(e_5) = a_1e_5, \quad \Delta(e_6) = a_1e_6, \\ \Delta(e_7) = a_{13}e_3 + a_{14}e_4 + a_{15}e_5 + a_{16}e_6 + a_{17}e_7; \end{cases}$$

- for algebra  $g_{(n,1)}^3$  ( $n \geq 8$ ) :

$$\begin{cases} \Delta(e_1) = a_1e_1 + \sum_{i=3}^{n-1} a_i e_i, \\ \Delta(e_2) = a_1e_2 + \sum_{i=5}^{n-1} b_i e_i, \\ \Delta(e_3) = a_1e_3 + a_2e_{n-1}, \\ \Delta(e_i) = a_1e_i, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = \sum_{i=4}^{n-1} c_i e_i + a_1e_n; \end{cases}$$

- for algebra  $g_7^1$  :

$$\begin{cases} \Delta(e_1) = a_1e_1 + a_2e_3 + a_3e_4 + a_4e_5 + a_5e_6 + a_6e_7, \\ \Delta(e_2) = a_1e_2 + a_7e_4 + a_8e_5 + a_9e_6 + a_{10}e_7, \\ \Delta(e_3) = a_1e_3 + a_{11}e_5 + a_{12}e_6 + a_{13}e_7, \\ \Delta(e_4) = a_1e_4 + a_{14}e_6 + a_{15}e_7, \\ \Delta(e_5) = a_1e_5 + a_{16}e_7, \\ \Delta(e_6) = a_1e_6, \\ \Delta(e_7) = a_1e_7; \end{cases}$$

- for algebra  $g_9^2$  :

$$\begin{cases} \Delta(e_1) = a_1e_1 + a_2e_5 + a_3e_6 + a_4e_7 + a_5e_8 + a_6e_9, \\ \Delta(e_2) = a_1e_2 + a_7e_6 + a_8e_7 + a_9e_8 + a_{10}e_9, \\ \Delta(e_3) = a_1e_3 + a_{11}e_7 + a_{12}e_8 + a_{13}e_9, \\ \Delta(e_4) = a_1e_4 + a_{14}e_8 + a_{15}e_9, \\ \Delta(e_5) = a_1e_5 + a_{16}e_9, \\ \Delta(e_i) = a_1e_i, \quad 6 \leq i \leq 9; \end{cases}$$

- for algebra  $g_{11}^3$  :

$$\begin{cases} \Delta(e_1) = a_1e_1 + a_2e_6 + a_3e_7 + a_4e_8 + a_5e_9 + a_6e_{10} + a_7e_{11}, \\ \Delta(e_2) = a_1e_2 + a_8e_7 + a_9e_8 + a_{10}e_9 + a_{11}e_{10} + a_{12}e_{11}, \\ \Delta(e_3) = a_1e_3 + a_{13}e_{10} + a_{14}e_{11}, \\ \Delta(e_4) = a_1e_4 + a_{15}e_{10} + a_{16}e_{11}, \\ \Delta(e_5) = a_1e_5 + a_{17}e_{10} + a_{18}e_{11}, \\ \Delta(e_6) = a_1e_6 + a_{19}e_{11}, \\ \Delta(e_i) = a_1e_i, \quad 7 \leq i \leq 11. \end{cases}$$

*Proof.* We prove the theorem for the algebra  $g_{(n,1)}^1 (n \geq 7)$ , and for the algebras  $g_{(5,1)}^1$ ,  $g_{(5,1)}^2$ ,  $g_{(6,1)}^2$ ,  $g_{(n,1)}^2 (n \geq 7)$ ,  $g_{(7,1)}^3$ ,  $g_{(n,1)}^3 (n \geq 8)$ ,  $g_7^1$ ,  $g_9^2$  and  $g_{11}^3$  the proofs are similar.

Let  $\Delta$  be an arbitrary local  $\frac{1}{2}$ -derivation on  $g_{(n,1)}^1 (n \geq 7)$ . Since  $\Delta$  is a local  $\frac{1}{2}$ -derivation, for any element  $x = \sum_{i=1}^n x_i e_i \in g_{(n,1)}^1 (n \geq 7)$ , there exists a  $\frac{1}{2}$ -derivation  $\varphi_x$ , such that  $\Delta(x) = \varphi_x(x)$ .

Choosing subsequently  $x = e_0, x = e_1, \dots, x = e_n$ , we obtain the general form of  $\Delta$  as follows:

$$\begin{cases} \Delta(e_1) = \sum_{i=1}^n a_i e_i, \\ \Delta(e_2) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n, \\ \Delta(e_3) = c_1 e_3 + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_i) = d_i e_i + c_i e_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = d_n e_n; \end{cases}$$

Consider

$$\Delta(e_2 + e_3) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n + c_1 e_3 + c_2 e_{n-1} + c_3 e_n.$$

On the other hand,

$$\begin{aligned} \Delta(e_2 + e_3) &= \varphi_{e_2+e_3}(e_2 + e_3) = \varphi_{e_2+e_3}(e_2) + \varphi_{e_2+e_3}(e_3) \\ &= \alpha_{1,e_2+e_3} e_2 + \beta_{n-2,e_2+e_3} e_{n-2} + \beta_{n-1,e_2+e_3} e_{n-1} \\ &\quad + \beta_{n,e_2+e_3} e_n + \alpha_{1,e_2+e_3} e_3 + \frac{1}{2} \beta_{n-2,e_2+e_3} e_{n-1} - \frac{1}{2} \alpha_{n-2,e_2+e_3} e_n. \end{aligned}$$

Comparing the coefficients at the basis elements  $e_2$  and  $e_3$ , we get  $\alpha_{1,e_2+e_3} = b_1$ ,  $\alpha_{1,e_2+e_3} = c_1$ , which implies

$$b_1 = c_1.$$

Now for  $4 \leq i \leq n-3$ , we consider

$$\Delta(e_2 + e_i) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n + d_i e_i + c_i e_n.$$

On the other hand, there exists  $\frac{1}{2}$ -derivation  $\varphi_{e_2+e_i}$  such that

$$\begin{aligned} \Delta(e_2 + e_i) &= \varphi_{e_2+e_i}(e_2 + e_i) = \varphi(e_2) + \varphi(e_i) \\ &= \alpha_{1,e_2+e_i} e_2 + \beta_{n-2,e_2+e_i} e_{n-2} + \beta_{n-1,e_2+e_i} e_{n-1} + \beta_{n,e_2+e_i} e_n \\ &\quad + \alpha_{1,e_2+e_i} e_i + \frac{(-1)^i}{2} \alpha_{n-i+1,e_2+e_i} e_n. \end{aligned}$$

Comparing the coefficients of  $e_2$  and  $e_i$ , we obtain that  $d_i = b_1$  for all  $4 \leq i \leq n-3$ .

Similarly, for  $n-2 \leq i \leq n-1$  from

$$\Delta(e_4 + e_i) = b_1 e_4 + c_4 e_n + d_i e_i + c_i e_n.$$

On the other hand, there exists  $\frac{1}{2}$ -derivation  $\varphi_{e_4+e_i}$  such that

$$\begin{aligned} \Delta(e_4 + e_i) &= \varphi_{e_4+e_i}(e_4 + e_i) = \varphi(e_4) + \varphi(e_i) \\ &= \alpha_{1,e_4+e_i} e_4 + \frac{1}{2} \alpha_{n-3,e_4+e_i} e_n + \alpha_{1,e_4+e_i} e_i + \frac{(-1)^i}{2} \alpha_{n-i+1,e_4+e_i} e_n. \end{aligned}$$

Comparing the coefficients of  $e_4$  and  $e_i$ , we obtain that  $d_i = b_1$  for all  $n-2 \leq i \leq n-1$ .

Now, we show that any linear operator of the form

$$\begin{cases} \Delta(e_1) = \sum_{i=1}^n a_i e_i, \\ \Delta(e_2) = b_1 e_2 + b_2 e_{n-2} + b_3 e_{n-1} + b_4 e_n, \\ \Delta(e_3) = b_1 e_3 + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_i) = b_1 e_i + c_i e_n, \quad 4 \leq i \leq n-1, \\ \Delta(e_n) = d e_n; \end{cases}$$

is a local  $\frac{1}{2}$ -derivation on  $g_{(n,1)}^1 (n \geq 7)$ .

Considering  $\Delta(x) = \varphi_x(x)$  for any  $x = \sum_{i=1}^n x_i e_i$ , we obtain the following system of equality:

$$\begin{aligned} a_1 x_1 &= \alpha_{1,x} x_1, \\ a_i x_1 + b_1 x_i &= \alpha_{i,x} x_1 + \alpha_{1,x} x_i, \quad 2 \leq i \leq n-3, \\ a_{n-2} x_1 + b_2 x_2 + b_1 x_{n-2} &= \alpha_{n-2,x} x_1 + \beta_{n-2,x} x_2 + \alpha_{1,x} x_{n-2}, \\ a_{n-1} x_1 + b_3 x_2 + c_2 x_3 + b_1 x_{n-1} &= \alpha_{n-1,x} x_1 + \beta_{n-1,x} x_2 + \frac{\beta_{n-2,x}}{2} x_3 + \alpha_{1,x} x_{n-1}, \\ a_n x_1 + b_4 x_2 + \sum_{i=3}^{n-1} c_i x_i + d x_n &= \alpha_{n,x} x_1 + \beta_{n,x} x_2 + \sum_{i=3}^{n-1} (-1)^i \frac{\alpha_{n-i+1,x}}{2} x_i + \alpha_{1,x} x_n. \end{aligned} \tag{2}$$

Let us consider the following cases:

Case 1. Let  $x_0 \neq 1$ , then one can take

$$\begin{aligned} \alpha_{1,x} &= a_1, \\ \alpha_{i,x} &= \frac{a_i x_1 + b_1 x_i - \alpha_{1,x} x_i}{x_1}, \quad 2 \leq i \leq n-3, \\ \alpha_{n-2,x} &= \frac{a_{n-2} x_1 + b_2 x_2 + b_1 x_{n-2} - \beta_{n-2,x} x_2 - \alpha_{1,x} x_{n-2}}{x_1}, \\ \alpha_{n-1,x} &= \frac{a_{n-1} x_1 + b_3 x_2 + c_2 x_3 + b_1 x_{n-1} - \beta_{n-1,x} x_2 - \frac{\beta_{n-2,x}}{2} x_3 - \alpha_{1,x} x_{n-1}}{x_1}, \\ \alpha_{n,x} &= \frac{a_n x_1 + b_4 x_2 + \sum_{i=3}^{n-1} c_i x_i + d x_n - \beta_{n,x} x_2 - \sum_{i=3}^{n-1} (-1)^i \frac{\alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x} x_n}{x_1}, \end{aligned}$$

where  $\beta_{n-2}$ ,  $\beta_{n-1}$ ,  $\beta_n$  defined arbitrary.

Case 2. Let  $x_1 = 0$  and  $x_2 \neq 0$ , then we may choose

$$\begin{aligned}\alpha_{1,x} &= b_1, \\ \beta_{n-2,x} &= \frac{b_2x_2 + b_1x_{n-2} - \alpha_{1,x}x_{n-2}}{x_2}, \\ \beta_{n-1,x} &= \frac{b_3x_2 + c_2x_3 + b_1x_{n-1} - \frac{\beta_{n-2,x}}{2}x_3 - \alpha_{1,x}x_{n-1}}{x_2}, \\ \beta_{n,x} &= \frac{b_4x_2 + \sum_{i=3}^{n-1} c_i x_i + dx_n - \sum_{i=3}^{n-1} (-1)^i \frac{\alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x}x_n}{x_2},\end{aligned}$$

where  $\alpha_2, \alpha_3, \dots, \alpha_{n-2}$  defined arbitrary.

Case 3. Let  $x_1 = x_2 = 0$  and  $x_3 \neq 0$ , then we may choose

$$\begin{aligned}\alpha_{1,x} &= b_1, \\ \beta_{n-2,x} &= \frac{2(c_2x_3 + b_1x_{n-1} - \alpha_{1,x}x_{n-1})}{x_3}, \\ \alpha_{n-2,x} &= 2 \cdot \frac{\sum_{i=3}^{n-1} c_i x_i + dx_n - \sum_{i=4}^{n-1} (-1)^i \frac{\alpha_{n-i+1,x}}{2} x_i - \alpha_{1,x}x_n}{x_3},\end{aligned}$$

where  $\alpha_2, \alpha_3, \dots, \alpha_{n-3}$  defined arbitrary.

Case 4. Let  $x_1 = x_2 = \dots = x_{i-1} = 0$ ,  $x_i \neq 0$ ,  $4 \leq i \leq n-1$ , then we may take

$$\begin{aligned}\alpha_{1,x} &= b_1, \\ \alpha_{n-i+1,x} &= 2 \cdot (-1)^i \cdot \frac{\sum_{j=i}^{n-1} c_j x_j + dx_n - \sum_{j=i}^{n-2} \frac{(-1)^{j+1} \alpha_{n-j,x} x_{j+1}}{2} - \alpha_{1,x}x_n}{x_i},\end{aligned}$$

where  $\alpha_2, \alpha_3, \dots, \alpha_{n-i}$  defined arbitrary.

Case 5. Let  $x_1 = x_2 = \dots = x_{n-1} = 0$ ,  $x_n \neq 0$ , then we may take  $\alpha_{1,x} = b_1$ .

Thus, we have that for any  $a_i, b_1, b_2, b_3, b_4, c_2, \dots, c_{n-1}$  and  $d$  there exist  $\alpha_{i,x}, \beta_{n-2,x}, \beta_{n-1,x}, \beta_{n,x}$ , such that the equalities (2) hold. Hence any linear transformation of the form (1) is a local  $\frac{1}{2}$ -derivation on  $g_{(n,1)}^1$  ( $n \geq 7$ ).  $\square$

In the following table, we give the dimensions of the spaces of  $\frac{1}{2}$ -derivations and local  $\frac{1}{2}$ -derivations of quasi-filiform Lie algebras of maximum length:

Algebra	$\dim \text{Der}_{\frac{1}{2}}$	$\dim \text{LocDer}_{\frac{1}{2}}$
$g_{(5,1)}^1$	10	17
$g_{(n,1)}^1 (n \geq 7)$	$n + 3$	$2n + 4$
$g_{(5,1)}^2$	10	15
$g_{(6,1)}^2$	9	19
$g_{(n,1)}^2 (n \geq 7)$	$n + 2$	$2n - 1$
$g_{(7,1)}^3$	9	17
$g_{(n,1)}^3 (n \geq 8)$	$n + 1$	$3n - 10$
$g_7^1$	9	16
$g_9^2$	9	16
$g_{11}^3$	10	19

**Corollary 1.** *The quasi-filiform Lie algebras of maximum length admit local  $\frac{1}{2}$ -derivations which are not  $\frac{1}{2}$ -derivations.*

**3.2. 2-local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.** Now we investigate 2-local  $\frac{1}{2}$ -derivations on quasi-filiform Lie algebras of maximum length.

**Theorem 4.** *The quasi-filiform Lie algebras of maximum length admits a 2-local  $\frac{1}{2}$ -derivation which is not a  $\frac{1}{2}$ -derivation.*

*Proof.* We prove the theorem for the algebra  $g_{(n,1)}^1 (n \geq 7)$ , and for the algebras  $g_{(5,1)}^1$ ,  $g_{(5,1)}^2$ ,  $g_{(6,1)}^2$ ,  $g_{(n,1)}^2 (n \geq 7)$ ,  $g_{(7,1)}^3$ ,  $g_{(n,1)}^3 (n \geq 8)$ ,  $g_7^1$ ,  $g_9^2$  and  $g_{11}^3$  the proofs are similar.

Let us define a homogeneous non additive function  $f$  on  $\mathbb{C}^2$  as follows

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2}{x_2}, & \text{if } x_2 \neq 0, \\ 0, & \text{if } x_2 = 0, \end{cases}$$

where  $(x_1, x_2) \in \mathbb{C}^2$ .

Define the operator  $\nabla$  on  $g_{(n,1)}^1 (n \geq 7)$ , such that

$$\nabla(x) = f(x_1, x_2)e_n, \quad (3)$$

for any element  $x = \sum_{i=1}^n x_i e_i$ ,

The operator  $\nabla$  is not a  $\frac{1}{2}$ -derivation, since it is not linear.

Let us show that,  $\nabla$  is a 2-local  $\frac{1}{2}$ -derivation. For this purpose, define a  $\frac{1}{2}$ -derivation  $\varphi$  on  $g_{(n,1)}^1 (n \geq 7)$  by

$$\varphi(x) = (\alpha_n x_1 + \beta_n x_2)e_n.$$

For each pair of elements  $x$  and  $y$ , we choose  $\alpha_n$  and  $\beta_n$ , such that  $\nabla(x) = \varphi(x)$  and  $\nabla(y) = \varphi(y)$ . Let us rewrite the above equalities as system of linear equations with respect to unknowns  $\alpha_n, \beta_n$  as follows

$$\begin{cases} x_1\alpha_n + x_2\beta_n = f(x_1, x_2), \\ y_1\alpha_n + y_2\beta_n = f(y_1, y_2). \end{cases} \quad (4)$$

**Case 1.**  $x_1y_2 - x_2y_1 = 0$ . In this case, since the right-hand side of the system (4) is homogeneous, it has infinitely many solutions.

**Case 2.**  $x_1y_2 - x_2y_1 \neq 0$ . In this case, the system (4) has a unique solution.  $\square$

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