

ON MODAL PRESENTATION OF EXPLOSIVE AND PARACONSISTENT EQUILIBRIUM LOGIC

S.P. ODINTSOV 

Communicated by ???

Abstract: Fariñas del Cerro, Herzig and Su proved that the non-monotonic consequence relation determined by Answer Set Semantics (ASP) for logic programs with negation-as-failure can be embedded into a monotonic modal logic via a variation of Gödel-Tarski Translation. This article generalizes the mentioned result to ASP for logic programs with two kinds of negation: negation-as-failure and strong negation and to PAS, the paraconsistent version of ASP admitting answer sets that are inconsistent w.r.t. the strong negation.

Keywords: logic programs, negation-as-failure, strong negation, equilibrium logic, deductive base, temporal logic, equilibrium modal theory.

The stable model (answer set) semantics for logic programs with negation-as-failure \neg suggested by M. Gelfond and V. Lifschitz [7] gives rise to a separate paradigm in the setting of Logic Programming, so called Answer Set Programming (ASP). An important fact was established by D. Pearce [19], who proved that the intermediate logic *HT* of “here-and-there”, which is known also as the Gödel-Smetanich logic and can be determined by a Kripke frame with two worlds, can serve as a tool for reasoning about answer

ODINTSOV S.P., ON MODAL PRESENTATION OF PARACONSISTENT EQUILIBRIUM LOGIC.

© 2025 ODINTSOV S.P..

The work was executed in Mathematical Institute RAN and supported by is supported by the Russian Science Foundation (Project No. No. 23-11-00104, <https://rscf.ru/project/23-11-00104/>).

Received ???, Published ???

sets. The main property involved is that answer sets can be viewed as a certain kind of minimal *HT*-models, which are called equilibrium models. The same holds for logic programs in the extended language, which includes not only the negation-as-failure \neg , but also the strong negation \sim (which was originally introduced in Logic Programming under the name of classical negation [8]). In this case [19] answer sets are in one-to one correspondence with equilibrium \mathbf{N}_5 -models, where \mathbf{N}_5 can be considered as *HT* enriched with the strong negation. More exactly, \mathbf{N}_5 is a finite-valued extension of the explosive Nelson logic $\mathbf{N3}$, which can be determined via a 5-element algebra. On the other hand, \mathbf{N}_5 is the least conservative extension of *HT* in the lattice of $\mathbf{N3}$ -extensions. The logic $\mathbf{N3}$ is based on the concept of constructible falsity, which was introduced into logic by D. Nelson [13] via his system of constructive arithmetic with strong negation. The propositional fragment of Nelson's arithmetic, which is denoted now as $\mathbf{N3}$, was subsequently axiomatised by N. Vorob'ev [24, 25].

A second key property relating non-classical logics with ASP was established in [10]: programs are strongly equivalent wrt answer set semantics if and only if they are equivalent viewed as propositional theories in *HT* (in \mathbf{N}_5 if \sim occurs in the language). Here, two programs Π_1 and Π_2 are called strongly equivalent if for any program Π , $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same answer sets. This shows that *HT* and \mathbf{N}_5 can be used for program transformation and optimisation.

Paraconsistent version of answer set semantics (PAS) admits answer sets that are inconsistent w.r.t. the strong negation. PAS was studied as a logic programming semantics by C. Sakama and K. Inoue [21]. Later, the work [1] has made some progress towards a logical, declarative style of characterization for PAS. However, [1] does not axiomatize or otherwise syntactically characterize the underlying (monotonic) logic of PAS. In [16], it was proved that semantical frames for the substructural logics used in [1] can be reduced to a simpler Routley frames [20] with additional falsity constant. This provides a description of paraconsistent answer sets as a special kind of minimal Routley models. It is proved in [16] that these Routley models determine an extension of the paraconsistent Nelson logic $\mathbf{N4}^\perp$ [14]. This extension was denoted $\mathbf{N9}$ due to the reason that it can be determined via a 9-element algebra. Again, $\mathbf{N9}$ is the least conservative extension of *HT* in the class of $\mathbf{N4}^\perp$ -extensions. Finally, the strong equivalence theorem was proved [16]: two programs are strongly equivalent iff they are equivalent as $\mathbf{N9}$ -theories.

The next step towards the declarative treatment of ASP was done by L. Fariñas del Cerro, A. Herzig and E. Su [5]. They proved that the non-monotonic consequence relation determined by Answer Set Semantics (ASP) for logic programs with negation-as-failure can be embedded into a monotonic modal logic **MEM** via a variation of Gödel-Tarski Translation [9]. This article generalizes the mentioned result to the ordinary and paraconsistent

versions of ASP for logic programs with two kinds of negation: negation-as-failure and strong negation. To this end we need the possibility to embed the constructive logic with strong negation into a suitable modal logic.

Belnapian version of normal modal logic **BS4** [17] relates to **S4** in exactly the same way as the logic $\mathbf{N4}^\perp$ relates to intuitionistic logic. Its semantics can be obtained from that of **S4** via replacement of two-valued valuations by four valued ones. In each of possible worlds a formula may have one of four truth values *True*, *False*, *Neither*, *Both* of Belnap-Dunn matrix **BD4** [2], which provide a semantics for First Degree Entailment **FDE** [4]. In [17] it was proved that $\mathbf{N4}^\perp$ is faithfully embedded into the logic **BS4** via the translation T_B , a natural modification of Gödel-Tarski translation. This result shows that modal companions of Nelson's logic extensions defined via T_B belong to the lattice of **BS4**-extensions (see [23] for details). So logics based on **BS4**-extensions looks suitable for the goals of this article.

In our reasoning we will essentially follow the line depicted in [5], but we make one serious modification. Following [6] we understand a logic as a structural Tarskian consequence relation defined over some propositional language. According to this definition **MEM** of [5] is rather a theory than a logic because it is not closed under substitutions. In our work we try maximally distinguish 'logical' and 'theoretical' parts of construction. First we define a kind of Belnapian temporal logic **BSK_{t2}** and prove that the Belnapian version T_B of Gödel-Tarski translation faithfully embeds the logic **N₉** (the deductive base of PAS) into **BSK_{t2}**, and the same holds for **N₅** and the explosive version of **BSK_{t2}**. Further, we define theories over **BSK_{t2}** and its explosive extension and prove that equilibrium entailments over **N₉** and **N₅** can be embedded into this theories.

The paper is structured as follows. Section 1 contains necessary information on constructive logics, Belnapian modal logics and semantics of logic programs with negations. In Section 2 we define a special temporal logic **BSK_t** such that future and past modalities of **BSK_t** are defined via accessibility relations that are not mutually inverse. We introduce also the logic **BSK_{t2}** that extends **BSK_t** imposing further restrictions on both accessibility relations. Section 3 investigates the \blacksquare -free fragment of **BSK_{t2}** (\blacksquare stands for the necessity in the past). We prove that this fragment of **BSK_{t2}** is a modal companion of the deductive base of the equilibrium entailment. Finally, in Section 4 we embed the equilibrium entailment into **BSK_{t2}**.

1 Preliminaries

As usual by a *propositional language* \mathcal{L} we mean a finite tuple of logical connectives and constants. The set $\text{Form}_{\mathcal{L}}$ of \mathcal{L} -formulas is constructed in a usual way from the fixed countable set Prop of propositional variables and the constants of \mathcal{L} with the help of \mathcal{L} -connectives. The languages we consider will include the implication connective \rightarrow . We will define logics in different propositional languages via Hilbert style deductive systems. And

we assume that every deductive system under consideration includes the standard axioms of intuitionistic logic **Int** in the list of its axioms and the rules of *modus ponens* (MP) and of substitution (SUB)

$$\text{(MP)} \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \quad \text{SUB} \quad \frac{\varphi(p_1, \dots, p_n)}{\varphi(\psi_1, \dots, \psi_n)}$$

in the set of its inference rules. With every logic L defined in the language \mathcal{L} we associate the inference relation \vdash_L . For a subset $\Gamma \cup \{\varphi\} \subseteq \text{Form}_{\mathcal{L}}$, $\Gamma \vdash_L \varphi$ means that φ can be obtained from the elements of Γ and the theorems of L with the help of (MP). Recall that a theorem of L is a formula, which can be inferred from the axioms of L with the help of all inference rules, not only (MP). We write $\varphi \in L$ instead of ‘ φ is a theorem of L ’.

For a logic L in the language \mathcal{L} , we denote by $\mathcal{E}L$ the family of all axiomatic extensions of L in the same language.

If L_i is a logic in the language \mathcal{L}_i , $i = 1, 2$, $L_1 \cup L_2$ denotes a logic in the language $\mathcal{L}_1 \cup \mathcal{L}_2$ defined by the union of axioms of L_1 and L_2 and the union of rules of these logics.

A proper subset $\Gamma \subseteq \text{Form}_{\mathcal{L}}$ is said to be a *prime L -theory* if (i) Γ contains all L -theorems; (ii) Γ is closed under (MP) ($\varphi, \varphi \rightarrow \psi \in \Gamma$ implies $\psi \in \Gamma$); (iii) Γ satisfies the disjunction property ($\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$). Notice that the axioms of **Int** and (MP) allow to prove in a standard way the Extension Lemma for every logic L considered in the article.

Lemma 1. *Let $\Gamma \not\vdash_L \varphi$. Then there exists a prime L -theory Σ such that $\Gamma \subseteq \Sigma$ and $\Sigma \not\vdash_L \varphi$.*

1.1. Constructive logics with strong negation. The paraconsistent version $\mathbf{N4}^\perp$ [14] of Nelson’s constructive logic with strong negation is defined in the propositional language \mathcal{L}^\sim including the absurdity constant \perp and logical connectives $\wedge, \vee, \rightarrow, \sim$, standing respectively for conjunction, disjunction, weak implication and strong negation. The set Lit^\sim of *literals* is defined as $\text{Prop} \cup \{\sim p \mid p \in \text{Prop}\}$. Arbitrary $\mathbf{S} \subseteq \text{Lit}^\sim$ can be represented as $\mathbf{S} = (\mathbf{S}^+, \mathbf{S}^-)$, where

$$\mathbf{S}^+ = \mathbf{S} \cap \text{Prop} \quad \text{and} \quad \mathbf{S}^- = \{p \mid \sim p \in \mathbf{S}\}$$

We say that \mathbf{S} is *consistent* if $\mathbf{S}^+ \cap \mathbf{S}^- = \emptyset$.

The Hilbert style deductive system for $\mathbf{N4}^\perp$ has (SUB) and (MP) as its only inference rules. The axioms include the standard list of axioms of intuitionistic logic in the language $\{\wedge, \vee, \rightarrow, \perp\}$:

- | | |
|--|---|
| I1. $p \rightarrow (q \rightarrow p)$ | I2. $(p \wedge q) \rightarrow p$ |
| I3. $(p \rightarrow (q \rightarrow \gamma)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow \gamma))$ | I4. $(p \wedge q) \rightarrow q$ |
| I5. $(p \rightarrow q) \rightarrow ((p \rightarrow \gamma) \rightarrow (p \rightarrow (q \wedge \gamma)))$ | I6. $p \rightarrow (p \vee q)$ |
| I7. $(p \rightarrow \gamma) \rightarrow ((q \rightarrow \gamma) \rightarrow ((p \vee q) \rightarrow \gamma))$ | I8. $q \rightarrow (p \vee q)$ |
| I9. $\perp \rightarrow p$ | |

plus the following strong negation axioms (where $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$):

$$\begin{array}{ll}
\mathbf{N1.} \sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q) & \mathbf{N2.} \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q) \\
\mathbf{N3.} \sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q) & \mathbf{N4.} \sim\sim p \leftrightarrow p \\
\mathbf{N5.} \sim\perp &
\end{array}$$

The explosive logic $\mathbf{N3}^\perp$ is obtained via adding $(p \wedge \sim p) \rightarrow q$ to the list of $\mathbf{N4}^\perp$ -axioms, symbolically $\mathbf{N3}^\perp = \mathbf{N4}^\perp + \{(p \wedge \sim p) \rightarrow q\}$. Notice that intuitionistic logic \mathbf{Int} coincides with the \sim -free fragment of both logics, $\mathbf{N4}^\perp$ and $\mathbf{N3}^\perp$.

Kripke style semantics for Nelson's Logics is defined as follows. We say that a pair $\mathcal{W} = \langle W, \leq \rangle$ is a *frame* if \leq is a preorder on W , i.e., a reflexive and transitive relation. An $\mathbf{N4}^\perp$ -*model* (*4-model* over frame \mathcal{W}) is a tuple $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$, where \mathcal{W} is a frame and valuations $v^+, v^- : \text{Prop} \rightarrow \langle W, \leq \rangle^+$ are such that for $x, y \in W$, $p \in \text{Prop}$, and $\epsilon \in \{+, -\}$ we have

$$(x \in v^\epsilon(p) \text{ and } x \leq y) \text{ implies } y \in v^\epsilon(p). \quad (1)$$

In other words, both $v^+(p)$ and $v^-(p)$ are *cones* w.r.t. \leq .

An $\mathbf{N4}^\perp$ -model $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is said to be an $\mathbf{N3}^\perp$ -*model* (*3-model* over \mathcal{W}) if

$$v^+(p) \cap v^-(p) = \emptyset \text{ for all } p \in \text{Prop}. \quad (2)$$

Now we define two different relations \models^+ and \models^- for verification and falsification of formulas in worlds of the model. Naturally, we use v^+ and v^- to define verification and falsification of propositional variables:

$$\mathcal{M}, x \models^+ p \Leftrightarrow x \in v^+(p); \quad \mathcal{M}, x \models^- p \Leftrightarrow x \in v^-(p)$$

Verification and falsification of complex formulas are defined as follows:

$$\begin{array}{ll}
\mathcal{M}, x \models^+ \alpha \wedge \beta & \text{iff } \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^+ \beta \\
\mathcal{M}, x \models^- \alpha \wedge \beta & \text{iff } \mathcal{M}, x \models^- \alpha \text{ or } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \models^+ \alpha \vee \beta & \text{iff } \mathcal{M}, x \models^+ \alpha \text{ or } \mathcal{M}, x \models^+ \beta \\
\mathcal{M}, x \models^- \alpha \vee \beta & \text{iff } \mathcal{M}, x \models^- \alpha \text{ and } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \models^+ \alpha \rightarrow \beta & \text{iff } \forall y \geq x (\mathcal{M}, y \not\models^+ \alpha \text{ or } \mathcal{M}, y \models^+ \beta) \\
\mathcal{M}, x \models^- \alpha \rightarrow \beta & \text{iff } \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^- \beta \\
\mathcal{M}, x \not\models^+ \perp & \text{and } \mathcal{M}, x \models^- \perp \\
\mathcal{M}, x \models^+ \sim \alpha & \text{iff } \mathcal{M}, x \models^- \alpha \\
\mathcal{M}, x \models^- \sim \alpha & \text{iff } \mathcal{M}, x \models^+ \alpha
\end{array}$$

The persistence condition (1) can be generalized to arbitrary formulas, i.e., for every $\varphi \in \text{Form}_{\mathcal{L}^\sim}$ and $\epsilon \in \{+, -\}$ we have

$$(\mathcal{M}, x \models^\epsilon \varphi \text{ and } x \leq y) \text{ implies } \mathcal{M}, y \models^\epsilon \varphi. \quad (3)$$

If \mathcal{M} is an $\mathbf{N3}^\perp$ -model, the consistency condition (2) also can be generalized to arbitrary formulas, i.e., for any φ and x we have

$$\mathcal{M}, x \not\models^+ \varphi \text{ or } \mathcal{M}, x \not\models^- \varphi. \quad (4)$$

We say that φ is *true* on \mathcal{M} and write $\mathcal{M} \models \varphi$ if $\mathcal{M}, x \models^+ \varphi$ for all $x \in W$. We write $\mathcal{W} \models_4 \varphi$ if φ is true on every 4-model over \mathcal{W} , and $\mathcal{W} \models_3 \varphi$ if $\mathcal{M} \models \varphi$ for every 3-model \mathcal{M} over \mathcal{W} . For $\Gamma \subseteq \text{Form}_{\mathcal{L}^\sim}$ and a world x of \mathcal{M} ,

we write $\mathcal{M} \models \Gamma$ ($\mathcal{M}, x \models^\epsilon \Gamma$, $\epsilon \in \{+, -\}$) if $\mathcal{M} \models \varphi$ ($\mathcal{M}, x \models^\epsilon \varphi$) for all $\varphi \in \Gamma$. If $\mathcal{M} \models \Gamma$, we say that \mathcal{M} is a *model* of Γ . In a similar way, we write $\mathcal{W} \models_\epsilon \Gamma$ and say that \mathcal{W} is an ϵ -*model* of Γ , where $\epsilon \in \{3, 4\}$, if $\mathcal{W} \models_\epsilon \varphi$ for all $\varphi \in \Gamma$. Finally, we write $\Gamma \models_{\mathcal{M}} \varphi$ if for every world x of \mathcal{M} we have $\mathcal{M}, x \models \varphi$, whenever $\mathcal{M}, x \models \Gamma$.

For $L \in \mathcal{E}\mathbf{N4}^\perp$ and a class of frames \mathcal{K} we say that

- L is *weakly 3-complete* (*4-complete*) w.r.t. \mathcal{K} if for every $\varphi \in \text{Form}_{\mathcal{L}^\sim}$
 $\varphi \in L$ iff $\mathcal{M} \models \varphi$ for every 3-model (4-model) \mathcal{M} over $\mathcal{W} \in \mathcal{K}$.
- L is *strongly 3-complete* (*4-complete*) w.r.t. \mathcal{K} if for every $\Gamma \subseteq \text{Form}_{\mathcal{L}^\sim}$
and $\varphi \in \text{Form}_{\mathcal{L}^\sim}$ we have

$\Gamma \vdash_L \varphi$ iff $\Gamma \models_{\mathcal{M}} \varphi$ for every 3-model (4-model) \mathcal{M} over $\mathcal{W} \in \mathcal{K}$.

Obviously, the strong 4- or 3-completeness implies the weak 4- or 3-completeness w.r.t. the same class of frames.

The following characterization of $\mathbf{N4}^\perp$ [11] and $\mathbf{N3}^\perp$ [22] is well known:

- $\mathbf{N4}^\perp$ is strongly 4-complete w.r.t. the class of all frames;
- $\mathbf{N3}^\perp$ is strongly 3-complete w.r.t. the class of all frames.

Recall that $HT = \mathbf{Int} + \{p \vee (p \rightarrow q) \vee \neg q\}$, where $\neg q$ abbreviates $q \rightarrow \perp$, is the greatest extension of \mathbf{Int} different from the classical logic \mathbf{CL} . HT is known as Gödel-Smetanich logic or "hear-and-there" logic. We are interested in $\mathbf{N4}^\perp$ - and $\mathbf{N3}^\perp$ -extensions via the same axiom:

$$\mathbf{N9} = \mathbf{N4}^\perp + \{p \vee (p \rightarrow q) \vee \neg q\} \quad \text{and} \quad \mathbf{N5} = \mathbf{N3}^\perp + \{p \vee (p \rightarrow q) \vee \neg q\}.$$

Both logics $\mathbf{N9}$ and $\mathbf{N5}$ are determined by the same two-element partially ordered frame $\mathcal{W}^{HT} = \langle W^{HT}, \leq \rangle$, where $W^{HT} = \{h, t\}$ and $h \leq t$. More exactly, we have:

- $\mathbf{N9}$ is strongly 4-complete w.r.t. the class $\{\mathcal{W}^{HT}\}$;
- $\mathbf{N5}$ is strongly 3-complete w.r.t. the class $\{\mathcal{W}^{HT}\}$.

The choice of notation $\mathbf{N9}$ and $\mathbf{N5}$ is conditioned by the facts that $\mathbf{N9}$ can be determined by a 9-element algebra [16], and $\mathbf{N5}$ by a 5-element algebra [19].

Since we have only two worlds, an $\mathbf{N9}$ -model \mathcal{M} (over \mathcal{W}^{HT}) is completely determined by sets of literals verified in the worlds h and t , so it can be identified with a pair $\langle \mathbf{H}, \mathbf{T} \rangle$, where $\mathbf{H}, \mathbf{T} \subseteq \text{Lit}^\sim$ and

$$\mathbf{H}^+ = \{p \mid \mathcal{M}, h \models^+ p\}, \quad \mathbf{H}^- = \{p \mid \mathcal{M}, h \models^- p\},$$

$$\mathbf{T}^+ = \{p \mid \mathcal{M}, t \models^+ p\}, \quad \mathbf{T}^- = \{p \mid \mathcal{M}, t \models^- p\}.$$

In view of (1) we have $\mathbf{H} \subseteq \mathbf{T}$. If \mathcal{M} is an $\mathbf{N5}$ -model, the pair $\langle \mathbf{H}, \mathbf{T} \rangle$ satisfies additionally the condition that \mathbf{H} and \mathbf{T} are consistent.

Further, we put

$$\mathbf{B4} = \mathbf{N4}^\perp + \{p \vee \neg p\} \quad \text{and} \quad \mathbf{B3} = \mathbf{N3}^\perp + \{p \vee \neg p\}.$$

These logics can be considered as expansions of four- and three-valued Belnap-Dunn logics (see [15] and [12] for details) via connectives \rightarrow and \perp . They are characterized by a one-element frame $\mathcal{W}^T = \langle \{t\}, \leq \rangle$:

- **B4** is strongly 4-complete w.r.t. the class $\{\mathcal{W}^T\}$;
- **B3** is strongly 3-complete w.r.t. the class $\{\mathcal{W}^T\}$.

Naturally, every **B4**-model \mathcal{M} (over \mathcal{W}^T) can be identified with the set \mathbf{T} of literals verified at t , i.e.

$$\mathbf{T}^+ = \{p \mid \mathcal{M}, h \models^+ p\}, \quad \mathbf{T}^- = \{p \mid \mathcal{M}, h \models^- p\},$$

If \mathcal{M} is a **B3**-model, then \mathbf{T} must be consistent.

1.2. Belnapian modal logics. The Belnapian versions **BK** and **BS4** of normal modal logics **K** and, respectively, **S4** were defined in [17]. We define **BK** in the language $\mathcal{L}^\square = \mathcal{L}^\sim \cup \{\square\}$ as it was done in [18]. The possibility operator is defined as $\diamond\varphi := \sim\square\sim\varphi$. We also need the following abbreviations: $\neg\varphi := \varphi \rightarrow \perp$, $\varphi \Leftrightarrow \psi := (\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$. The list of axioms of **BK** includes the following groups of axioms:

- I. The axioms of classical logic in the language $\{\wedge, \vee, \rightarrow, \perp\}$.
- II. The strong negation axioms of **N4**[⊥] plus

$$\neg\sim\square p \leftrightarrow \square\neg\sim p$$

- III. The modal axiom of **K**: $\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)$

The list of inference rules includes (SUB), (MP) and the normalization rule (NR_□):

$$\frac{\varphi}{\square\varphi}$$

The following formulas are **BK**-theorems:

$$\neg\square p \leftrightarrow \diamond\neg p, \quad \neg\diamond p \leftrightarrow \square\neg p, \quad \diamond(p \wedge q) \rightarrow (\diamond p \wedge \diamond q) \quad (5)$$

Logic **BS4** is an extension of **BK** obtained via adding the modal axioms of **S4**, i.e.,

$$\mathbf{BS4} = \mathbf{BK} + \{\square p \rightarrow p, \square p \rightarrow \square\square p\}.$$

The explosive extensions of **BK** and **BS4** are defined as follows:

$$\mathbf{B3K} = \mathbf{BK} + \{(p \wedge \sim p) \rightarrow q\}, \quad \mathbf{B3S4} = \mathbf{BS4} + \{(p \wedge \sim p) \rightarrow q\}$$

To define Kripke style semantics for **BK** we use the same frames as for **K**. Namely, we say that a pair $\mathcal{W} = \langle W, R \rangle$ is an **K-frame** if R is a binary relation on W . A **BK-model** (4-model over \mathcal{W}) is a tuple $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$, where $v^+, v^- : \text{Prop} \rightarrow 2^W$.

An **S4-frame** $\mathcal{W} = \langle W, R \rangle$ is a **K-frame**, where R is preorder. A **BS4-model** is a **BK-model** over an **S4-frame**.

A **B3K-model** $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ (3-model over \mathcal{W}) is a **BK-model** satisfying the consistency condition (2). A **B3S4-model** $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is a **B3K-model** over an **S4-frame**.

The verification \models^+ and falsification \models^- relations between worlds and formulas are defined in exactly the same way as for $\mathbf{N4}^\perp$ -models in case of propositional variables, constant \perp , and connectives \vee, \wedge, \sim . For \rightarrow and \Box we have:

$$\begin{aligned} \mathcal{M}, x \models^+ \alpha \rightarrow \beta & \text{ iff } \mathcal{M}, x \not\models^+ \alpha \text{ or } \mathcal{M}, x \models^+ \beta \\ \mathcal{M}, x \models^- \alpha \rightarrow \beta & \text{ iff } \mathcal{M}, x \models^+ \alpha \text{ and } \mathcal{M}, x \models^- \beta \\ \mathcal{M}, x \models^+ \Box \alpha & \text{ iff } \forall y(xRy \text{ implies } \mathcal{M}, y \models^+ \alpha) \\ \mathcal{M}, x \models^- \Box \alpha & \text{ iff } \exists y(xRy \text{ and } \mathcal{M}, y \models^- \alpha) \end{aligned}$$

It is easy to see that for possibility operator we have then:

$$\begin{aligned} \mathcal{M}, x \models^+ \Diamond \alpha & \text{ iff } \exists y(xRy \text{ and } \mathcal{M}, y \models^+ \alpha) \\ \mathcal{M}, x \models^- \Diamond \alpha & \text{ iff } \forall y(xRy \text{ implies } \mathcal{M}, y \models^- \alpha) \end{aligned}$$

Again for a **B3K**-model \mathcal{M} the consistency condition (2) can be generalized to arbitrary formulas, i.e. $\mathcal{M}, x \not\models^+ \varphi$ or $\mathcal{M}, x \not\models^- \varphi$ for any φ and x .

The truth of a formulas in a **BK**-model is defined via the verification relation, i.e., $\mathcal{M} \models \varphi$ means that $\mathcal{M}, x \models^+ \varphi$ for all $x \in W$. For a **K**-frame \mathcal{W} we write $\mathcal{W} \models_4 \varphi$ if $\mathcal{M} \models \varphi$ for every 4-model \mathcal{M} over \mathcal{W} , and $\mathcal{W} \models_3 \varphi$ if $\mathcal{M} \models \varphi$ for every 3-model \mathcal{M} over \mathcal{W} . For $\Gamma \cup \{\varphi\} \subseteq \text{Form}_{\mathcal{L}\Box}$, the relations $\mathcal{M} \models \Gamma, \Gamma \models_{\mathcal{M}} \varphi, \mathcal{W} \models_4 \Gamma$, and $\mathcal{W} \models_3 \Gamma$ are defined in an obvious way. For $L \in \mathcal{EBK}$ and a class of **BK**-frames \mathcal{K} , the sense of expressions ‘ L is weakly 4-complete (3-complete) w.r.t. the class \mathcal{K} ’ and ‘ L is strongly 4-complete (3-complete) w.r.t. the class \mathcal{K} ’ is defined in exactly the same way as for $\mathbf{N4}^\perp$ -extensions.

If $\mathcal{W} = \langle W, R \rangle$ is an **S4**-frame, $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is a 4-model over \mathcal{W} , and $K \subseteq W$ is a cone w.r.t. R ($x \in K$ and xRy imply $y \in K$), then

$$\mathcal{W}^K := \langle K, R \cap K^2 \rangle, \quad \mathcal{M}^K = \langle \mathcal{W}^K, v_K^+, v_K^- \rangle,$$

where $v_K^+(p) = v^+(p) \cap K$ and $v_K^-(p) = v^-(p) \cap K$. For any $x \in K, \varphi \in \text{Form}_{\mathcal{L}\Box}$, and $\epsilon \in \{+, -\}$ we have

$$\mathcal{M}, x \models^\epsilon \varphi \text{ iff } \mathcal{M}^K, x \models^\epsilon \varphi \quad (6)$$

In [17] the following results were proved:

- **BK** is strongly 4-complete w.r.t. the class of all **K**-frames;
- **B3K** is strongly 3-complete w.r.t. the class of all **K**-frames;
- **BS4** is strongly 4-complete w.r.t. the class of all **S4**-frames;
- **B3S4** is strongly 3-complete w.r.t. the class of all **S4**-frames.

Moreover, it was proved in [17] that $\mathbf{N4}^\perp$ and $\mathbf{N3}^\perp$ are faithfully embedded into **BS4** and, respectively, into **B3S4** via an analog T_B of the Gödel-Tarski translation that embeds **Int** into **S4**. The translation $T_B : \text{Form}_{\mathcal{L}\sim} \rightarrow \text{Form}_{\mathcal{L}\Box}$ is defined as follows:

$$\begin{array}{ll}
T_B p = \Box p & T_B \sim p = \Box \sim p \\
T_B(\varphi \vee \psi) = T_B \varphi \vee T_B \psi & T_B \sim(\varphi \vee \psi) = T_B \sim \varphi \wedge T_B \sim \psi \\
T_B(\varphi \wedge \psi) = T_B \varphi \wedge T_B \psi & T_B \sim(\varphi \wedge \psi) = T_B \sim \varphi \vee T_B \sim \psi \\
T_B(\varphi \rightarrow \psi) = \Box(T_B \varphi \rightarrow T_B \psi) & T_B \sim(\varphi \rightarrow \psi) = T_B \varphi \wedge T_B \sim \psi \\
T_B \perp = \perp & T_B \sim \sim \varphi = T_B \varphi \\
& T_B \sim \perp = \sim \perp
\end{array}$$

A logic $M \in \mathcal{EBS4}$ is said to be a *modal companion* of $L \in \mathcal{EN4}^\perp$ if T_B faithfully embeds L into M , i.e.

$$\varphi \in L \text{ iff } T_B \varphi \in M$$

for all $\varphi \in \text{Form}_{\mathcal{L}^\sim}$. According to this definition **BS4** is a modal companion of $\mathbf{N4}^\perp$, and **B3S4** is a modal companion of $\mathbf{N3}^\perp$.

Let $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ be a **BS4**-model. Define the new valuations $v'^+(p) = \{w \in W \mid \mathcal{M}, w \models^+ \Box p\}$ and $v'^-(p) = \{w \in W \mid \mathcal{M}, w \models^+ \Box \sim p\}$. It is obvious that $\mathcal{M}' = \langle W, R, v'^+, v'^- \rangle$ is an $\mathbf{N4}^\perp$ -model too.

Lemma 2. [17] *Let $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ be a **BS4**-model, $x \in W$, and $\varphi \in \text{Form}_{\mathcal{L}^\sim}$. Then*

$$\mathcal{M}', x \models^+ \varphi \Leftrightarrow \mathcal{M}, x \models^+ T_B \varphi.$$

This simple fact allows to prove (see [17]) that **BS4** is a modal companion of $\mathbf{N4}^\perp$, and **B3S4** is a modal companion of $\mathbf{N3}^\perp$.

1.3. Logic programming preliminaries.

By a *logic program* Π we mean a set of rules of the form

$$(r) \quad \alpha_1 \vee \dots \vee \alpha_k \leftarrow \beta_1 \wedge \dots \wedge \beta_n \wedge \neg \beta_{n+1} \wedge \dots \wedge \neg \beta_{n+m},$$

where $\alpha_i, \beta_j \in \text{Lit}^\sim$. We say that logic program Π is *normal* if $k = 1$ for all rules in Π , and that Π is *positive* (w.r.t. \neg) if $m = 0$ for all rules in Π .

Thus, the programs under consideration may contain two kinds of negation: the default negation, or negation-as-failure, denoted as \neg (usually written as ‘not’) and the strong or explicit negation [8] that may occur in α_i and β_j . In what follows we will identify a rule of the form (r) with a formula

$$(\beta_1 \wedge \dots \wedge \beta_n \wedge \neg \beta_{n+1} \wedge \dots \wedge \neg \beta_{n+m}) \rightarrow (\alpha_1 \vee \dots \vee \alpha_k) \in \text{Form}_{\mathcal{L}^\sim},$$

where $\neg \beta_j$ is understood as $\beta_j \rightarrow \perp$. A set $\mathbf{H} \subseteq \text{Lit}^\sim$ is a model of a logic program Π if \mathbf{H} is a **B4**-model of the set of all formulas corresponding to the rules of Π .

Now we recall the definition of stable models. Notice that originally Gelfond and Lifschitz [7] defined stable models for positive normal programs.

Let Π be a logic program and $\mathbf{T} \subseteq \text{Lit}^\sim$. The *Gelfond-Lifschitz reduct* (*GL-reduct*) of Π w.r.t. \mathbf{T} is a positive program obtained from Π in two

steps. First, we exclude from Π all rules containing $\neg\beta_i$ for $\beta_i \in \mathbf{T}$. Second, we delete all conjunctive terms of the form $\neg\beta_i$ from the rest of rules.

We say that $\mathbf{T} \subseteq \text{Lit}^\sim$ is a *stable model* of Π if $\mathbf{T} \models \Pi^{\mathbf{T}}$, and $\mathbf{H} \models \Pi^{\mathbf{T}}$ for $\mathbf{H} \subseteq \mathbf{T}$ implies $\mathbf{H} = \mathbf{T}$. In other words, \mathbf{T} is a minimal w.r.t. inclusion **B4**-model of $\Pi^{\mathbf{T}}$.

Now we define the relation \leq among \mathbf{N}_9 -models as follows. Let $\langle \mathbf{H}_1, \mathbf{T}_1 \rangle$ and $\langle \mathbf{H}_2, \mathbf{T}_2 \rangle$ be \mathbf{N}_9 -models. We set

$$\langle \mathbf{H}_1, \mathbf{T}_1 \rangle \leq \langle \mathbf{H}_2, \mathbf{T}_2 \rangle \text{ iff } \mathbf{T}_1 = \mathbf{T}_2 \text{ and } \mathbf{H}_1 \subseteq \mathbf{H}_2.$$

An \mathbf{N}_9 -model of the form $\langle \mathbf{T}, \mathbf{T} \rangle$ is called *total*.

For an arbitrary subset $\Gamma \subseteq \text{Form}_{\mathcal{L}^\sim}$, a total model $\langle \mathbf{T}, \mathbf{T} \rangle$ is said to be an *equilibrium model* of Γ if $\langle \mathbf{T}, \mathbf{T} \rangle \models \Gamma$ and there is no $\mathbf{H} \subseteq \text{Lit}^\sim$ such that $\mathbf{H} \neq \mathbf{T}$ and $\langle \mathbf{H}, \mathbf{T} \rangle \models \Gamma$. In other words an equilibrium model of Γ is a total model of Γ , which is \leq -minimal in the class of \mathbf{N}_9 -models of Γ .

For logic programs, there is a close connection between stable and equilibrium models.

Theorem 1. [16] *For a logic program Π , a set $\mathbf{T} \subseteq \text{Lit}^\sim$ is a stable model of Π iff $\langle \mathbf{T}, \mathbf{T} \rangle$ is an equilibrium model of Π .*

Originally [16] this statement was proved via the reduction to the results of [1], a short direct proof can be found in [12].

In what follows, $\mathcal{E}l_9(\Gamma)$ denotes the set of all equilibrium models of Γ , and $\mathcal{E}l_5(\Gamma)$ denotes the set of all consistent equilibrium models of Γ , i.e. the set of those equilibrium models of Γ that are \mathbf{N}_5 -models. We define equilibrium consequence relations as follows:

$$\Gamma \sim_{el}^9 \varphi \text{ iff } \langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \text{ for every } \langle \mathbf{H}, \mathbf{T} \rangle \in \mathcal{E}l_9(\Gamma).$$

$$\Gamma \sim_{el}^5 \varphi \text{ iff } \langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \text{ for every } \langle \mathbf{H}, \mathbf{T} \rangle \in \mathcal{E}l_5(\Gamma).$$

2 Special temporal logic

Similar to [5] we define a special temporal logic, where the future (\square, \diamond) and past ($\blacksquare, \blacklozenge$) modalities are defined via accessibility relations that are not mutually inverse, but however are closely connected. Prior to do it we recall the definition of a fusion of modal logics, and of the temporary version of **BS4** defined in [17].

Let $\mathcal{L}^\blacksquare := \mathcal{L}^\sim \cup \{\blacksquare\}$, $\mathcal{L}^\blacklozenge := \mathcal{L}^\square \cup \{\blacklozenge\}$, and $\blacklozenge\varphi := \sim\blacksquare\sim\varphi$.

For $L \in \mathcal{E}BK$, we denote by L_\blacksquare the logic in the language \mathcal{L}^\blacksquare defined via the same axioms and rules as L but with \square replaced by \blacksquare . Clearly, $L_\blacksquare \in \mathcal{E}BK_\blacksquare$.

For $L^1, L^2 \in \mathcal{E}BK$, we put $L^1 * L^2 := L^1 \cup L^2_\blacksquare$. We say that $L^1 * L^2$ is a *fusion* of logics L^1 and L^2 .

The temporal version **BS4_t** of **BS4** was defined in [17] as

$$\mathbf{BS4}_t = \mathbf{BS4} * \mathbf{BS4} + \{p \rightarrow \square\blacklozenge p, p \rightarrow \blacksquare\blacklozenge p\}.$$

Frames and models are defined for $\mathbf{BS4}_t$ in the same way as for $\mathbf{BS4}$. For connectives of \mathcal{L}^\square , the verification and falsification also are defined as for $\mathbf{S4}$ -frames. For \blacksquare we have:

$$\begin{aligned}\mathcal{M}, x \models^+ \blacksquare\alpha &\text{ iff } \forall y(yRx \text{ implies } \mathcal{M}, y \models^+ \alpha) \\ \mathcal{M}, x \models^- \blacksquare\alpha &\text{ iff } \exists y(yRx \text{ and } \mathcal{M}, y \models^- \alpha)\end{aligned}$$

As a consequence for \blacklozenge we have:

$$\begin{aligned}\mathcal{M}, x \models^+ \blacklozenge\alpha &\text{ iff } \exists y(yRx \text{ and } \mathcal{M}, y \models^+ \alpha) \\ \mathcal{M}, x \models^- \blacklozenge\alpha &\text{ iff } \forall y(yRx \text{ implies } \mathcal{M}, y \models^- \alpha)\end{aligned}$$

All related notions are modified for the language \mathcal{L}^t in an obvious way. Naturally, $\mathbf{BS4}_t$ is strongly 4-complete w.r.t. the class of all $\mathbf{S4}$ -frames, and $\mathbf{B3S4}_t = \mathbf{B3S4}_t + \{(p \wedge \sim p) \rightarrow q\}$ is strongly 3-complete w.r.t. the class of all $\mathbf{S4}$ -frames.

Further, let us consider the fusion

$$\mathbf{BSK} = \mathbf{BS4} * \mathbf{BK}$$

and its explosive extension $\mathbf{B3SK} = \mathbf{BSK} + \{(p \wedge \sim p) \rightarrow q\}$.

A \mathbf{BSK} -frame is a tuple $\mathcal{W} = \langle W, R, S \rangle$, where R is a preorder on W and $S \subseteq W^2$. A \mathbf{BSK} -model $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ is a \mathbf{BSK} -frame \mathcal{W} augmented with two valuations $v^+, v^- : \text{Prop} \rightarrow 2^W$. One can combine in an obvious way the completeness results for $\mathbf{BS4}$, \mathbf{BK} and their explosive extensions from [17] to obtain the following

Theorem 2. *Logic \mathbf{BSK} ($\mathbf{B3SK}$) is strongly 4-complete (3-complete) w.r.t. the class of \mathbf{BSK} -frames.*

We denote by id_W the diagonal of W , i.e., $\text{id}_W = \{(a, a) \mid a \in W\}$.

Proposition 1. *Let $\mathcal{W} = \langle W, R, S \rangle$ be a \mathbf{BSK} -frame and $\epsilon \in \{3, 4\}$. The following equivalences hold:*

- 1) $\mathcal{W} \models_\epsilon p \rightarrow \blacksquare\blacklozenge p$ iff $S \subseteq R^{-1}$;
- 2) $\mathcal{W} \models_\epsilon p \rightarrow \square(p \vee \blacklozenge p)$ iff $R \subseteq S^{-1} \cup \text{id}_W$.

Proof. We fix some \mathbf{BSK} -frame $\mathcal{W} = \langle W, R, S \rangle$.

1) This is one of standard axioms of temporal logic, and since \sim does not occur in this formula the three- or four-valued case should not differ from the ordinary one. However we provide this proof to be self contained. Let $S \subseteq R^{-1}$, and let $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ be a model over \mathcal{W} . Assume that $\mathcal{M}, x \models^+ p$ and choose some y with xSy . We have then yRx by $S \subseteq R^{-1}$, and so $\mathcal{M}, y \models^+ \blacklozenge p$. Since y is an arbitrary S -successor of x , we have $\mathcal{M}, x \models^+ \blacksquare\blacklozenge p$. We have thus proved that $S \subseteq R^{-1}$ implies $\mathcal{W} \models_4 p \rightarrow \blacksquare\blacklozenge p$ and, in particular, $\mathcal{W} \models_3 p \rightarrow \blacksquare\blacklozenge p$.

To prove the inverse implication we assume that $S \not\subseteq R^{-1}$ and $x, y \in W$ are such that xSy but $\neg(yRx)$. Let $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ be such that $v^+(p) = \{x\}$ and $v^-(q) = \emptyset$ for all $q \in \text{Prop}$. Obviously, \mathcal{M} is a $\mathbf{B3SK}$ -model. In this case we have $\mathcal{M}, x \models^+ p$ and $\mathcal{M}, y \not\models^+ \blacklozenge p$, whence $\mathcal{M}, x \not\models^+ \blacksquare\blacklozenge p$. Thus, $S \not\subseteq R^{-1}$ implies $\mathcal{W} \not\models_3 p \rightarrow \blacksquare\blacklozenge p$, moreover, $\mathcal{W} \not\models_4 p \rightarrow \blacksquare\blacklozenge p$.

2) First we assume that $R \subseteq S^{-1}\text{Uid}_W$. Let $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ be a model over \mathcal{W} and $x, y \in W$ be such that $\mathcal{M}, x \models^+ p$ and xRy . If $x = y$, then $\mathcal{M}, y \models^+ p \vee \blacklozenge p$. If $x \neq y$, then ySx , and we again have $\mathcal{M}, y \models^+ p \vee \blacklozenge p$. Consequently, $\mathcal{M}, x \models^+ \Box(p \vee \blacklozenge p)$. We proved thus $\mathcal{W} \models_4 p \rightarrow \Box(p \vee \blacklozenge p)$, moreover, $\mathcal{W} \models_3 p \rightarrow \Box(p \vee \blacklozenge p)$.

Now we assume that $R \not\subseteq S^{-1}\text{Uid}_W$ and choose $x, y \in W$ such that xRy , $x \neq y$, and $\neg(ySx)$. As in Item 1 we take a **B3SK**-model $\mathcal{M} = \langle \mathcal{W}, v^+, v^- \rangle$ such that $v^+(p) = \{x\}$ and $v^-(q) = \emptyset$ for all $q \in \text{Prop}$. We have $\mathcal{M}, x \models^+ p$. At the same time the conditions $x \neq y$ and $\neg(ySx)$ imply $\mathcal{M}, y \not\models^+ p \vee \blacklozenge p$, whence $\mathcal{M}, x \not\models^+ \Box(p \vee \blacklozenge p)$. Thus, $\mathcal{W} \not\models_3 p \rightarrow \Box(p \vee \blacklozenge p)$ and $\mathcal{W} \not\models_4 p \rightarrow \Box(p \vee \blacklozenge p)$. □

We define a weak version of **BS4_t** as follows:

$$\mathbf{BSK}_t := \mathbf{BSK} + \{p \rightarrow \blacksquare \lozenge p, p \rightarrow \Box(p \vee \blacklozenge p)\}.$$

We put also $\mathbf{B3SK}_t := \mathbf{BSK}_t + \{(p \wedge \sim p) \rightarrow q\}$.

To prove the completeness of **BSK_t** and of its axiomatic extensions via different classes of frames we will use the canonical model method.

First we notice that every prime L -theory Γ over $L \in \mathcal{EBSK}_t$ is complete and consistent w.r.t. \neg . Indeed, $\varphi \vee \neg\varphi \in \Gamma$ since **BSK_t** contains axioms of classical logic in the language $\{\vee, \wedge, \rightarrow, \perp\}$. Consequently, the disjunction property of Γ implies $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. In particular, any two prime L -theories are incomparable w.r.t. set-theoretical inclusion \subseteq , i.e. for prime L -theories Γ and Δ we have:

$$\Gamma \neq \Delta \text{ implies } \Gamma \setminus \Delta \neq \emptyset. \quad (7)$$

If $\varphi, \neg\varphi \in \Gamma$, then $(\varphi \wedge \neg\varphi) \rightarrow \psi \in \Gamma$ implies that $\Gamma = \text{Form}_{\mathcal{L}^t}$.

For $L \in \mathcal{EBSK}_t$, the *canonical L -frame* is defined as $\mathcal{W}^L = \langle W^L, R^L, S^L \rangle$, where

- W^L is the set of all prime L -theories;
- $\Gamma R^L \Delta$ iff $\Gamma_{\Box} \subseteq \Delta$, where $\Gamma_{\Box} = \{\varphi \mid \Box\varphi \in \Gamma\}$;
- $\Gamma S^L \Delta$ iff $\Gamma_{\blacksquare} \subseteq \Delta$, where $\Gamma_{\blacksquare} = \{\varphi \mid \blacksquare\varphi \in \Gamma\}$;

The *canonical L -model* has the form $\mathcal{M}^L = \langle \mathcal{W}, v_L^+, v_L^- \rangle$, where

$$v_L^+(p) = \{\Gamma \in W^L \mid p \in \Gamma\} \text{ and } v_L^-(p) = \{\Gamma \in W^L \mid \sim p \in \Gamma\}.$$

The abbreviations $\lozenge\varphi := \sim\Box\sim\varphi$ and $\blacklozenge\varphi := \sim\lozenge\sim\varphi$ easily imply that

$$\Gamma R^L \Delta \text{ iff } \Delta^{\lozenge} \subseteq \Gamma; \quad \Gamma S^L \Delta \text{ iff } \Delta^{\blacklozenge} \subseteq \Gamma,$$

where $\Delta^{\lozenge} = \{\lozenge\varphi \mid \varphi \in \Delta\}$ and $\Delta^{\blacklozenge} = \{\blacklozenge\varphi \mid \varphi \in \Delta\}$. Further, by induction on the structure of formulas one can easily prove the canonical model lemma:

Lemma 3. *Let $L \in \mathcal{EBSK}_t$. For every prime L -theory Γ and formula φ , the following equivalences hold:*

$$\mathcal{M}^L, \Gamma \models^+ \varphi \text{ iff } \varphi \in \Gamma; \quad \mathcal{M}^L, \Gamma \models^- \varphi \text{ iff } \sim\varphi \in \Gamma.$$

Theorem 3. *Logic \mathbf{BSK}_t ($\mathbf{B3SK}_t$) is strongly 4-complete (3-complete) w.r.t. the class of \mathbf{BSK} -frames $\langle W, R, S \rangle$ such that*

$$R = S^{-1} \cup \text{id}_W. \quad (8)$$

Proof. We omit the routine correctness proof and check the completeness. Let $\Gamma \not\vdash_{\mathbf{BSK}_t} \varphi$. By Lemma 1 there is a prime \mathbf{BSK}_t -theory Δ with $\Gamma \subseteq \Delta$ and $\varphi \notin \Delta$. According to Lemma 3, we have

$$\mathcal{M}^{\mathbf{BSK}_t}, \Delta \models^+ \psi \text{ for all } \psi \in \Gamma \text{ and } \mathcal{M}^{\mathbf{BK}}, \Delta \not\models^+ \varphi.$$

It remains to check that the canonical \mathbf{BSK}_t -frame $\mathcal{W}^{\mathbf{BSK}_t}$ is a \mathbf{BSK} -frame, i.e., that the relation $R^{\mathbf{BSK}_t}$ is reflexive and transitive, and that

$$R^{\mathbf{BSK}_t} = (S^{\mathbf{BSK}_t})^{-1} \cup \text{id}_{\mathcal{W}^{\mathbf{BSK}_t}}. \quad (9)$$

As well as in case of normal modal logics (see [3, Theorem 5.16]) we can check that the axiom $\Box p \rightarrow p$ implies that $R^{\mathbf{BSK}_t}$ is reflexive, and that $\Box p \rightarrow \Box \Box p$ implies that $R^{\mathbf{BSK}_t}$ is transitive.

Let us check (9). For brevity we will omit the upper index $(\cdot)^{\mathbf{BSK}_t}$. Assume that $R \not\subseteq S^{-1} \cup \text{id}_W$. In this case there are $\Gamma, \Delta \in W$ such that $\Gamma R \Delta$, $\Gamma \neq \Delta$, and $\neg(\Delta S \Gamma)$. The latter is equivalent to $\Delta \blacksquare \not\subseteq \Gamma$. Let $\blacksquare \varphi \in \Delta$ and $\varphi \notin \Gamma$. The completeness of Γ implies $\neg \varphi \in \Gamma$. By (7) and $\Gamma \neq \Delta$ there is $\psi \in \Gamma \setminus \Delta$, so $\neg \varphi \wedge \psi \in \Gamma$. The axiom $p \rightarrow \Box(p \blacklozenge p)$ implies

$$(\neg \varphi \wedge \psi) \rightarrow \Box((\neg \varphi \wedge \psi) \vee \blacklozenge(\neg \varphi \wedge \psi)) \in \Gamma.$$

By (MP) and $\Gamma R \Delta$ we obtain $(\neg \varphi \wedge \psi) \vee \blacklozenge(\neg \varphi \wedge \psi) \in \Delta$. Since $\psi \notin \Delta$, we have $\neg \varphi \wedge \psi \notin \Delta$, so $\blacklozenge(\neg \varphi \wedge \psi) \in \Delta$. From $\blacklozenge(\neg \varphi \wedge \psi) \rightarrow (\blacklozenge \neg \varphi \wedge \blacklozenge \psi) \in \mathbf{BSK}_t$, we conclude $\blacklozenge \neg \varphi \in \Delta$. By (5) $\neg \blacksquare \varphi \in \Delta$, which contradicts to the \neg -consistency of Δ . We have thus proved $R \subseteq S^{-1} \cup \text{id}_W$.

Now we prove that $S \subseteq R^{-1}$. Let $\Gamma, \Delta \in W$ be such that $\Gamma S \Delta$, i.e., $\Delta \blacklozenge \subseteq \Gamma$. If $\Box \varphi \in \Delta$, then $\blacklozenge \Box \varphi \in \Gamma$. By axiom $p \rightarrow \blacksquare \blacklozenge p$ we have $\neg \varphi \rightarrow \blacksquare \blacklozenge \neg \varphi \in \Gamma$. By (5) $\blacksquare \blacklozenge \neg \varphi \leftrightarrow \neg \blacklozenge \Box \varphi \in \Gamma$. Consequently, $\neg \varphi \rightarrow \neg \blacklozenge \Box \varphi \in \Gamma$, whence $\blacklozenge \Box \varphi \rightarrow \varphi \in \Gamma$. Finally, $\varphi \in \Gamma$. We have thus proved that $\Gamma S \Delta$ implies $\Delta R \Gamma$, which completes the proof of 4-completeness for \mathbf{BSK}_t .

In case of $\mathbf{B3SK}_t$ we only have to check that the canonical $\mathbf{B3SK}_t$ -model is a 3-model, i.e., that $\{p, \sim p\} \subseteq \Gamma$ does not hold for any $p \in \text{Prop}$ and $\Gamma \in W^{\mathbf{B3SK}_t}$. This fact readily follows from the $\mathbf{B3SK}_t$ -axiom $(p \wedge \sim p) \rightarrow q$. \square

Now it is natural to say that $\mathcal{W} = \langle W, R, S \rangle$ is a \mathbf{BSK}_t -frame, if \mathcal{W} is a \mathbf{BSK} -frame and $R = S^{-1} \cup \text{id}_W$.

We consider some further conditions on the accessibility relations of a \mathbf{BSK}_t -frame $\langle W, R, S \rangle$:

- (1^R) $\forall x, y, z \in W((xRy \ \& \ xRz \ \& \ x \neq y \ \& \ x \neq z) \Rightarrow y = z)$;
- (2^S) $\forall x, y(xSy \Rightarrow ySy)$;
- (3^S) $\forall x, y, z \in W((xSy \ \& \ ySz \Rightarrow y = z)$.

We will need also \mathcal{L}^t -formulas:

$$\mathbf{alt}_2 : \Box p \vee \Box(p \rightarrow q) \vee \Box((p \wedge q) \rightarrow r), \quad \blacksquare \cdot \mathbf{T}_\blacksquare : \blacksquare(\blacksquare p \rightarrow p).$$

Proposition 2. *Let $\mathcal{W} = \langle W, R, S \rangle$ be a \mathbf{BSK}_t -frame and $\epsilon \in \{3, 4\}$. Then the following equivalences hold:*

- (1) $\mathcal{W} \models_\epsilon \mathbf{alt}_2$ iff R satisfies (1^R) ;
- (2) $\mathcal{W} \models_\epsilon \blacksquare \cdot \mathbf{T}_\blacksquare$ iff S satisfies (2^S) ;
- (3) $\mathcal{W} \models_\epsilon \blacksquare(p \rightarrow \blacksquare p)$ iff S satisfies (3^S) .

Proof. (1) It is clear that the validity of a formula which does not contain \sim and \blacksquare on a frame $\langle W, R, S \rangle$ is equivalent to the validity of this formulas on a frame $\langle W, R \rangle$ for normal modal logics. It is also known (see, e.g. [3, Prop. 3.45]) that the validity of \mathbf{alt}_2 is equivalent to the condition that every world has at most 2 different R -successors. In view of reflexivity of R in \mathbf{BSK}_t -frames we obtain that the validity of \mathbf{alt}_2 in a \mathbf{BSK}_t -frame $\langle W, R, S \rangle$ is equivalent to the condition that every world has at most one proper R -successor, i.e., to (1^R) .

(2) Again, it is known [3, Prop. 3.30] that the validity of $\blacksquare p \rightarrow p$ on $\langle W, R, S \rangle$ is equivalent to the reflexivity of S . The additional \blacksquare in front of this formula restricts this condition to worlds that are S -successors. So the validity of $\blacksquare \cdot \mathbf{T}_\blacksquare$ is equivalent to (2^S) .

(3) Assume that $\mathcal{W} = \langle W, R, S \rangle$ satisfies (3^S) , i.e., every S -successor has no proper S -successors, and that \mathcal{M} is a 4-model over \mathcal{W} . Check that $\mathcal{M} \models \blacksquare(p \rightarrow \blacksquare p)$. Let $x, y \in W$ and xSy . If $\mathcal{M}, y \models^+ p$ and ySz , then $y = z$ by (3^S) and $\mathcal{M}, z \models^+ p$. So $\mathcal{M}, y \models^+ p \rightarrow \blacksquare p$ and $\mathcal{M}, x \models^+ \blacksquare(p \rightarrow \blacksquare p)$.

Assume that $\mathcal{W} = \langle W, R, S \rangle$ is such that xSy, ySz , and $y \neq z$. Consider a 3-model $c\mathcal{M}$ over \mathcal{W} such that $v^+(p) = \{y\}$ and $v^-(y) = \emptyset$. Then $\mathcal{M}, y \models^+ p$, $\mathcal{M}, y \not\models^+ \blacksquare p$, and so $\mathcal{M}, x \not\models^+ \blacksquare(p \rightarrow \blacksquare p)$. \square

Now we put

$$\mathbf{BSK}_{t2} := \mathbf{BSK}_t + \{\mathbf{alt}_2, \blacksquare(p \leftrightarrow \blacksquare p)\}, \quad \mathbf{B3SK}_{t2} := \mathbf{BSK}_{t2} + \{(p \wedge \sim p) \rightarrow q\}.$$

A \mathbf{BSK}_t -frame $\mathcal{W} = \langle W, R, S \rangle$ is said to be a \mathbf{BSK}_{t2} -frame if it satisfies the conditions (1^R) , (2^S) , (3^S) .

Theorem 4. *Logic \mathbf{BSK}_{t2} ($\mathbf{B3SK}_{t2}$) is strongly 4-complete (3-complete) w.r.t. the class of \mathbf{BSK}_{t2} -frames.*

Proof. Since $\blacksquare(p \leftrightarrow \blacksquare p)$ is equivalent to the conjunction of $\blacksquare \cdot \mathbf{T}_\blacksquare$ and $\blacksquare(p \rightarrow \blacksquare p)$, the correctness part follows from Proposition 2. For the completeness part, it would be enough to check that the canonical \mathbf{BSK}_{t2} -frame, which we denote for brevity as $\mathcal{W}_2 = \langle W_2, R_2, S_2 \rangle$ is a \mathbf{BSK}_{t2} -frame. That \mathcal{W}_2 is a \mathbf{BSK}_t -frame can be checked as in the proof of Theorem 3. The axiom \mathbf{alt}_2 implies that every $\Gamma \in W_2$ has at most two R_2 -successors. This fact and the reflexivity of R_2 imply the condition (1^R) .

Now we check that $\Gamma S_2 \Delta$ implies $\Delta S_2 \Delta$ for all $\Gamma, \Delta \in W_2$. Let $\Gamma S_2 \Delta$, i.e., $\Gamma_\blacksquare \subseteq \Delta$. Assume that $\blacksquare \varphi \in \Delta$. Since $\blacksquare(\blacksquare \varphi \rightarrow \varphi)$ is a theorem of \mathbf{BSK}_{t2} we have $\blacksquare(\blacksquare \varphi \rightarrow \varphi) \in \Gamma$, whence $\blacksquare \varphi \rightarrow \varphi \in \Delta$. Applying (MP) we obtain $\varphi \in \Delta$. So $\Delta_\blacksquare \subseteq \Delta$, and the condition (2^S) holds for \mathcal{W}_2 .

Finally, we assume that $\Gamma, \Delta, \Sigma \in W_2$ are such that $\Gamma_{\blacksquare} \subseteq \Delta$ and $\Delta_{\blacksquare} \subseteq \Sigma$. We have to check that $\Delta = \Sigma$. Let $\varphi \in \Delta$. Since $\blacksquare(\varphi \rightarrow \blacksquare\varphi) \in \Gamma$, we have $\varphi \rightarrow \blacksquare\varphi \in \Delta$. By (MP) $\blacksquare\varphi \in \Delta$, and so $\varphi \in \Sigma$. We proved $\Delta \subseteq \Sigma$. By (7) we conclude $\Delta = \Sigma$. Thus, the condition (3^S) holds for the canonical \mathbf{BSK}_{t_2} -frame too. \square

3 \blacksquare -free fragments of \mathbf{BSK}_{t_2} and $\mathbf{B3SK}_{t_2}$

We define

$$\mathbf{BS4}_2 = \mathbf{BS4} + \{\mathbf{alt}_2\}, \quad \mathbf{B3S4}_2 = \mathbf{BS4}_2 + \{(p \wedge \sim p) \rightarrow q\}$$

Proposition 3. (1) *Logic $\mathbf{BS4}_2$ ($\mathbf{B3S4}_2$) is strongly 4-complete (3-complete) w.r.t. the class of $\mathbf{BS4}$ -frames satisfying (1^R) .*

(2) *Logic $\mathbf{BS4}_2$ ($\mathbf{B3S4}_2$) is strongly 4-complete (3-complete) w.r.t. the class $\{\mathcal{W}^{HT}\}$.*

Proof. (1) As above we use [3, Theorem 5.16] to check that the canonical frames of logics $\mathbf{BS4}_2$ and $\mathbf{B3S4}_2$ are $\mathbf{BS4}$ -frames satisfying (1^R) . According to this theorem the axiom $\Box p \rightarrow p$ implies the reflexivity of canonical frames. In a similar way, $\Box p \rightarrow \Box\Box p$ and \mathbf{alt}_2 imply the transitivity and the condition (1^R) respectively.

(2) We consider the case of $\mathbf{BS4}_2$. It is clear that \mathcal{W}^{HT} is a $\mathbf{BS4}$ -frame satisfying (1^R) , so $\Gamma \vdash_{\mathbf{BS4}_2} \varphi$ implies $\Gamma \models_{\mathcal{M}} \varphi$ for every 4-model \mathcal{M} over \mathcal{W}^{HT} .

Let $\Gamma \not\vdash_{\mathbf{BS4}_2} \varphi$. Then $\mathcal{M}, x \models^+ \Gamma$ and $\mathcal{M}, x \not\models^+ \varphi$ for some 4-model \mathcal{M} over a $\mathbf{BS4}$ -frame $\mathcal{W} = \langle W, R \rangle$ satisfying (1^R) . By (6) we have

$$\mathcal{M}^{x\uparrow}, x \models^+ \Gamma \quad \text{and} \quad \mathcal{M}^{x\uparrow}, x \not\models^+ \varphi,$$

where $x \uparrow = \{y \in W \mid xRy\}$. The condition (1^R) implies that $|x \uparrow| \leq 2$, i.e., $\mathcal{W}^{x\uparrow}$ is isomorphic to \mathcal{W}^{HT} or \mathcal{W}^T . Since \mathcal{W}^T can be identified with the upper world of \mathcal{W}^{HT} , we obtain $\mathcal{M}', x \models^+ \Gamma$ and $\mathcal{M}', x \not\models^+ \varphi$ for some 4-model \mathcal{M}' over \mathcal{W}^{HT} and $x \in \{h, t\}$. \square

Corollary 1. *Logic $\mathbf{BS4}_{t_2}$ ($\mathbf{B3S4}_{t_2}$) is a conservative extension of $\mathbf{BS4}_2$ ($\mathbf{BS4}_2$).*

Proof. This statement readily follows from the observation that $\langle W, R, S \rangle$ is a \mathbf{BSK}_{t_2} -frame iff $R = S^{-1} \cup \text{id}_W$ and $\langle W, R \rangle$ is a $\mathbf{BS4}$ -frame satisfying (1^R) . \square

Theorem 5. *The logic $\mathbf{BS4}_2$ ($\mathbf{B3S4}_2$) is a modal companion of \mathbf{N}_9 (\mathbf{N}_5), i.e., for every $\varphi \in \text{Form}_{\mathcal{L}^{\sim}}$ the following two equivalences hold:*

$$\begin{aligned} \varphi \in \mathbf{N}_9 & \quad \text{iff} \quad T_B\varphi \in \mathbf{BS4}_2, \\ \varphi \in \mathbf{N}_5 & \quad \text{iff} \quad T_B\varphi \in \mathbf{B3S4}_2. \end{aligned}$$

Proof. We consider the case of \mathbf{N}_9 . If $T_B\varphi \notin \mathbf{BS4}_2$, then by Item 2 of Proposition 3 $\mathcal{M}, x \not\models^+ T_B\varphi$ for some 4-model \mathcal{M} over \mathcal{W}^{HT} and $x \in \{h, t\}$.

By Lemma 2 we have $\mathcal{M}', x \not\models \varphi$. Moreover, \mathcal{M}' is obviously an \mathbf{N}_9 -model. Consequently, $\varphi \notin \mathbf{N}_9$.

If $\varphi \notin \mathbf{N}_9$. Then $\mathcal{M}, x \not\models^+ \varphi$ for a suitable \mathbf{N}_9 -model and $x \in \{h, t\}$. Obviously, \mathcal{M} is a $\mathbf{BS4}$ -model over \mathcal{W}^{HT} . Since $v^+(p)$ and $v^-(p)$ are cones for any $p \in \text{Prop}$, we have

$$x \in v^+(p) \Leftrightarrow \mathcal{M}, x \models^+ \Box p \text{ and } w \in v^-(p) \Leftrightarrow \mathcal{M}, w \models^+ \Box \sim p,$$

which implies $\mathcal{M}' = \mathcal{M}$. We have then $\mathcal{M}, w \not\models^+ T_B \varphi$ by Lemma 2, and so $T_B \varphi \notin \mathbf{BS4}$. □

This theorem together with Proposition 1 yields

Corollary 2. *The translation T_B faithfully embeds the logic \mathbf{N}_9 (\mathbf{N}_5) into the special temporal logic \mathbf{BSK}_{t2} ($\mathbf{B3SK}_{t2}$).*

4 Equilibrium theory over $\mathbf{BS4}_{t2}$

We define a theory \mathbf{BKE} as the least set of formulas closed under the rules of $\mathbf{BS4}_{t2}$, containing the axioms of $\mathbf{BS4}_{t2}$ as well as the following formulas:

$$\Diamond \left(\bigwedge_{\alpha \in A} \Box \alpha \wedge \bigwedge_{\beta \in B} \Box \beta \wedge \bigwedge_{\gamma \in C} \Box \neg \gamma \right) \rightarrow \Diamond \Diamond \left(\bigwedge_{\alpha \in A \cup C} \neg \alpha \wedge \bigwedge_{\beta \in B} \beta \right), \quad (10)$$

where A, B, C are disjoint finite subsets of Lit^\sim , and $A \neq \emptyset$.

It is clear that \mathbf{BKE} contains all theorems of $\mathbf{BS4}_{t2}$ and is closed under the rules (NR_\Box) and (NR_\blacksquare) , but it need not be closed under (SUB) . Substituting to the axioms (10) we obtain formulas which are not of the form (10). So \mathbf{BKE} is namely a theory over $\mathbf{BS4}_{t2}$, not an axiomatic extension.

The theory $\mathbf{B3KE}$ is defined in exactly the same way but over $\mathbf{B3S4}_{t2}$.

Now we describe models of theories \mathbf{BKE} and $\mathbf{B3KE}$.

First, for a model $\mathcal{M} = \langle W, R, v^+, v^- \rangle$ and $x \in W$ we denote by At_x the set of all literals true at x :

$$At_x = \{ \alpha \mid \alpha \in \text{Lit}^\sim \text{ and } \mathcal{M}, x \models^+ \alpha \}.$$

We say that $\mathcal{M} = \langle W, R, S, v^+, v^- \rangle$ is a \mathbf{BKE} -model if $\langle W, R, S \rangle$ is a $\mathbf{BS4}_{t2}$ -frame, i.e., R and S are related by (8), R is a preorder on W satisfying (1^R) , S satisfies (2^S) , (3^S) , and for every $x \in W$ the following two conditions are satisfied:

- (1^E) if x is an *isolated point*, i.e., there is no $y \in W$ such that $y \neq x$, and yRx or xRy , then $At_x = \emptyset$.
- (2^E) if x is a proper R -successor of some $y \in W$ and $At_x \neq \emptyset$, then for every proper subset $B \subsetneq At_x$ there is $z \in W$ such that xSz and $At_z = B$.

$\mathbf{B3KE}$ -models are \mathbf{BKE} -models satisfying the consistency condition (2).

Theorem 6. *Let $\varphi \in \text{Form}_{\mathcal{L}t}$. The following equivalences are true:*

- (1) $\varphi \in \mathbf{BKE}$ iff $\mathcal{M} \models \varphi$ for every **BKE**-model;
(2) $\varphi \in \mathbf{B3KE}$ iff $\mathcal{M} \models \varphi$ for every **B3KE**-model;

Proof. (1) First we prove that $\varphi \in \mathbf{BKE}$ implies the truth of φ on all **BKE**-models. Since **BKE**-models are based on **BS4_{t2}**-frames, Theorem 4 implies that it would be enough to check the truth of axioms (10) on **BKE**-models. Let $\mathcal{M} = \langle W, R, S, v^+, v^- \rangle$ be a **BKE**-model and $x \in W$. Assume that

$$\mathcal{M}, x \models^+ \diamond \left(\bigwedge_{\alpha \in A} \Box \alpha \wedge \bigwedge_{\beta \in B} \Box \beta \wedge \bigwedge_{\gamma \in C} \Box \neg \gamma \right),$$

where A, B, C are disjoint finite subsets of Lit^\sim , and $A \neq \emptyset$. Then there is $y \in W$ such that xRy and

$$\mathcal{M}, y \models^+ \bigwedge_{\alpha \in A} \Box \alpha \wedge \bigwedge_{\beta \in B} \Box \beta \wedge \bigwedge_{\gamma \in C} \Box \neg \gamma.$$

Reflexivity of R implies $A \cup B \subseteq At_y$. Since $\mathcal{M}, y \models^+ \neg \alpha$ is equivalent to $\mathcal{M}, y \not\models^+ \alpha$, we have $C \cap At_y = \emptyset$. If x has no proper R -successors, then $x = y$. In this case $A \neq \emptyset$ and (1^E) imply that there is z such that $x \neq z$ and zRx . By (2^E) there is an S -successor u of x such that $At_u = B$. So we have $\mathcal{M}, x \models^+ \diamond (\bigwedge_{\alpha \in A \cup C} \neg \alpha \wedge \bigwedge_{\beta \in B} \beta)$ and

$$\mathcal{M}, x \models^+ \diamond \diamond \left(\bigwedge_{\alpha \in A \cup C} \neg \alpha \wedge \bigwedge_{\beta \in B} \beta \right). \quad (11)$$

Otherwise we can assume that y is a proper R -successor of x . From (2^E) we again obtain that $\diamond (\bigwedge_{\alpha \in A \cup C} \neg \alpha \wedge \bigwedge_{\beta \in B} \beta)$ is verified at y , and (11) as a consequence.

To prove the inverse implication we define the canonical frame \mathcal{W}^E for the theory **BKE** in exactly the same way as it was done earlier for axiomatic extensions of **BSK_t**. We put $\mathcal{W}^E = \langle W^E, R^E, S^E \rangle$, where W^E is the set of all prime **BSK_{t2}**-theories containing **BKE**,

$$\Gamma R^E \Delta \text{ iff } \Gamma_{\Box} \subseteq \Delta, \quad \Gamma S^E \Delta \text{ iff } \Gamma_{\blacksquare} \subseteq \Delta.$$

If $\varphi \in \text{Form}_{\mathcal{L}^t}$, we denote the set of its propositional variables as $\text{var}(\varphi)$ and put

$$\text{Lit}_{\varphi} = \{p, \sim p \mid p \in \text{var}(\varphi)\}.$$

For $\varphi \notin \mathbf{BKE}$ we define a kind of canonical model $\mathcal{M}_{\varphi}^E = \langle \mathcal{W}^E, v_E^+, v_E^- \rangle$ putting

$$v_{\varphi}^+(p) = \{\Gamma \in W^L \mid p \in \Gamma \cap \text{var}(\varphi)\}, \quad v_{\varphi}^-(p) = \{\Gamma \in W^L \mid \sim p \in \Gamma \cap \text{Lit}_{\varphi}\}.$$

The analog of canonical model lemma (Lemma 3) can also be proved for \mathcal{M}_{φ}^E by induction on the structure of formulas. For every $\psi \in \text{Form}_{\mathcal{L}^t}$ and $\Gamma \in W^E$ we have

$$\mathcal{M}_{\varphi}^E, \Gamma \models^+ \psi \text{ iff } \psi \in \Gamma; \quad \mathcal{M}_{\varphi}^E, \Gamma \models^- \psi \text{ iff } \sim \psi \in \Gamma. \quad (12)$$

That \mathcal{W}^E is a **BSK_{t2}**-frame can be proved in exactly the same way as in Theorem 4. Let us check that \mathcal{M}_{φ}^E is a **BKE**-model.

We take some $\Gamma \in W^E$ that has no proper R^E -successors. If $\Gamma \cap \text{Lit}_\varphi \neq \emptyset$, we choose finite disjoint $A, B, C \subseteq \text{Lit}_\varphi$ such that

$$A \cup B = \Gamma \cap \text{Lit}_\varphi, \quad A \neq \emptyset, \quad C = \text{Lit}_\varphi \setminus \Gamma.$$

Then $\mathcal{M}_\varphi^E, \Gamma \models^+ \bigwedge_{\alpha \in A} \alpha \wedge \bigwedge_{\beta \in B} \beta \wedge \bigwedge_{\gamma \in C} \neg\gamma$. Since R^E is reflexive and Γ has no other R^E -successors, we have

$$\mathcal{M}_\varphi^E, \Gamma \models^+ \bigwedge_{\alpha \in A} \Box\alpha \wedge \bigwedge_{\beta \in B} \Box\beta \wedge \bigwedge_{\gamma \in C} \Box\neg\gamma.$$

The reflexivity of R^E yields $\mathcal{M}_\varphi^E, \Gamma \models^+ \Diamond(\bigwedge_{\alpha \in A} \Box\alpha \wedge \bigwedge_{\beta \in B} \Box\beta \wedge \bigwedge_{\gamma \in C} \Box\neg\gamma)$. Finally, by (12) we obtain $\Diamond(\bigwedge_{\alpha \in A} \Box\alpha \wedge \bigwedge_{\beta \in B} \Box\beta \wedge \bigwedge_{\gamma \in C} \Box\neg\gamma) \in \Gamma$. Since Γ contains all **BKE**-axioms, we have $\Diamond\Diamond(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta) \in \Gamma$, which is equivalent by (12) to

$$\mathcal{M}_\varphi^E, \Gamma \models^+ \Diamond\Diamond\left(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta\right). \quad (13)$$

The lack of proper R^E -successors implies $\mathcal{M}_\varphi^E, \Gamma \models^+ \Diamond\Diamond(\bigwedge_{\alpha \in A \cup C} \neg\alpha \wedge \bigwedge_{\beta \in B} \beta)$. This fact together with $A \cup B \cup C = \text{Lit}_\varphi$ implies in turn that Γ has an S^E -successor Δ with $At_\Delta = B$. Since $At_\Gamma \neq At_\Delta$, Δ is a proper S^E -successor or, equivalently, R^E -predecessor of Γ . We have thus proved that $At_\Gamma \neq \emptyset$ implies that Γ is not an isolated point. So (1^E) holds for \mathcal{M}_φ^E .

Let $\Gamma, \Delta \in W^E$ be such that $\Gamma R^E \Delta$, $\Gamma \neq \Delta$, $\Delta \cap \text{Lit}_\varphi \neq \emptyset$, and $B \subsetneq \Delta \cap \text{Lit}_\varphi$. In this case Δ has no proper R^E -successors, and arguing as above we obtain that Δ has an S^E -successor Σ with $At_\Sigma = B$. Thus, \mathcal{M}_φ^E satisfies (2^E) too.

We have proved that \mathcal{M}_φ^E is a **BKE**-model. Since $\varphi \notin \mathbf{BKE}$, by extension lemma there is $\Gamma \in W^E$ such that $\varphi \notin \Gamma$, whence $\mathcal{M}_\varphi^E, \Gamma \not\models^+ \varphi$.

(2) That **BKE**-axioms are true on **B3KE**-models readily follows from the previous item. It is obvious too that a canonical model

$$\mathcal{M}_\varphi^{3E} = \langle W^{3E}, R^{3E}, S^{3E}, v_{3E}^+, v_{3E}^- \rangle$$

defined as above, but with W^{3E} consisting of all prime theories extending **B3KE**, is a 3-model. □

Theorem 7. *Let $\varphi, \psi \in \text{Form}_{\mathcal{L}^\sim}$. Then*

$$\varphi \vdash_{el}^9 \psi \quad \text{iff} \quad (T_B\varphi \wedge \blacksquare\neg T_B\varphi) \rightarrow T_B\psi \in \mathbf{BKE}.$$

$$\varphi \vdash_{el}^5 \psi \quad \text{iff} \quad (T_B\varphi \wedge \blacksquare\neg T_B\varphi) \rightarrow T_B\psi \in \mathbf{B3KE}.$$

Proof. We consider only the case of paraconsistent equilibrium consequence. Assume that $\varphi \not\vdash_{el}^9 \psi$, i.e., there is an equilibrium model $\langle \mathbf{T}, \mathbf{T} \rangle$ of φ such that $\langle \mathbf{T}, \mathbf{T} \rangle \not\models \psi$. Recall that an equilibrium model of φ is a total **N_g**-model $\langle \mathbf{T}, \mathbf{T} \rangle$ such that

$$\langle \mathbf{T}, \mathbf{T} \rangle \models \varphi \quad \text{and} \quad \langle \mathbf{H}, \mathbf{T} \rangle \not\models \varphi \quad \text{for all} \quad \mathbf{H} \subsetneq \mathbf{T}.$$

Let us consider a **BKE**-model $\mathcal{M}^{\mathbf{T}} = \langle W^{\mathbf{T}}, R^{\mathbf{T}}, S^{\mathbf{T}}, v_{\mathbf{T}}^+, v_{\mathbf{T}}^- \rangle$, where

- $W^{\mathbf{T}} = \{\mathbf{H} \mid \mathbf{H} \subseteq \mathbf{T}\}$;
- $R^{\mathbf{T}} = \{(\mathbf{H}, \mathbf{H}) \mid \mathbf{H} \in W^{\mathbf{T}}\} \cup \{(\mathbf{H}, \mathbf{T}) \mid \mathbf{H} \subseteq \mathbf{T}\}$;
- $S^{\mathbf{T}} = \{(\mathbf{H}, \mathbf{H}) \mid \mathbf{H} \in W^{\mathbf{T}} \setminus \{\mathbf{T}\}\} \cup \{(\mathbf{T}, \mathbf{H}) \mid \mathbf{H} \subsetneq \mathbf{T}\}$;
- $v_{\mathbf{T}}^+(p) = \{\mathbf{H} \mid p \in \mathbf{H}\}$, $v_{\mathbf{T}}^-(p) = \{\mathbf{H} \mid \sim p \in \mathbf{H}\}$, $p \in \text{Prop}$.

It is clear that $R^{\mathbf{T}}$ is a partial order of depth 2 with the greatest element \mathbf{T} , $S^{\mathbf{T}}$ satisfies (8) and has the only irreflexive point \mathbf{T} . So $\langle W^{\mathbf{T}}, R^{\mathbf{T}}, S^{\mathbf{T}} \rangle$ is a **BSK**_{t2}-frame. The valuations $v_{\mathbf{T}}^+$, $v_{\mathbf{T}}^-$ are defined so that $At_{\mathbf{H}} = \mathbf{H}$ for all $\mathbf{H} \in W^{\mathbf{T}}$.

Further, for every $\mathbf{H} \in W^{\mathbf{T}}$ with $\mathbf{H} \neq \mathbf{T}$, the generated submodel $\mathcal{M}^{\mathbf{H}\uparrow} := (\mathcal{M}^{\mathbf{T}})^{\mathbf{H}\uparrow}$ is of the form $\langle \mathbf{H}, \mathbf{T} \rangle$. Moreover, since $\mathbf{H} \subseteq \mathbf{T}$, we can consider $\langle \mathbf{H}, \mathbf{T} \rangle$ as an **N_g**-model, i.e. $(\mathcal{M}^{\mathbf{H}\uparrow})' = \mathcal{M}^{\mathbf{H}\uparrow}$. If $\mathbf{H} = \mathbf{T}$, then $\mathcal{M}^{\mathbf{T}\uparrow} = \mathbf{T}$ and it can be identified with the total model $\langle \mathbf{T}, \mathbf{T} \rangle$. By Lemma 2 and (6) we have for every $\chi \in \text{Form}_{\mathcal{L}^{\sim}}$ and $\mathbf{H} \in W^{\mathbf{T}}$,

$$\mathcal{M}^{\mathbf{T}}, \mathbf{H} \models^+ T_B \chi \text{ iff } \langle \mathbf{H}, \mathbf{T} \rangle \models \chi. \quad (14)$$

From this observation we obtain

$$\mathcal{M}^{\mathbf{T}}, \mathbf{T} \models^+ T_B \varphi \text{ and } \mathcal{M}^{\mathbf{T}}, \mathbf{H} \models^+ \neg T_B \varphi \text{ for } \mathbf{H} \subsetneq \mathbf{T}.$$

We also have $\mathcal{M}^{\mathbf{T}}, \mathbf{T} \models^+ \blacksquare \neg T_B \varphi$, since $\{\mathbf{H} \mid \mathbf{H} \subsetneq \mathbf{T}\}$ is the set of all $S^{\mathbf{T}}$ -successors of \mathbf{T} . From $\langle \mathbf{T}, \mathbf{T} \rangle \not\models \psi$ and (14) we have $\mathcal{M}^{\mathbf{T}}, \mathbf{T} \not\models^+ T_B \psi$, whence

$$\mathcal{M}^{\mathbf{T}}, \mathbf{T} \not\models^+ (T_B \varphi \wedge \blacksquare \neg T_B \varphi) \rightarrow T_B \psi.$$

We have thus proved the right-to-left implication.

Let us assume now that $(T_B \varphi \wedge \blacksquare \neg T_B \varphi) \rightarrow T_B \psi \notin \mathbf{BKE}$, i.e., that there are a **BKE**-model $\mathcal{M} = \langle W, R, S, v^+, v^- \rangle$ and $x \in W$ such that

$$\mathcal{M}, x \not\models^+ (T_B \varphi \wedge \blacksquare \neg T_B \varphi) \rightarrow T_B \psi.$$

Then we have

$$\mathcal{M}, x \models^+ T_B \varphi, \quad \mathcal{M}, y \not\models^+ T_B \varphi \text{ whenever } xSy, \quad \mathcal{M}, x \not\models^+ T_B \psi.$$

If there is y such that xRy and $x \neq y$, then ySx by (8), and xSx by (2^S). So we have simultaneously $\mathcal{M}, x \models^+ T_B \varphi$ and $\mathcal{M}, x \not\models^+ T_B \varphi$. Consequently, x has no proper R -successors. Assume that $At_x = \emptyset$. By Lemma 2 and (6) we have $\emptyset \models \varphi$ and $\emptyset \not\models \psi$. Thus, $\langle \emptyset, \emptyset \rangle$ is an equilibrium model of φ that refutes $\varphi \sim_{el}^9 \psi$.

In case $At_x \neq \emptyset$ by (1^E) we obtain that x is not an isolated point. Since x has no proper R -successors there is y such that $y \neq x$ and yRx . Take some $\mathbf{H} \subsetneq At_x$. By (2^E) there is z such that $At_z = \mathbf{H}$ and xSz . Applying again Lemma 2 and (6) we obtain $\langle \mathbf{H}, At_x \rangle \not\models \varphi$. Thus, $\langle At_x, At_x \rangle$ is an equilibrium model of φ that refutes ψ . So, $\varphi \not\sim_{el}^9 \psi$. □

References

- [1] J. Alcántara, C. Demasio, L.M. Pereira, *A Declarative Characterisation of Disjunctive Paraconsistent Answer Sets*, R. López de Mántaras, L. Saitta (eds.), Proc. of ECAI 2004, IOS Press, 2004, 951-957.
- [2] N. Belnap, *A useful four-valued logic*, J.M. Dunn, G. Epstein (eds), Modern Uses of Multiple-Valued Logic, Dordrecht, D. Reidel, 1977, 5–37.
- [3] A. Chagrov, M. Zakhariashev, *Modal Logic*, Oxford, Clarendon Press, 1997.
- [4] J.M. Dunn, *Intuitive semantics for first-degree entailments and ‘coupled trees’*, Philosophical Studies, **29** (1976), 149–168.
- [5] L. Fariñas del Cerro, A. Herzig, E.I. Su, *Capturing equilibrium models in modal logic*, Journal of Applied Logic, **12:2** (2014), 192–207.
- [6] J. Font, *Abstract Algebraic Logic: An Introductory Textbook*, College Publications, 2016.
- [7] M. Gelfond, V. Lifschitz, *The stable model semantics for logic programming*, Proc. of ICLP’88, The MIT Press, 1988, 1070–1080.
- [8] M. Gelfond, V. Lifschitz, *Classical negation in logic programs and disjunctive databases*, New Generation Computing, **9** (1991), 365–385.
- [9] K. Gödel, *Eine Interpretation des intuitionistischen Aussagenkalküls*, Ergebnisse eines mathematischen Kolloquiums, **4** (1933), 39–40.
- [10] V. Lifschitz, D. Pearce, A. Valverde, *Strongly equivalent logic programs*, ACM Transactions on Computational Logic, **2:2** (2001), 526–541.
- [11] E.G.K. López-Escobar, *Refutability and Elementary Number Theory*, Indagationes Mathematicae, **34** (1972), 362–374.
- [12] N.V. Mayatski, S.P. Odintsov, *On deductive bases for paraconsistent answer set semantics*, Journal of Applied Non-Classical Logic, **23:1–2** (2013), 131–146.
- [13] D. Nelson, *Constructible falsity*, Journal of Symbolic Logic, **14** (1949):16–26.
- [14] S.P. Odintsov, *The class of extensions of Nelson’s paraconsistent logic*, Studia Logica, **80** (2005), 291–320.
- [15] S.P. Odintsov *Constructive Negations and Paraconsistency*, Trends in Logic, **26**, Dordrecht, Springer, 2008.
- [16] S. Odintsov, D. Pearce, (2005). *Routley Semantics for Answer Sets*, C. Baral, G. Greco, N. Leone, G. Terracina (eds) Logic Programming and Nonmonotonic Reasoning. LPNMR 2005. Lecture Notes in Computer Science, vol 3662. Springer, Berlin, Heidelberg, 2005, 343–355.
- [17] S.P. Odintsov, H. Wansing, *Modal logics with Belnapian truth values*, Journal of Applied Non-Classical Logics, **20:3** (2010), 279–301.
- [18] S.P. Odintsov, H. Wansing, *Disentangling FDE-Based Paraconsistent Modal Logics*, Studia Logica, **105:1** (2017), 1221–1254.
- [19] D. Pearce, *A new logical characterization of stable models and answer sets*, Proceedings of Non-Monotonic Extensions of Logic Programming 1996, LNCS 1216, Springer, 1997, 57–70.
- [20] R. Routley. *Semantical Analyses of Propositional Systems of Fitch and Nelson*, Studia Logica, **33:3** (1974): 283-298.
- [21] C. Sakama, K. Inoue, *Paraconsistent Stable Semantics for Extended Disjunctive Programs*, Journal of Logic and Computation, **5** (1995), 265–285.
- [22] R.H. Thomason, *A Semantical Study of Constructible Falsity*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, **15** (1969), 247–257.
- [23] A.G. Vishneva, S.P. Odintsov *Modal companions of special extensions of constructive Nelson’s logic (in russian)*, Matematicheskie Zametki, **117:3** (2025), 344-364.
- [24] N.N. Vorob’ev, *A constructive propositional calculus with strong negation (in russian)*, Doklady Akademii Nauk SSSR, **85** (1952):465–468.

- [25] N.N. Vorob'ev, *The problem of deducibility in constructive propositional calculus with strong negation (in russian)*, Doklady Akademii Nauk SSSR, **85** (1952):689–692.

SERGEI PAVLOVICH ODINTSOV
MATHEMATICAL INSTITUTE RAN,
UL. GUBKINA, 8,
630090, MOSKVA, RUSSIA
Email address: `odintsov.sergey2013@yandex.ru`