T-PSEUDOFINITE ACTS OVER ABELIAN GROUP

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Abstract: The main results of the article concern the model theory of T-pseudofinite acts over abelian groups, where T is the theory of all acts over the group. A left act over a group G is a set on which G acts unitarily from the left. We give characteristic properties of an abelian group G that are necessary for the existence of a T-pseudofinite G-act in the class $K_{G,\bar{S}}$ and sufficient for the class $K_{G,\bar{S}}$ to be T-pseudofinite, where T is the theory of all G-acts, $\bar{S}$ is a finite set of subgroups of G, and $K_{G,\bar{S}}$ is the class of all coproducts of G-acts of the form $_G(G/G_1)$, $G_1 \in \bar{S}$. It follows from this result that for a finitely generated abelian group G the class $K_{G,\bar{S}}$ is T-pseudofinite; for the multiplicative group of rational numbers G the class $K_{G,\bar{S}}$ is also T-pseudofinite. It is noted that there exists an abelian group G and a G-act $_G A$ such that $_G A$ is not T-pseudofinite but is pseudofinite, where T is the theory of all G-acts, namely, for a quasicyclic group G as a divisible group, the G-act $_G G$ is not T-pseudofinite, but is pseudofinite.

Keywords: T-pseudofinite structure, act over abelian group, theory of all acts over abelian group, T-pseudofinite act over abelian group.

Introduction

The structure $\mathfrak{M}$ in language L is called pseudofinite if every sentence true in $\mathfrak{M}$ has a finite model. The theory of pseudofinite structures is a well-developed theory. A significant number of works are devoted to the theory of pseudofinite structures ([1]-[4]). In [5] the authors of this work considered the issues of pseudofiniteness of connected unars without cycles, in [6] the pseudofinite acts over monoid with finite number of isomorphism types of finite cyclic subacts are studied; in particular, it is proven that a coproduct of finite acts over monoid is pseudofinite; as a consequence, it is shown that every act over finite group is pseudofinite.

In [7], the concept of T-pseudofiniteness is introduced for the model of theory T. A model $\mathfrak{M}$ of a theory T is called T-pseudofinite if every sentence true in $\mathfrak{M}$ is also true in a finite model of the theory T. It is clear that T-pseudofiniteness implies pseudofiniteness for every theory T, and pseudofiniteness implies T-pseudofiniteness for every finite axiomatizable theory T. In [7], necessary and sufficient conditions for the T-pseudofiniteness of acts over an abelian group with a finite number of isomorphism types of connected subacts are given.

It is known that a structure $\mathfrak{M}$ in a language L is pseudofinite if and only if $\mathfrak{M}$ is elementarily equivalent to an ultraproduct of finite structures of L. In this paper, a similar result is proved for T-pseudofinite structures (Theorem 1).

For an abelian group G and a finite set $\bar{S}$ of subgroups of G, we define the class $K_{G,\bar{S}}$ of all coproducts of G-acts of the form $_G(G/G_1)$, where $G_1 \in \bar{S}$. This class is axiomatizable (Corollary 1). Note that for any theory T in the language L and any class K of models of T, if there exists a T-pseudofinite structure $\mathfrak{M} \in K$, then K is a T-pseudofinite class. In this paper, we give characteristic properties of the abelian group G that are necessary for the existence of a T-pseudofinite G-act in the class $K_{G,\bar{S}}$ and sufficient for the class $K_{G,\bar{S}}$ to be T-pseudofinite, where T is the theory of all G-acts and $\bar{S}$ is a finite set of subgroups of G (Theorem 2). As a consequence of this theorem we obtain the following result: if for every subgroup $G_1 \in \bar{S}$ of group G and every finite set $F \subseteq G \setminus G_1$ there exists a finite index subgroup G_2 of group G such that $G_1 \subseteq G_2$ and $F \cap G_2 = \varnothing$, then the class $K_{G,\bar{S}}$ is T-pseudofinite (Corollary 2). It follows from this result that for a finitely generated abelian group G the class $K_{G,\bar{S}}$ is T-pseudofinite (Proposition 2); for the multiplicative group of rational numbers G the class $K_{G,\bar{S}}$ is also T-pseudofinite (Proposition 1).

It is noted that there exists an abelian group G and an act over G such that this act is not T-pseudofinite but is pseudofinite, where T is the theory of all G-acts (Corollary 4), namely, for a quasicyclic group G as a divisible group, the act G over $_G G$ is not T-pseudofinite (Proposition 3), but is pseudofinite (Corollary 3).

1   Preliminary information

Let us recall some definitions and facts from act theory and model theory (see [8, 9, 10]). Let S be a monoid with identity 1. A structure $\langle A; s\rangle_{s \in S}$ in the language $L_S = \{s \mid s \in S\}$ consisting of unary operation symbols is a (left) *S-act* if $s_1(s_2 a) = (s_1 s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$. An S-act $\langle A; s\rangle_{s \in S}$ is denoted by $_S A$. Elements x, y of an S-act $_S A$ are called *connected* (denoted by $x \sim y$) if there exist $n \in \omega$, $a_0, \dots, a_n \in A$, $s_1, \dots, s_n \in S$ such that $x = a_0$, $y = a_n$, and $a_i = s_i a_{i-1}$ or $a_{i-1} = s_i a_i$. An S-act $_S A$ is called *connected* if we have $x \sim y$ for every $x, y \in {}_S A$. It is easy to check that $\sim$ is a congruence relation on the S-act $_S A$. The classes of this relation are called *connected components* of the S-act $_S A$. *A coproduct* of S-acts $_S A_i$ is a disjunctive union of this S-acts. The coproduct of S-acts $_S A_i$ is denoted by $\coprod_{i \in I} {}_S A_i$. It is known [8] that every S-act $_S A$ can be uniquely represented as a coproduct of connected components. Let G be a group and H be a subgroup of G. By $_G G/H$ we denote G-act $_G\{gH \mid g \in G\}$ with unary operations defined as follows: $g(aH) = (ga)H$ for every $g, a \in G$. Each connected G-act has the form $_G G/H$ for some subgroup H of G and it has no proper subacts.

The structure $\mathfrak{M}$ in language L is called *pseudofinite* if every sentence true in $\mathfrak{M}$ has a finite model. It is known that the structure $\mathfrak{M}$ in language L is pseudofinite iff $\mathfrak{M}$ is elementary equivalent to an ultraproduct of finite structures in language L ([11]).

Let T be a consistent (but possibly incomplete) theory in the language L. The model $\mathfrak{M}$ of the theory T is called *T-pseudofinite* if every sentence true in $\mathfrak{M}$ is also true in some finite model of the theory T. From the proof of Theorem 1 [6] we get

Fact 1. *Every coproduct of finite S-acts is a T-pseudofinite S-act, where T is the theory of all S-acts.*

A class K of models of the theory T is called *T-pseudofinite* if whenever $K \vDash \varphi$, then φ is true in some finite model of the theory T.

Fact 2. *Let a group G have only a finite number of finite index subgroups and T be the theory of all G-acts. Then a G-act $_G A$ is T-pseudofinite if and only if $_G A$ is elementarily equivalent to the coproduct of finite G-acts, i.e. by Fact 1, $_G A$ is T-pseudofinite.*

Fact 3. [12] *Every finitely generated abelian group is isomorphic to a group of the form*

$$\mathbb{Z}^n \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_U\mathbb{Z},$$

where $n \geq 0$, and the numbers $q_1, \dots, q_U$ are powers of (not necessarily distinct) prime numbers.

For an arbitrary set of sentences in language L and a structure $\mathfrak{M}$ in language L, instead of writing $\mathfrak{M} \in Mod(\Gamma)$ we will use the notation $\mathfrak{M} \vDash \Gamma$.

We will distinguish between the notations $A \subseteq B$ and $A \subset B$.

Theorem 1. *Let T be a theory in language L and $\mathfrak{M}$ be a model of T. Then $\mathfrak{M}$ is a T-pseudofinite structure if and only if $\mathfrak{M}$ is elementary equivalent to the ultraproduct of finite models of the theory T.*

Proof. Necessity. Let $\mathfrak{M}$ be a T-pseudofinite model of T. We denote the set $\{\varphi \mid \mathfrak{M} \vDash \varphi\}$ by Φ. By the definition of a T-pseudofinite structure, for every sentence $\varphi \in \Phi$ there exists a finite model $\mathfrak{N}_\varphi$ of the theory T such that $\mathfrak{N}_\varphi \vDash \varphi$. Let $D_\varphi = \{\psi \in \Phi \mid \psi \vdash \varphi\}$, where $\varphi \in \Phi$, and $D' = \{D_\varphi \mid \varphi \in \Phi\}$. It is clear that D' is the family of sets with finite intersection property. Then D' is contained in some ultrafilter D on the set Φ. We will prove that $\mathfrak{M}$ is elementary equivalent to the ultraproduct $\prod_{\psi \in \Phi} \mathfrak{N}_\psi / D$. Let $\mathfrak{M} \vDash \varphi$. Note that $\psi \in D_\varphi$ implies $\mathfrak{N}_\psi \vDash \varphi$. Then $D_\varphi \subseteq \{\psi \in \Phi \mid \mathfrak{N}_\psi \vDash \varphi\}$. Since $D_\varphi \in D$, then $\{\psi \in \Phi \mid \mathfrak{N}_\psi \vDash \varphi\} \in D$. By the definition of ultraproduct, we have $\prod_{\psi \in \Phi} \mathfrak{N}_\psi / D \vDash \varphi$. Thus, $\mathfrak{M} \equiv \prod_{\psi \in \Phi} \mathfrak{N}_\psi / D$.

Sufficiency. Let $\mathfrak{M}$ is elementary equivalent to the ultraproduct $\prod_{i \in I} \mathfrak{N}_i / D$ of finite models of the theory T and $\mathfrak{M} \vDash \varphi$. Then $\prod_{i \in I} \mathfrak{N}_i / D \vDash \varphi$. By Łos's Theorem ([13], chapter 3, § 17) $I_0 = \{i \in I \mid \mathfrak{N}_i \vDash \varphi\} \in D$. If $i_0 \in I_0$ then $\mathfrak{N}_{i_0} \vDash \varphi$. Thus $\mathfrak{M}$ is a T-pseudofinite model of T. $\qquad\square$

Let G be an abelian group, $S_1, \ldots, S_n$ be pairwise distinct subgroups of G. For sets $\bar{S} = \{S_1, \ldots, S_n\}$, $\bar{S}_1 \subseteq \bar{S}$, $\bar{S}_2 \subseteq \bar{S}$, $S \in \bar{S}$ and elements $g_1 \notin \bigcup \bar{S}$, $g_2 \notin \bigcup \bar{S}_1$, we define the functions

$$\varphi_1^{\bar{S}\bar{S}_1 S} : \overline{U}_1^{\bar{S}\bar{S}_1 S} \to G, \ \overline{U}_1^{\bar{S}\bar{S}_1 S} = \{U \in \bar{S}_1 \mid S \setminus U \neq \varnothing\}, \ \varphi_1^{\bar{S}\bar{S}_1 S}(U) \in S \setminus U,$$

$$\varphi_2^{\bar{S}\bar{S}_2 S} : \overline{U}_2^{\bar{S}\bar{S}_2 S} \to G, \ \overline{U}_2^{\bar{S}\bar{S}_2 S} = \{U \in \bar{S}_2 \mid U \setminus S \neq \varnothing\}, \ \varphi_2^{\bar{S}\bar{S}_2 S}(U) \in U \setminus S,$$

$$\varphi_3^{\bar{S}} : \overline{U}_3^{\bar{S}} \to G, \ \overline{U}_3^{\bar{S}} = \{U \in \bar{S} \mid \textstyle\bigcup \bar{S} \setminus U \neq \varnothing\}, \ \varphi_3^{\bar{S}}(U) \in \textstyle\bigcup \bar{S} \setminus U,$$

$$\varphi_4^{\bar{S}\bar{S}_1} : \overline{U}_4^{\bar{S}\bar{S}_1} \to G, \ \overline{U}_4^{\bar{S}\bar{S}_1} = \{U \in \bar{S} \mid U \notin \bar{S}_1\}, \ \varphi_4^{\bar{S}\bar{S}_1}(U) \in U,$$

$$\varphi_5^{\bar{S}\bar{S}_1} : \overline{U}_5^{\bar{S}\bar{S}_1} \to G, \ \overline{U}_5^{\bar{S}\bar{S}_1} = \{U \in \bar{S} \mid \textstyle\bigcup \bar{S}_1 \setminus U \neq \varnothing\}, \ \varphi_5^{\bar{S}\bar{S}_1}(U) \in \textstyle\bigcup \bar{S}_1 \setminus U,$$

$$\varphi_6^{\bar{S}\bar{S}_1} : \overline{U}_6^{\bar{S}\bar{S}_1} \to G, \ \overline{U}_6^{\bar{S}\bar{S}_1} = \{U \in \bar{S} \mid U \setminus \textstyle\bigcup \bar{S}_1 \neq \varnothing\}, \ \varphi_6^{\bar{S}\bar{S}_1}(U) \in U \setminus \textstyle\bigcup \bar{S}_1,$$

and the formulae

$$\alpha_{\bar{S}}(g_1) \leftharpoonup \forall x (g_1 x \neq x),$$

$$\beta_{\bar{S}}(\bar{S}_1, g_2, \varphi_4^{\bar{S}\bar{S}_1}) \leftharpoonup \forall x \Big((g_2 x = x) \to \bigvee_{U \in \overline{U}_4^{\bar{S}\bar{S}_1}} (\varphi_4^{\bar{S}\bar{S}_1}(U) x = x) \Big),$$

$$\gamma_{\bar{S}}(\bar{S}_1, \bar{S}_2, S, \varphi_1^{\bar{S}\bar{S}_1 S}, \varphi_2^{\bar{S}\bar{S}_2 S}) \leftharpoonup$$
$$\leftharpoonup \exists x \Big(\bigwedge_{U \in \overline{U}_1^{\bar{S}\bar{S}_1 S}} (\varphi_1^{\bar{S}\bar{S}_1 S}(U) x = x) \wedge \bigwedge_{U \in \overline{U}_2^{\bar{S}\bar{S}_2 S}} (\varphi_2^{\bar{S}\bar{S}_2 S}(U) x \neq x) \Big),$$

$$\delta_{\bar{S}}(\varphi_3^{\bar{S}}) \leftharpoonup \forall x \bigvee_{U \in \overline{U}_3^{\bar{S}}} (\varphi_3^{\bar{S}}(U) x \neq x),$$

$$\varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5{}^{\bar{S}\bar{S}_1}, \varphi_6{}^{\bar{S}\bar{S}_1}) \leftrightharpoons$$

$$\leftrightharpoons \forall x (\bigwedge_{U \in \overline{U}_5^{\bar{S}\bar{S}_1}} (\varphi_5{}^{\bar{S}\bar{S}_1}(U)x = x) \rightarrow \bigvee_{U \in \overline{U}_6^{\bar{S}\bar{S}_1}} (\varphi_6{}^{\bar{S}\bar{S}_1}(U)x = x)).$$

Let us introduce one more notation:

$$\Gamma_{\bar{S}} = \{\alpha_{\bar{S}}(g) \mid g \notin \bigcup \bar{S}\} \cup$$

$$\cup \{\beta_{\bar{S}}(\bar{S}_1, g, \varphi_4{}^{\bar{S}\bar{S}_1}) \mid \bar{S}_1 \subset \bar{S}, g \notin \bigcup \bar{S}_1\} \cup$$

$$\cup \{\gamma_{\bar{S}}(\bar{S}_1, \bar{S}_2, S, \varphi_1{}^{\bar{S}\bar{S}_1 S}, \varphi_2{}^{\bar{S}\bar{S}_2 S}) \mid \bar{S}_1, \bar{S}_2 \subseteq \bar{S}\} \cup$$

$$\cup \{\delta_{\bar{S}}(\varphi_3{}^{\bar{S}}) \mid \bigcup \bar{S} \notin \bar{S}\} \cup$$

$$\cup \{\varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5{}^{\bar{S}\bar{S}_1}, \varphi_6{}^{\bar{S}\bar{S}_1}) \mid \bar{S}_1 \subset \bar{S}, \bigcup \bar{S}_1 \notin \bar{S}\}.$$

Lemma 1. *Let G be an abelian group, $S_1, \ldots, S_n$ be pairwise distinct subgroups of G, $\bar{S} = \{S_1, \ldots, S_n\}$. Then ${}_G A \vDash \Gamma_{\bar{S}}$ iff*

$$_G A \cong \coprod_{i \in I_1} {}_G(G/S_1)_i \sqcup \ldots \sqcup \coprod_{i \in I_n} {}_G(G/S_n)_i$$

for some nonempty sets $I_1, \ldots, I_n$.

Proof. Necessity. Let ${}_G A \vDash \Gamma_{\bar{S}}$ and ${}_G A \cong \coprod_{j \in J} {}_G(G/K_j)$, where K_j are subgroups of G ($j \in J$).

Let us show that $K_j \in \bar{S}$. We consider two cases.

(1) $K_j \neq \bigcup \bar{S}$. Let $\bar{S}_1 = \{S \in \bar{S} \mid S \subseteq K_j\}$. Then $\bigcup \bar{S}_1 \subseteq K_j$.

We will show that $\bigcup \bar{S}_1 = K_j$. Suppose that there exists $g \in K_j \setminus \bigcup \bar{S}_1$. Then $gK_j = K_j$. Let $\bar{S}_1 \subset \bar{S}$, that is, $\bar{S} \setminus \bar{S}_1 \neq \varnothing$. By the definition of the set $\bar{S}_1$, we have $U \nsubseteq K_j$ for every group $U \in \bar{S} \setminus \bar{S}_1$, that is, there is $g_U \in U \setminus K_j$ and $g_U K_j \neq K_j$. We define the function $\varphi_4{}^{\bar{S}\bar{S}_1}$ as follows: $\varphi_4{}^{\bar{S}\bar{S}_1}(U) = g_U$ for every $U \in \overline{U}_4^{\bar{S}\bar{S}_1}$. Since $\bar{S}_1 \subset \bar{S}$, then by condition, ${}_G A \vDash \beta_{\bar{S}}(\bar{S}_1, g, \varphi_4{}^{\bar{S}\bar{S}_1})$, contradiction. Therefore, $\bar{S} = \bar{S}_1$, $g \in K_j \setminus \bigcup \bar{S}$ and $gK_j = K_j$, but ${}_G A \vDash \alpha_{\bar{S}}(g)$, and again a contradiction. Thus, $\bigcup \bar{S}_1 = K_j$.

Now we will show that $K_j \in \bar{S}$. Suppose the opposite. Then $\bigcup \bar{S}_1 \notin \bar{S}$. Note that $U \subset \bigcup \bar{S}_1$ for every $U \in \bar{S}_1$. Therefore, for every $U \in \bar{S}$ there is $g_U \in \bigcup \bar{S}_1 \setminus U$. Since $\bigcup \bar{S}_1 = K_j$, then $g_U \in K_j$ and $g_U K_j = K_j$. Let us define the function $\varphi_5{}^{\bar{S}\bar{S}_1}$ as follows: $\varphi_5{}^{\bar{S}\bar{S}_1}(U) = g_U$ for every $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$. Since $K_j \neq \bigcup \bar{S}$, then $\bigcup \bar{S}_1 \subset \bigcup \bar{S}$ and there is a group $U' \in \bar{S}$ such that $U' \setminus \bigcup \bar{S}_1 \neq \varnothing$. Then $\bar{S}_1 \subset \bar{S}$ and $U' \in \overline{U}_6^{\bar{S}\bar{S}_1}$. For every $U \in \bar{S}$ such that $U \setminus \bigcup \bar{S}_1 \neq \varnothing$, we choose $r_U \in U \setminus \bigcup \bar{S}_1$. Since $\bigcup \bar{S}_1 = K_j$, then $r_U K_j \neq K_j$. Let us define the function $\varphi_6{}^{\bar{S}\bar{S}_1}$ as follows: $\varphi_6{}^{\bar{S}\bar{S}_1}(U) = r_U$ for every $U \in \overline{U}_6^{\bar{S}\bar{S}_1}$. As noted above, $\overline{U}_6^{\bar{S}\bar{S}_1} \neq \varnothing$. Since $\bigcup \bar{S}_1 \notin \bar{S}$, then by condition, ${}_G A \vDash \varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5{}^{\bar{S}\bar{S}_1}, \varphi_6{}^{\bar{S}\bar{S}_1})$. Consequently, there exists a group $U \in \overline{U}_6^{\bar{S}\bar{S}_1}$ such that $r_U \in K_j$, contradiction. Therefore, $K_j \in \bar{S}$.

(2) $K_j = \bigcup \bar{S}$. Let us assume that $K_j \notin \bar{S}$, that is, $\bigcup \bar{S} \notin \bar{S}$. Then $\bigcup \bar{S} \neq U$ for every $U \in \bar{S}$. Therefore, for every $U \in \bar{S}$ there is $g_U \in \bigcup \bar{S} \setminus U$. We define the function $\varphi_3{}^{\bar{S}}$ as follows: $\varphi_3{}^{\bar{S}}(U) = g_U$ for every $U \in \overline{U}_3^{\bar{S}}$. Since $K_j = \bigcup \bar{S}$, then $g_U K_j = K_j$ for every g_U. Since $\bigcup \bar{S} \notin \bar{S}$, then by condition, $_G A \vDash \delta_{\bar{S}}(\varphi_3{}^{\bar{S}})$, contradiction. Therefore, $K_j \in \bar{S}$.

Let us show that for every $S \in \bar{S}$ there is $j \in J$ such that $S = K_j$. Suppose that there is $S \in \bar{S}$ such that $S \neq K_j$ for every $j \in J$. Then for every $j \in J$ either $K_j \subset S$ or $K_j \setminus S \neq \varnothing$. Let $\bar{S}_1 = \{K_j \mid K_j \subset S\}$, and $\bar{S}_2 = \{K_j \mid K_j \setminus S \neq \varnothing\}$. Since $K_j \in \bar{S}$ for every $j \in J$, then $\bar{S}_1, \bar{S}'_1 \subseteq \bar{S}$. It is clear that $\bigcup_{j \in J} K_j = \bar{S}_1 \cup \bar{S}'_1$. For every $U \in \bar{S}_1$ we choose the element $g_U \in S \setminus U$. We define the function $\varphi_1{}^{\bar{S}\bar{S}_1 S}$ as follows: $\varphi_1{}^{\bar{S}\bar{S}_1 S}(U) = g_U$ for every $U \in \overline{U}_1^{\bar{S}\bar{S}_1 S}$. For every $U \in \bar{S}_2$ we choose the element $r_U \in U \setminus S$. We define the function $\varphi_2{}^{\bar{S}\bar{S}_2 S}$ as follows: $\varphi_2{}^{\bar{S}\bar{S}_2 S}(U) = r_U$ for every $U \in \overline{U}_2^{\bar{S}\bar{S}_2 S}$. Let $j \in J$. If $K_j \subset S$, that is, $K_j \in \bar{S}_1$, then $g_{K_j} K_j \neq K_j$; if $K_j \setminus S \neq \varnothing$, that is, $K_j \in \bar{S}_2$, then $r_{K_j} K_j = K_j$. By condition,

$$_G A \vDash \gamma_{\bar{S}}(\bar{S}_1, \bar{S}'_1, S, \varphi_1{}^{\bar{S}\bar{S}_1 S}, \varphi_2{}^{\bar{S}\bar{S}_2 S}),$$

contradiction. Consequently, for every $S \in \bar{S}$ there is $j \in J$ such that $S = K_j$.

Thus, $K_j \in \bar{S}$ for every $j \in J$.

Sufficiency. Let $_G A \cong {}_G B$ and $_G B = \coprod_{i \in I_1} {}_G(G/S_1)_i \sqcup \ldots \sqcup \coprod_{i \in I_n} {}_G(G/S_n)_i$. Let us show that $_G A \vDash \Gamma_{\bar{S}}$.

Let $g \notin \bigcup \bar{S}$. Then $gS \neq S$ for every $S \in \bar{S}$. Therefore, $_G A \vDash \alpha_{\bar{S}}(g)$.

Let $\bar{S}_1 \subset \bar{S}$, $g \notin \bigcup \bar{S}_1$, $S \in \bar{S}$ and the function $\varphi_4{}^{\bar{S}\bar{S}_1} : \overline{U}_4^{\bar{S}\bar{S}_1} \to G$ is defined. We assume that $gS = S$. Then $S \notin \bar{S}_1$, that is, $S \in \bar{S} \setminus \bar{S}_1$. Therefore, $\varphi_4{}^{\bar{S}\bar{S}_1}(S)S = S$ and $_G A \vDash \beta_{\bar{S}}(\bar{S}_1, g, \varphi_4{}^{\bar{S}\bar{S}_1}, \varphi_4{}^{\bar{S}\bar{S}_1})$.

Let $\bar{S}_1, \bar{S}_2 \subseteq \bar{S}$, $S \in \bar{S}$ and the functions $\varphi_1{}^{\bar{S}\bar{S}_1 S} : \overline{U}_1^{\bar{S}\bar{S}_1 S} \to G$ and $\varphi_2{}^{\bar{S}\bar{S}_2 S} : \overline{U}_2^{\bar{S}\bar{S}_1 S} \to G$ are defined. Then $\varphi_1{}^{\bar{S}\bar{S}_1 S}(U)S = S$ for every $U \in \overline{U}_1^{\bar{S}\bar{S}_1 S}$ and $\varphi_2{}^{\bar{S}\bar{S}_1}(U)S \neq S$ for every $U \in \overline{U}_2^{\bar{S}\bar{S}_2 S}$. Therefore,

$$_G A \vDash \gamma_{\bar{S}}(\bar{S}_1, \bar{S}', S, \varphi_1{}^{\bar{S}\bar{S}_1 S}, \varphi_2{}^{\bar{S}\bar{S}_2 S}).$$

Let $\bigcup \bar{S} \notin \bar{S}$, $S \in \bar{S}$ and the function $\varphi_3{}^{\bar{S}} : \overline{U}_3^{\bar{S}} \to G$ is defined. Then $S \subset \bigcup \bar{S}$ and $S \in U_3^{\bar{S}}$. Therefore, $\varphi_3{}^{\bar{S}}(S)S \neq S$ and $_G A \vDash \delta_{\bar{S}}(\varphi_3{}^{\bar{S}})$.

Let $\bar{S}_1 \subset \bar{S}$, $\bigcup \bar{S}_1 \notin \bar{S}$ and the functions $\varphi_5{}^{\bar{S}\bar{S}_1} : \overline{U}_5^{\bar{S}\bar{S}_1} \to G$ and $\varphi_6{}^{\bar{S}\bar{S}_1} : \overline{U}_6^{\bar{S}\bar{S}_1} \to G$ are defined. Suppose that $S \in \bar{S}$ and $\varphi_5{}^{\bar{S}\bar{S}_1}(U)S = S$ for all $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$, that is, for all $U \in \bar{S}$ such that $\bigcup \bar{S}_1 \setminus U \neq \varnothing$. Let us show that $S \in \overline{U}_6^{\bar{S}\bar{S}_1}$, that is, $S \setminus \bigcup \bar{S}_1 \neq \varnothing$. Assume that $S \subseteq \bigcup \bar{S}_1$. If $S = \bigcup \bar{S}_1$, then $\bigcup \bar{S}_1 \in \bar{S}$, a contradiction. Therefore, $S \subset \bigcup \bar{S}_1$, $S \in \overline{U}_5^{\bar{S}\bar{S}_1}$ and $\varphi_5{}^{\bar{S}\bar{S}_1}(S) \in S$, but $\varphi_5{}^{\bar{S}\bar{S}_1}(S) \in \bigcup \bar{S}_1 \setminus S$, a contradiction. So we proven $S \in \overline{U}_6^{\bar{S}\bar{S}_1}$. Therefore,

$\varphi_6{}^{\bar{S}\bar{S}_1}(S) \in S \setminus \bigcup \bar{S}_1$. This implies the equality $\varphi_6{}^{\bar{S}\bar{S}_1}(S)S = S$ and

$$_G A \vDash \varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5{}^{\bar{S}\bar{S}_1}, \varphi_6{}^{\bar{S}\bar{S}_1}).$$

$\square$

Let $K_{G,\bar{S}}$ denote the class of all G-acts of the form

$$\coprod_{i \in I_1} {}_G(G/S_1)_i \sqcup \ldots \sqcup \coprod_{i \in I_n} {}_G(G/S_n)_i,$$

where $I_1, \ldots, I_n$ are some nonempty sets and $\bar{S} = \{S_1, \ldots, S_n\}$. From Lemma 1 we obtain the following corollary.

Corollary 1. *Let G be an abelian group, $S_1, \ldots, S_n$ be pairwise distinct subgroups of G. Then the class $K_{G,\bar{S}}$ is axiomatizable.*

2 *T*-**pseudofiniteness of class** $K_{G,\bar{S}}$

Note that for any theory T in the language L and any class K of models of T, if there exists a T-pseudofinite structure $\mathfrak{M} \in K$, then K is a T-pseudofinite class. Theorem 2 gives us characteristic properties of the abelian group G that are necessary for the existence of a T-pseudofinite G-act in the class $K_{G,\bar{S}}$ and sufficient for the class $K_{G,\bar{S}}$ to be T-pseudofinite, where T is the theory of all G-acts.

Theorem 2. *Let G be an abelian group, T be the theory of all G-acts, $S_1, \ldots, S_n$ be pairwise distinct subgroups of the group G, $\bar{S} = \{S_1, \ldots, S_n\}$. Then*

$$(a) \Longrightarrow (b) \Longrightarrow (c),$$

where

(a) there exists $_G A \in K_{G,\bar{S}}$ such that $_G A$ is T-pseudofinite;

(b) for every finite set $F \subseteq G$ there are finite index subgroups $U_1, \ldots, U_k$ of group G such that following conditions are satisfied:

1) $F \cap \bigcup_{1 \le i \le k} U_i \subseteq \bigcup \bar{S}$;

2) for every set $\bar{S}_1 \subset \bar{S}$ and for every group U_i

$$(F \cap U_i) \setminus \bigcup \bar{S}_1 \ne \varnothing \Rightarrow \exists U \in \bar{S} \setminus \bar{S}_1 \ (U \cap F \subseteq U_i);$$

3) for every $S \in \bar{S}$ there exists U_{i_0}, $i_0 \in \{1, \ldots, k\}$, such that $S \cap F \subseteq U \cup U_{i_0}$ and $U \cap F \cap U_{i_0} \subseteq S$ for every $U \in \bar{S}$;

4) if $\bigcup \bar{S} \notin \bar{S}$, then for every group U_i there is a group $U \in \bar{S}$ such that $\bigcup \bar{S} \cap U_i \cap F \subseteq U$;

5) for every set $\bar{S}_1 \subset \bar{S}$ such that $\bigcup \bar{S}_1 \notin \bar{S}$, and for every group U_i

$$\forall U \in \bar{S}((\bigcup \bar{S}_1 \cap U_i \cap F) \setminus U \ne \varnothing) \Rightarrow$$

$$\Rightarrow \exists U \in \bar{S}((U \cap F) \setminus \bigcup \bar{S}_1 \ne \varnothing \wedge (U \cap F) \setminus \bigcup \bar{S}_1 \subseteq U_i);$$

(c) the class $K_{G,\bar{S}}$ is T-pseudofinite.

Proof. Let the conditions of Theorem be satisfied.

$(a) \Longrightarrow (b)$ Let $_GA$ be a T-pseudofinite G-act,

$$_GA = \coprod_{i \in I_1} {}_G(G/S_1)_i \sqcup \ldots \sqcup \coprod_{i \in I_n} {}_G(G/S_n)_i,$$

where $_G(G/S_k)_i$ are the copies of $_GG/S_k$, $I_1, \ldots, I_n$ are nonempty sets, F be a finite subset of G, Γ be a set of sentences from $\Gamma_{\bar{S}}$ in language

$$L_F = \{f \mid f \in F\} \subseteq L_G.$$

It is clear that Γ is a finite set. Let Θ denote the conjunction of all sentences from Γ. By Lemma 1, $_GA \vDash \Gamma_{\bar{S}}$. Therefore, $_GA \vDash \Theta$. Since $_GA$ is a T-pseudofinite G-act, there is a finite G-act $_GB$ such that $_GB \vDash \Theta$. Then $_GB = {}_GG/U_1 \sqcup \ldots \sqcup {}_GG/U_k$, where $U_1, \ldots, U_k$ are finite index subgroups of group G.

Let us show that condition 1) is satisfied. Suppose that there is an element $g \in (F \cap \bigcup_{1 \leq i \leq k} U_i) \setminus \bigcup \bar{S}$. Since $_GB \vDash \Theta$, $g \in G \setminus \bigcup \bar{S}$ and $g \in F$, then $_GB \vDash \alpha_{\bar{S}}(g)$. So $g\bar{U}_i \neq U_i$, that is, $g \notin U_i$, for every i, $1 \leq i \leq k$. Contradiction. Condition 1) has been proven.

Let us show that condition 2) is satisfied. Let $\bar{S}_1 \subset \bar{S}$, $1 \leq i \leq k$ and $(F \cap U_i) \setminus \bigcup \bar{S}_1 \neq \varnothing$. Then there exists $g \in (F \cap U_i) \setminus \bigcup \bar{S}_1$ and $gU_i = U_i$. Let us show that there exists a group $U \in \bar{S} \setminus \bar{S}_1$ such that $U \cap F \subseteq U_i$. Assume the converse. Then for every group $U \in \bar{S} \setminus \bar{S}_1$ (that is, for every group $U \in \overline{U}_4^{\bar{S}\bar{S}_1}$) there is $g_U \in (U \cap F) \setminus U_i$. We define the function $\varphi_4^{\bar{S}\bar{S}_1} : \overline{U}_4^{\bar{S}\bar{S}_1} \to G$ as follows: $\varphi_4^{\bar{S}\bar{S}_1}(U) = g_U$ for every group $U \in \overline{U}_4^{\bar{S}\bar{S}_1}$. Since $_GB \vDash \Theta$, then $_GB \vDash \beta_{\bar{S}}(\bar{S}_1, g, \varphi_4^{\bar{S}\bar{S}_1})$. Consequently, $g_U U_i = U_i$ for some group $U \in \overline{U}_4^{\bar{S}\bar{S}_1}$, that is, $g_U \in U_i$, contradiction. Thus, there is a group $U \in \bar{S} \setminus \bar{S}_1$ such that $U \cap F \subseteq U_i$. Condition 2) has been proven.

Let us show that condition 3) is satisfied. Assume the converse. Then there is $S \in \bar{S}$ such that for every U_i, $1 \leq i \leq k$, there exists a group $R_i \in \bar{S}$ satisfying the relation $S \cap F \not\subseteq R_i \cup U_i$, or there exists a group $Q_i \in \bar{S}$ satisfying the relation $Q_i \cap F \cap U_i \not\subseteq S$. Let K_1 be a subset of $\{1, \ldots, k\}$ such that $i \in K_1$ if and only if the group R_i is defined, and let K_2 be a subset of $\{1, \ldots, k\}$ such that $i \in K_2$ if and only if the group Q_i is defined. Let $\bar{S}_1$ denote the set $\{R_i \mid i \in K_1\}$, and $\bar{S}_2$ denote the set $\{Q_i \mid i \in K_2\}$. Then $\overline{U}_1^{\bar{S}\bar{S}_1 S} = \bar{S}_1$, $\overline{U}_2^{\bar{S}\bar{S}_2 S} = \bar{S}_2$. For every $i \in K_1$ we choose $g_i \in (S \cap F) \setminus (R_i \cup U_i)$ and for every $i \in K_2$ we choose $r_i \in (Q_i \cap F \cap U_i) \setminus S$. We define the functions $\varphi_1^{\bar{S}\bar{S}_1 S} : \overline{U}_1^{\bar{S}\bar{S}_1 S} \to G$ and $\varphi_2^{\bar{S}\bar{S}_2 S} : \overline{U}_2^{\bar{S}\bar{S}_2 S} \to G$ as follows: $\varphi_1^{\bar{S}\bar{S}_1 S}(R_i) = g_i$ for every group $R_i \in \overline{U}_1^{\bar{S}\bar{S}_1 S}$, and $\varphi_2^{\bar{S}\bar{S}_2 S}(Q_i) = g_i$ for every group $Q_i \in \overline{U}_2^{\bar{S}\bar{S}_2 S}$. Since $_GB \vDash \Theta$, then $_GB \vDash \gamma_{\bar{S}}(\bar{S}_1, \bar{S}_2, S, \varphi_1^{\bar{S}\bar{S}_1 S}, \varphi_2^{\bar{S}\bar{S}_2 S})$. Consequently, there exists U_{i_0}, $i_0 \in \{1, \ldots, k\}$, such that $g_i \in U_{i_0}$ for every $i \in K_1$ and $r_i \notin U_{i_0}$ for every $i \in K_2$. If $i_0 \in K_1$ then $g_{i_0} \in U_{i_0}$, contradiction. If $i_0 \in K_2$ then $r_{i_0} \notin U_{i_0}$, contradiction. Condition 3) has been proven.

Let us show that condition 4) is satisfied. Let $\bigcup \bar{S} \notin \bar{S}$. Assume the converse, that is, there is a group U_i such that $\bigcup \bar{S} \cap U_i \cap F \subset U$, that is, $U \in \overline{U}_3^{\bar{S}}$, for every group $U \in \bar{S}$. Therefore, for every group $U \in \bar{S}$ there exists $g_U \in (\bigcup \bar{S} \cap U_i \cap F) \setminus U$. We define the function $\varphi_3^{\bar{S}} : \overline{U}_3^{\bar{S}} \to G$ as follows: $\varphi_3^{\bar{S}}(U) = g_U$. Since $_G B \vDash \Theta$ and $\bigcup \bar{S} \notin \bar{S}$, then $_G B \vDash \delta_{\bar{S}}(\varphi_3^{\bar{S}})$. Then for the group U_i there is a group $U \in \overline{U}_3^{\bar{S}}$ such that $g_U U_i \neq U_i$, that is, $g_U \notin U_i$, contradiction. Thus, condition 4) is proven.

Let us show that condition 5) is satisfied. Assume the converse. Then there exist a set $\bar{S}_1 \subset \bar{S}$ and a group U_i with the following properties: $\bigcup \bar{S}_1 \notin \bar{S}$, for every group $U \in \bar{S}$ the set $(\bigcup \bar{S}_1 \cap U_i \cap F) \setminus U$ is nonempty and for every group $U' \in \bar{S}$ such that $(U' \cap F) \setminus \bigcup \bar{S}_1 \neq \varnothing$, we have $(U' \cap F) \setminus \bigcup \bar{S}_1 \nsubseteq U_i$. Therefore, $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$ for every group $U \in \bar{S}$, and $(U' \cap F) \setminus (\bigcup \bar{S}_1 \cup U_i) \neq \varnothing$ for every group $U' \in \bar{S}$ such that $(U' \cap F) \setminus \bigcup \bar{S}_1 \neq \varnothing$, that is, for every group $U' \in \overline{U}_6^{\bar{S}\bar{S}_1}$. For every $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$ we choose the element $g_U \in (\bigcup \bar{S}_1 \cap U_i \cap F) \setminus U$, and for every $U' \in \overline{U}_6^{\bar{S}\bar{S}_1}$ we choose the element $r_{U'} \in (U' \cap F) \setminus (\bigcup \bar{S}_1 \cup U_i)$. Note that $g_U U_i = U_i$ and $r_U U_i \neq U_i$. Functions $\varphi_5^{\bar{S}\bar{S}_1} : \overline{U}_5^{\bar{S}\bar{S}_1} \to G$ and $\varphi_6^{\bar{S}\bar{S}_1} : \overline{U}_6^{\bar{S}\bar{S}_1} \to G$ is defined as follows: $\varphi_5^{\bar{S}\bar{S}_1}(U) = g_U$ for every $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$ and $\varphi_6^{\bar{S}\bar{S}_1}(U') = r_{U'}$ for every $U' \in \overline{U}_6^{\bar{S}\bar{S}_1}$. Since $_G B \vDash \Theta$, $\bar{S}_1 \subset \bar{S}$ and $\bigcup \bar{S}_1 \notin \bar{S}$, then $_G B \vDash \varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5^{\bar{S}\bar{S}_1}, \varphi_6^{\bar{S}\bar{S}_1})$. Consequently, there exists a group $U \in \overline{U}_6^{\bar{S}\bar{S}_1}$ such that $r_U U_i = U_i$. Contradiction. Thus, condition 5) is proven.

$(b) \implies (c)$ Assume that conditions 1)-5) of Theorem are satisfied and $K_{G,\bar{S}} \vDash \Theta$. By Lemma 1 and Corollary 1, $\Gamma_{\bar{S}} \vdash \Theta$. Then there is a finite subset Γ of the set $\Gamma_{\bar{S}}$ such that $\Gamma \vdash \Theta$. Let F denote a finite subset of G such that Θ is a formula in language L_F. By condition of Theorem, for F there exist finite index subgroups $U_1, \ldots, U_k$ of the group G such that conditions 1)-5) are satisfied. Let $_G B = {}_G G/U_1 \sqcup \ldots \sqcup {}_G G/U_k$. To prove that the class $K_{G,\bar{S}}$ is T-pseudofinite, we will show that $_G B \vDash \Theta$. It is enough to prove that $_G B \vDash \Gamma$.

Let $\alpha_{\bar{S}}(g) \in \Gamma$, where $g \in F$ and $g \notin \bigcup \bar{S}$. By condition 1), $g \notin U_i$ for every i, $1 \leq i \leq k$. Therefore, $_G B \vDash \alpha_{\bar{S}}(g)$.

Let $\beta_{\bar{S}}(\bar{S}_1, g, \varphi_4^{\bar{S}\bar{S}_1}) \in \Gamma$, where $\bar{S}_1 \subset \bar{S}$, $\varphi_4^{\bar{S}\bar{S}_1} : \overline{U}_4^{\bar{S}\bar{S}_1} \to G$, $\varphi_4^{\bar{S}\bar{S}_1}(U) \in U \cap F$ for every $U \in U_4^{\bar{S}\bar{S}_1}$, $g \in F \setminus \bigcup \bar{S}_1$. Assume that $g U_i = U_i$. Then we have $g \in (F \cap U_i) \setminus \bigcup \bar{S}_1$, that is, $(F \cap U_i) \setminus \bigcup \bar{S}_1 \neq \varnothing$. By condition 2), there is a group $U \in \bar{S} \setminus \bar{S}_1$ such that $U \cap F \subseteq U_i$. Consequently, $U \in U_4^{\bar{S}\bar{S}_1}$ and $\varphi_4^{\bar{S}\bar{S}_1}(U) \in U \cap F \subseteq U_i$, that is, $\varphi_4^{\bar{S}\bar{S}_1}(U) U_i = U_i$. Thus, we proved that $_G B \vDash \beta_{\bar{S}}(\bar{S}_1, g, \varphi_4^{\bar{S}\bar{S}_1})$.

Let $\gamma_{\bar{S}}(\bar{S}_1, \bar{S}_2, S, \varphi_1^{\bar{S}\bar{S}_1 S}, \varphi_2^{\bar{S}\bar{S}_2 S}) \in \Gamma$, where $\bar{S}_1, \bar{S}_2 \subset \bar{S}$, $S \in \bar{S}$, $\varphi_1^{\bar{S}\bar{S}_1 S} : \overline{U}_1^{\bar{S}\bar{S}_1 S} \to G$, $\varphi_2^{\bar{S}\bar{S}_2 S} : \overline{U}_2^{\bar{S}\bar{S}_2 S} \to G$. By condition 3), there exists U_{i_0}, $i_0 \in \{1, \ldots, k\}$, such that $S \cap F \subseteq U \cup U_{i_0}$ and $U \cap F \cap U_{i_0} \subseteq S$ for every $U \in \bar{S}$.

We show that

$$_G B \vDash \bigwedge_{U \in \overline{U}_1^{\bar{S}\bar{S}_1 S}} (\varphi_1^{\bar{S}\bar{S}_1 S}(U) U_{i_0} = U_{i_0}) \wedge \bigwedge_{U \in \overline{U}_2^{\bar{S}\bar{S}_2 S}} (\varphi_2^{\bar{S}\bar{S}_2 S}(U) U_{i_0} \neq U_{i_0}).$$

Let $U \in \overline{U}_1^{\bar{S}\bar{S}_1 S}$. Then $S \setminus U \neq \varnothing$. Besides, $\varphi_1^{\bar{S}\bar{S}_1 S}(U) \in (S \cap F) \setminus U$. Therefore, by condition 3), $\varphi_1^{\bar{S}\bar{S}_1 S}(U) \in Q \cup U_{i_0}$ for every $Q \in \bar{S}$, in particular, $\varphi_1^{\bar{S}\bar{S}_1 S}(U) \in U \cup U_i$, that is, $\varphi_1^{\bar{S}\bar{S}_1 S}(U) \in U_{i_0}$. Let $U \in \overline{U}_2^{\bar{S}\bar{S}_2 S}$. Then $U \setminus S \neq \varnothing$. Besides, $\varphi_2^{\bar{S}\bar{S}_1 S}(U) \in (U \cap F) \setminus S$. If $\varphi_2^{\bar{S}\bar{S}_2 S}(U) \in U_{i_0}$ then $\varphi_2^{\bar{S}\bar{S}_2 S}(U) \in U \cap F \cap U_i$, that is, $\varphi_2^{\bar{S}\bar{S}_2 S}(U) \in S$, contradiction. Therefore, $\varphi_2^{\bar{S}\bar{S}_2 S}(U) \notin U_{i_0}$. Thus, $_G B \vDash \gamma_{\bar{S}}(\bar{S}_1, \bar{S}_2, S, \varphi_1^{\bar{S}\bar{S}_1 S}, \varphi_2^{\bar{S}\bar{S}_2 S})$.

Let $\delta_{\bar{S}}(\varphi_3^{\bar{S}}) \in \Gamma$, where $\bigcup \bar{S} \notin \bar{S}$, $\varphi_3^{\bar{S}} : \overline{U}_3^{\bar{S}} \to G$ and $1 \leq i \leq k$. By condition 4), there exists a group $U \in \bar{S}$ such that $\bigcup \bar{S} \cap U_i \cap F \subseteq U$. Since $\bigcup \bar{S} \notin \bar{S}$ and $U \in \bar{S}$, then $\bigcup \bar{S} \setminus U \neq \varnothing$, that is, $U \in \overline{U}_3^{\bar{S}}$. So $\varphi_3^{\bar{S}}(U) \in (\bigcup \bar{S} \cap F) \setminus U$ and $\varphi_3^{\bar{S}}(U) \notin U_i$. Therefore, $_G B \vDash \delta_{\bar{S}}(\varphi_3^{\bar{S}})$.

Let $\varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5^{\bar{S}\bar{S}_1}, \varphi_6^{\bar{S}\bar{S}_1}) \in \Gamma$, where $\bar{S}_1 \subset \bar{S}$, $\bigcup \bar{S}_1 \notin \bar{S}$, $\varphi_5^{\bar{S}\bar{S}_1} : \overline{U}_5^{\bar{S}\bar{S}_1} \to G$, $\varphi_6^{\bar{S}\bar{S}_1} : \overline{U}_6^{\bar{S}\bar{S}_1} \to G$ and $1 \leq i \leq k$. Suppose that $\varphi_5^{\bar{S}\bar{S}_1}(U) U_i = U_i$ for every $U \in U_5^{\bar{S}\bar{S}_1}$. Since $\bigcup \bar{S}_1 \notin \bar{S}$, then $\bigcup \bar{S}_1 \setminus U \neq \varnothing$ and $U \in \overline{U}_5^{\bar{S}\bar{S}_1}$ for every $U \in \bar{S}_1$. Since $\varphi_5^{\bar{S}\bar{S}_1}(U) \in U_i \cap F$ and $\varphi_5^{\bar{S}\bar{S}_1} \in \bigcup \bar{S}_1 \setminus U$, then $(\bigcup \bar{S}_1 \cap U_i \cap F) \setminus U \neq \varnothing$ for every $U \in \bar{S}_1$. By condition 5), there exists $U \in \bar{S}$ such that $(U \cap F) \setminus \bigcup \bar{S}_1 \neq \varnothing$ and $(U \cap F) \setminus \bigcup \bar{S}_1 \subseteq U_i$. Since $\varphi_6^{\bar{S}\bar{S}_1}(U) \in (U \cap F) \setminus \bigcup \bar{S}_1$, then $\varphi_6^{\bar{S}\bar{S}_1}(U) \in U_i$, that is, $\varphi_6^{\bar{S}\bar{S}_1}(U) U_i = U_i$. Thus, $_G B \vDash \varepsilon_{\bar{S}}(\bar{S}_1, \varphi_5^{\bar{S}\bar{S}_1}, \varphi_6^{\bar{S}\bar{S}_1})$. $\square$

Corollary 2. *Let G be an abelian group, $S_1, \ldots, S_n$ be pairwise distinct subgroups of the group G, $\bar{S} = \{S_1, \ldots, S_n\}$, and T be the theory of all G-acts. If for every subgroup $G_1 \in \bar{S}$ of group G and every finite set $F \subseteq G \setminus G_1$ there exists a finite index subgroup G_2 of group G such that $G_1 \subseteq G_2$ and $F \cap G_2 = \varnothing$, then the class $K_{G, \bar{S}}$ is T-pseudofinite.*

Proof. Let the conditions of Corollary 2 be satisfied, $S_1, \ldots, S_k \in \bar{S}$ be infinite index subgroups of G, $S_{k+1}, \ldots, S_n \in \bar{S}$ be finite index subgroups of G, and F be a finite subset of G. If $k = 0$, i.e. the set $\bar{S}$ has no infinite index subgroups of G, then by Fact 1 the class $K_{G, \bar{S}}$ is T-pseudofinite. Let $k \geq 1$, F be a finite subset of G. By the condition of Corollary 2, for every group S_i, $1 \leq i \leq k$, there exists a finite index subgroup L_i of G such that $S_i \subseteq L_i$ and $F_i \cap L_i = \varnothing$, where $F_i = F \setminus S_i$. Let us prove that conditions 1)-5) of Theorem 2 are satisfied for the sets F and $\overline{U}$, where $\overline{U} = \{L_1, \ldots, L_k, S_{k+1}, \ldots, S_n\}$.

Let $f \in F \cap \bigcup \overline{U}$. If $f \in S_i$ for some $i \in \{k+1, \ldots, n\}$, then $f \in \bigcup \overline{S}$. If $f \in L_j$ for some $j \in \{1, \ldots, k\}$, then by construction of the group L_j, we have $S_j \subseteq L_j$ and $F_j \cap L_j = \varnothing$, that is, $f \in S_j$ and $f \in \bigcup \bar{S}$. Thus, $F \cap \bigcup \overline{U} \subseteq \bigcup \bar{S}$ and condition 1) is satisfied.

Let $\bar{S}_1 \subset \bar{S}$ and $U \in \overline{U}$ such that $(F \cap U) \setminus \bigcup \bar{S}_1 \neq \varnothing$. If $U = S_i$ for some $i \in \{k+1, \ldots, n\}$, then $U \notin \bar{S}_1$ and $U \cap F \subseteq S_i$. Let $U = L_j$ for

some $j \in \{1, \ldots, k\}$. By construction of the group L_j, we have $S_j \subseteq L_j$ and $F_j \cap L_j = \varnothing$. Therefore, $S_j \cap F = L_j \cap F$, $(F \cap S_j) \setminus \bigcup \bar{S}_1 \neq \varnothing$, $S_j \in \bar{S} \setminus \bar{S}_1$, and $S_j \cap F \subseteq L_j$. Thus, condition 2) is satisfied.

Let $S \in \bar{S}$. If $S = S_i \in \overline{U}$ for some $i \in \{k+1, \ldots, n\}$, then $S \cap F \subseteq U \cup S_i$ and $U \cap F \cap S_i \subseteq S$ for every $U \in \bar{S}$. Let $S = S_j$ for some $j \in \{1, \ldots, k\}$. By construction of the group L_j, we have $S_j \subseteq L_j$ and $F_j \cap L_j = \varnothing$. Then $S \cap F = L_j \cap F \subseteq U \cup L_j$ and $U \cap F \cap L_j \subseteq S$ for every $U \in \bar{S}$. Thus, condition 3) is satisfied.

Let $U \in \overline{U}$. If $U = S_i \in \bar{S}$ for some $i \in \{k+1, \ldots, n\}$, then $\bigcup \bar{S} \cap U \cap F \subseteq S_i$. Let $U = L_j$ for some $j \in \{1, \ldots, m\}$. By construction of the group L_j, we have $S_j \subseteq L_j$ and $F_j \cap L_j = \varnothing$. Then $S_j \cap F = L_j \cap F$ and $\bigcup \bar{S} \cap U \cap F \subseteq S_j$. Thus, condition 4) is satisfied.

Let $\bar{S}_1 \subset \bar{S}$ and $U \in \overline{U}$. If $U = S_i$ for some $i \in \{k+1, \ldots, n\}$, then $U \in \bar{S}$ and $(\bigcup \bar{S}_1 \cap U \cap F) \setminus S_i = \varnothing$. Let $U = L_j$ for some $j \in \{1, \ldots, k\}$. By construction of the group L_j, we have $S_j \subseteq L_j$ and $F_j \cap L_j = \varnothing$. Then $S_j \cap F = L_j \cap F$ and $(\bigcup \bar{S}_1 \cap U \cap F) \setminus S_j = \varnothing$. Therefore, condition 5) is satisfied.

Thus, by Theorem 2, the class $K_{G,\bar{S}}$ is T-pseudofinite. $\square$

By Fact 1 every coproduct of finite S-acts is a T-pseudofinite S-act, where T is the theory of all S-acts. Then we have

Remark 1. *Let G be a group and $\bar{S}$ consists of finite index subgroups of G. Then the class $K_{G,\bar{S}}$ is T-pseudofinite.*

The following examples show that the T-pseudofiniteness of the class $K_{G,\bar{S}}$ depends not only on the group G, but also on the choice of the set $\bar{S}$ of subgroups of G.

Example 1. *Let $G = \mathbb{C}_{p^\infty} \times \mathbb{C}_p$ and T be the theory of all G-acts. The unique finite index subgroups of G are G and $G_1 = \mathbb{C}_{p^\infty} \times \{1\}$.*

1) If $\bar{S}_1 = \{G_1\}$ then by Remark 1 the class $K_{G,\bar{S}_1}$ is T-pseudofinite.

2) If $\bar{S}_2 = \{G_2\}$, where $G_2 = \{(a,1) \mid a \in \mathbb{C}_{p^\infty}, a^p = 1\}$, then the class $K_{G,\bar{S}_2}$ is not T-pseudofinite. Indeed, $\coprod_{i \in I} {}_G(G/G_2)_i \vDash \Phi$ for every nonempty I, where $\Phi \leftrightharpoons \forall x \neg((b,1)x = x)$, $b^{p^2} = 1$, $b^p \neq 1$, $b \in \mathbb{C}_{p^\infty}$, but ${}_G G/G \vDash \neg\Phi$ and ${}_G G/G_1 \vDash \neg\Phi$.

Proposition 1. *Let $\mathbb{Q}^\star$ be a multiplicative group of rational numbers, $S_1, \ldots, S_n$ be pairwise distinct subgroups of the group $\mathbb{Q}^\star$, $\bar{S} = \{S_1, \ldots, S_n\}$ and T be the theory of all $\mathbb{Q}^\star$-acts. Then $K_{\mathbb{Q}^\star,\bar{S}}$ is T-pseudofinite.*

Proof. Let the conditions of Proposition be satisfied, $G \in \bar{S}$ be a infinite index subgroup of $\mathbb{Q}^\star$, and $F = F_1 \cup F_2$ be a finite subset of $\mathbb{Q}^\star$, where $F_1 \cap G = \varnothing$ and $F_2 \subseteq G$. By Zorn's Lemma, there is a maximal subgroup U of the group $\mathbb{Q}^\star$ with the property: $F_1 \cap U = \varnothing$ and $F_2 \subseteq U$. By Corollary 2, to prove Proposition it is sufficient to prove that U is a finite index subgroup of the group $\mathbb{Q}^\star$.

Note that every positive rational number, other than one, can be represented as $p_1^{k_1} \cdot \ldots \cdot p_n^{k_n}$, where $p_1 \ldots p_n$ are pairwise distinct prime numbers, $k_1, \ldots, k_n$ are nonzero integers. For a prime number p and a rational number a we will write $p \mid a$, if $|a| = p^k \cdot p_1^{k_1} \cdot \ldots \cdot p_n^{k_n}$, $p, p_1 \ldots p_n$ are pairwise distinct prime numbers, $k, k_1, \ldots, k_n$ are nonzero integers. By construction of the group U, we have $p \in U$ for every prime p such that $p \nmid a$ for all $a \in F_1$.

Let us prove that for every prime p, if $p \mid a$ for some $a \in F_1$, then there exists $k > 0$ such that $p^k \in U$. Assume the opposite. Let p be prime number, $a \in F_1$, $p \mid a$ and $p^k \notin U$ for every $k \neq 0$. There exists a maximal positive n such that $b = p^{\varepsilon n} \cdot c$ for some $b \in F_1$, $c \in \mathbb{Q}^\star$, $p \nmid c$, $\varepsilon \in \{1, -1\}$. By U_j, $j \geq 1$, we denote the group $\{p^{(n+j)i}d \mid d \in U, i \in \mathbb{Z}\}$. It is clear that $U \subset U_j \subset \mathbb{Q}^\star$ for every $j \geq 1$. Since U is maximal subgroup of the group $\mathbb{Q}^\star$ with the property: $F_1 \cap U = \varnothing$ and $F_2 \subseteq U$, then for every $j \geq 1$ there exists $f_j \in F_1 \cap U_j$, that is, $f_j = p^{(n+j)i_j}d_j$ for some $d_j \in U$, $i_j \in \mathbb{Z}$. Since the set F_1 is finite, then there exist $0 < j_1 < j_2 < \ldots$ such that $f_{j_1} = f_{j_2} = \ldots$. If $d_{j_k} \neq d_{j_l}$ for some different j_k and j_l, then $d_{j_k} = p^s d_{j_l}$ for some $s \neq 0$; therefore, $p^s \in U$, a contradiction. Hence $d_{j_k} = d_{j_l}$ for all j_k, j_l, that is, $(n + j_k)i_{j_k} = (n + j_l)i_{j_l}$ for all j_k, j_l. Since $0 < j_1 < j_2 < \ldots$, then $|i_{j_1}| > |i_{j_2}| > \ldots$, a contradiction. Since $f \notin U$, then $i \neq 0$. This contradicts the choice n. Thus, for every prime p, if $p \mid a$ for some $a \in F_2$, then there exists $k_p > 0$ such that $p^{k_p} \in U$.

Let us prove that U is a finite index subgroup of the group $\mathbb{Q}^\star$. By P we denote a set of prime numbers p such that $p \mid f$ for some $f \in F_1$. It is clear that P is a finite set. Let $U' = \{\prod_{p \in P} p^{k_p i}a \mid i \in \mathbb{Z}, a \in \mathbb{Q}^\star, \forall p \in P(p \nmid a)\}$. Then U' is a finite index subgroup of the group $\mathbb{Q}^\star$, $U' \subseteq U$. Therefore, U is a finite index subgroup of the group $\mathbb{Q}^\star$. Proposition has been proven. $\quad\square$

Proposition 2. *Let G be a finitely generated abelian group, $S_1, \ldots, S_n$ be pairwise distinct subgroups of the group G, $\bar{S} = \{S_1, \ldots, S_n\}$, and T be the theory of all G-acts. Then the class $K_{G,\bar{S}}$ is T-pseudofinite.*

Proof. Let the conditions of Proposition be satisfied, $G_1 \in \bar{S}$ be a infinite index subgroup of G, and $F = F_1 \cup F_2$ be a finite subset of G, where $F_1 \subseteq G_1$ and $F_2 \cap G_1 = \varnothing$. By Zorn's Lemma, there is a maximal proper subgroup U of the group G with the property: $F_1 \subseteq U$ and $F_2 \cap U = \varnothing$. By Lemma 2, to prove Proposition it is sufficient to prove that U is a finite index subgroup of the group G. Let

$$G = U \cup \bigcup_{f \in F_2} (f + U) \cup \bigcup_{i \in \omega} (a_i + U),$$

where $a_i \notin U$ for all $i \in \omega$.

Let us prove that $f + U$ is an element of the factor group G/U of finite order for every $f \in F_2$. Suppose the opposite, that is, $nf + U \neq mf + U$ for some $f \in F_2$ and every $n, m \in \omega$, $n \neq m$. There exists $k \in \omega$ such that $f' \notin k'f + U$ for all $f' \in F_2$ and $|k'| \geq k$. Then for the group U' generated by the set $U \cup \{kf\}$, we have $U \subset U' \subset G$. By the construction of group U,

$f' \in U'$ for some $f' \in F_2$, that is, $f' \in lkf + U$. Since $f \notin U$ then $l \neq 0$. This contradicts the choice of k. Thus, $f + U$ is an element of the factor group G/U of finite order for every $f \in F_2$.

Let us prove that $a_i + U$ is an element of the factor group G/U of finite order for every $i \in \omega$. By U'' we denote the group generated by the set $U \cup \{a_i\}$. By the construction of group U, we have $f \in U''$ for some $f \in F_2$, that is, $f \in la_i + U$. Since $f + U$ is an element of the factor group G/U of finite order then $a_i + U$ is an element of the factor group G/U of finite order too.

Note that G/U as a homomorphic image of a finitely generated abelian group is a finitely generated abelian group. Thus, G/U is a periodic finitely generated Abelian group. By Fact 3, G/H is a finite group, that is, U is a finite index group. Proposition has been proven. □

3 T-pseudofiniteness and pseudofiniteness of acts over abelian groups

From the facts that divisible groups, in particular the additive groups of rational numbers, of real numbers, of complex numbers, the multiplicative group of positive real numbers, a quasicyclic group, have no proper finite index subgroups, and the multiplicative group of real numbers has exactly two finite index subgroups, and from Fact 2 we obtain Propositions 3 and 4:

Proposition 3. *Let G be a divisible group and T be the theory of all G-acts. Then G-act $_GA$ is T-pseudofinite if and only if $_GA$ is a coproduct of one-element G-acts.*

Proposition 4. *Let G be the multiplicative group of real numbers and T be the theory of all G-acts. Then the G-act $_GA$ is T-pseudofinite if and only if $_GA$ is a coproduct of one-element and two-element G-acts.*

Proposition 5. *Let G be a locally finite group. Then every G-act is pseudofinite.*

Proof. Let G be a locally finite group, $_GA$ be a G-act, $_GA \vDash \Theta$, where Θ is a formula in the language L_G, and the set F consists of all symbols from L_G, that occur in Θ. By the condition of Proposition, the subgroup G_1 of the group G generated by F is a finite group. Then the G_1-act $_{G_1}A$, which is the restriction of the G-act $_GA$ to the language $L_{G_1} \subseteq L_G$, is a coproduct of finite G_1-acts and $_{G_1}A \vDash \Theta$. By Fact 1, $_{G_1}A$ is T-pseudofinite, where T is the theory of all G_1-acts. Therefore, there exists finite G_1-act $_{G_1}B$ such that $_{G_1}B \vDash \Theta$. Let us arbitrarily extend the structure $_{G_1}B$ in the language L_{G_1} to the structure $_GB$ in the language L_G. Thus, $_GB \vDash \Theta$ and G-act $_GA$ is pseudofinite. □

Corollary 3. *Let G be a quasicyclic group. Then every G-act is pseudofinite.*

Since a quasicyclic group G is a divisible then the G-act $_GG$ is not T-pseudofinite (see Proposition 3), where T is the theory of all G-acts. So we get

Corollary 4. *There exists an abelian group G and a G-act ${}_G A$ such that ${}_G A$ is pseudofinite but not T-pseudofinite, where T is the theory of all G-acts.*

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