S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN ...

Vol. ..., No. ..., pp. ...–... (...) УДК ???.??
https://............ MSC ??X??

SOME PROPERTIES ON THE CLASS OF σ-UN-DUNFORD-PETTIS OPERATORS IN BANACH LATTICE

ISSAM OUBA , ELMOSTAFA BENDIB, LARBI ZRAOULA AND BOUAZZA EL WAHBI

Abstract: In this work, we studied more properties concerning the class of σ-un-Dunford-Pettis operators on Banach lattices. Furthermore, we introduce a new property that generalizes the Dunford-Pettis property, which we call the σ-un-Dunford-Pettis property. After that, we investigate proprieties about this new property. Finally, we present a necessary and sufficient condition under which for each σ-un-Dunford-Pettis operator is u-M-weakly compact and under which for each AM-σ-un-compact is σ-un-Dunford-Pettis operator.

Keywords: Order continuous, atomic, σ-un-Dunford-Pettis operator, u-M-weakly compact operator, AM-σ-un-compact.

1 Introduction and preliminaries

The notion of **σ-un-Dunford-Pettis** operators was introduced by N.Hafidi's et al in [1], which generalizes the class of Dunford-Pettis operators. Indeed, an operator $T : X \longrightarrow F$ from a Banach space X into a Banach lattice F is said to be sequentially un-Dunford-Pettis (abb, σ-un-DP) if $T(x_n) \xrightarrow{un} 0$ for every weakly null sequence (x_n) in X and an operator $T : X \longrightarrow Y$ from a Banach space X into a Banach space Y is called Dunford-Pettis if $T(x_n) \xrightarrow{\|\cdot\|} 0$ for every weakly null sequence (x_n) in X. Note that each Dunford-Pettis operator is σ-un-Dunford-Pettis, but the converse is not always true. In fact, let the identity operator Id_{ℓ^2}. Clearly, Id_{ℓ^2} is σ-un-Dunford-Pettis but it is not Dunford-Pettis. In their article, some properties of this class of operators have been studied. The purpose of this paper is to investigate additional properties of this classification of operators and to establish some results about the relations between the class of σ-un-Dunford-Pettis operators and other classes of operators as u-M-weakly-compact and AM-σ-un-compact.

- ➤ An operator T from a Banach lattice E into a Banach lattice F is called **unbounded M-weakly compact** (u-M-w-compact for short) if $T(x_n) \xrightarrow{un} 0$ for every norm bounded disjoint sequence (x_n) of E [9].

This work is organized as follows: In section 2 we present an extension of the characterization of σ-un-Dunford-Pettis operators obtained in Proposition 3.2 by N.Hafidi's [1] (Theorem 1) and we identify an additional condition under which every σ-un-Dunford-Pettis operator is σ-un-compact (Proposition 1), after that we present another characterization Banach lattices under which each σ-un-Dunford-Pettis operators $T : E \to E$ is σ-un-compact (res, σ-uaw-compact), T^2 σ-un-compact operators (Proposition 2) and for which each σ-un-Dunford-Pettis operators $T : E \to F$ is weakly compact (Theorem 2). Additionally, we present a sufficient condition for which for each σ-un-Dunford-Pettis operator T iff $|T|$ is σ-un-Dunford-Pettis (Proposition 3). On the other side, we introduce a new property that generalizes the Dunford-Pettis property (Definition 1). Following this, we present a result that each reflexive Banach lattice has σ-unDPP is finite dimensional (Theorem 5) and demonstrated that each Banach lattice has σ-unDPP whenever its dual has σ-unDPP (Proposition 6). In section 3, we establish the relationship between σ-un-Dunford-Pettis operators and u-M-weakly compact and as well AM-σ-un-compact (Proposition 9). We also present a necessary and sufficient condition under which for each σ-un-Dunford-Pettis operator is u-M-weakly compact (Theorem 7 and 8) and we present necessary conditions under which for each AM-σ-un-compact is σ-un-Dunford-Pettis operator (Proposition 9)

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|.\|$ of a Banach lattice E is order continuous if for each nets (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. The lattice operations in Banach lattice E are weakly sequentially continuous, if the sequence $(|x_n|)$ converges to 0 in the weak topology, whenever the sequence (x_n) converges weakly to 0 in E. A Banach lattice E is said to be KB-space, if every increasing norm bounded sequence of E^+ is norm convergent. A Banach lattice E is said to have the property (b) if each subset A of E is order bounded whenever it is b-order bounded. Note that, E is a KB-space iff E has the property (b) and order continuous norm. A nonzero element x of a vector lattice E is atomic if the order ideal generated by x equals the vector subspace generated by x. The vector lattice E is atomic (discrete), if it admits a complete disjoint system of discrete elements. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that inf(x,y)=0 we have $\|x+y\| = \max\{\|x\|\ ,\ \|y\|\}$. A Banach lattice E is said to be an AL-space if for each $x, y \in E^+$ such that inf(x,y)=0 we have $\|x+y\| = \|x\| + \|y\|$. A Banach space E has the positive Schur property if $\|x_n\| \longrightarrow 0$, for every (x_n) weak null in E^+. A Banach space X is said to be weakly sequentially complete whenever every weak Cauchy sequence of X converges weakly to some vector of X. A net (x_α) in a Banach lattice E is unbounded norm convergent (abb, un-norm convergent) to x if $\| |x_\alpha - x| \wedge u\| \longrightarrow 0$ (abb. $x_\alpha \xrightarrow{un} x$), for each $u \in E^+$. We note that the norm convergence implies the un-norm convergence and that un-norm convergence coincides with norm convergence on a Banach lattice with order unit. A net (x_α) in a Banach lattice E is unbounded absolutely weakly convergent (abb, uaw-convergent) to x if $|x_\alpha - x| \wedge u \xrightarrow{w} 0$ (abb. $x_\alpha \xrightarrow{w} x$), for each $u \in E^+$. We recall that a subset A of a Banach lattice E is said to be un-compact (res, sequentially un-compact), if every net (x_α) (respectively, every sequence (x_n)) in A has a subnet (respectively, subsequence) which is un-norm convergent. We recall that a subset A of a Banach lattice E is said to be uaw-compact (respectively, sequentially uaw-compact), if every net (x_α) (respectively, every sequence (x_n) in A has a subnet (respectively, subsequence) which is uaw-convergent. We will use the term operator $T : E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \in E^+$. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive then, its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

2 Space of σ-un-Dunford-Pettis operators

Let recall that in [6] an operator T from a Banach space X into a Banach lattice F is (sequentially) un-compact if $T(B_X)$ is relatively (sequentially) un-compact in F. Equivalently, for every bounded net (x_α) (respectively, every bounded sequence (x_n)) its image has a subnet (respectively, subsequence), which is un-convergent. and [2] an operator T from a Banach space X into a Banach space Y is called weakly compact whenever T carries the closed unit ball of X to a weakly relatively compact subset of Y.

In the following result, we present another characterization of σ-un-Dunford-Pettis operators. It is an extension of Proposition 3.2 obtained in [1].

Theorem 1. *Let $T : X \longrightarrow F$ be an operator from a Banach space X into a Banach lattice F, the following statements are equivalent:*

(1) *The operator T is σ-un-Dunford-Pettis.*
(2) *$T(A)$ is relatively σ-un-compact for each weakly compact set A of X.*

(3) *For an arbitrary Banach space Z and every weakly compact operator $R : Z \to X$, the operator TR is a σ-un-compact operator*
(4) *The operator TR is σ-un-compact for every weakly compact operator $R : \ell^1 \longrightarrow X$.*

Proof. 1) $\Longleftrightarrow$ 2) [1, Proposition 3.2.]
2) $\Rightarrow$ 3) Let B_Z be the closed unit ball of Z. Since R is weakly compact operator, implies that $R(B_Z)$ is weakly relatively compact. By assumption $TR(B_Z)$ is relatively σ-un-compact, it follows that TR is σ-un-compact.
3) $\Rightarrow$ 4) obvious.
4) $\Rightarrow$ 2) Let A be a weakly compact subset of X and (x_n) be a sequence in A. then the sequence (x_n) has subsequence (y_n) weakly convergent to y. Assume that $y = 0$ and consider the operator $R : \ell^1 \longrightarrow X$ defined by:

$$R(\lambda_1, \lambda_2, ...) = \sum_{n=1}^{\infty} \lambda_n y_n$$

From [2, Theorem 5.26] the operator R is weakly compact, and by hypothesis TR is σ-un-compact. Consider (e_n) the standard basis of ℓ^1, then the sequence $TR(e_n) = T(y_n)$ has a subsequence norm unbounded convergent in F. It follows that $T(A)$ is relatively sequentially un-compact. □

As consequence:

Corollary 1. *Let F be a Banach lattice. The following assertions are equivalent:*

(1) *F is atomic order continuous.*
(2) *Every weakly compact operator from an arbitrary Banach space X into F is σ-un-compact.*

Now, we use weak Cauchy sequences to offer another characterization of σ-un-Dunford-Pettis operators.

Proposition 1. *Let E and F be pair Banach lattices such that E is reflexive. Then the following assertions are equivalent:*

(1) *The operator $T : E \to F$ is σ-un-compact.*
(2) *The operator $T : E \to F$ is σ-un-Dunford-Pettis.*
(3) *The operator TR is σ-un-compact for every weakly compact operator $R : \ell^1 \longrightarrow E$.*
(4) *Every weak Cauchy sequence of E its image has a subsequence norm unbounded convergent in F.*

Proof. 1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Theorem 1.
3) $\Rightarrow$ 4) Let $T : E \to F$ and (x_n) be a weak Cauchy sequence of E. By [2, Theorem 4.70 and Theorem 4.60.] implies that E is a weakly sequentially complete, we infer that the sequence (x_n) is weakly convergent to x. Assume

that $x = 0$ and consider the operator $R : \ell^1 \to E$ given by :

$$R(\lambda_1, \lambda_2, ...) = \sum_{n=1}^{\infty} \lambda_n x_n$$

From [2, Theorem 5.26] the operator R is weakly compact and by hypotheses $T \circ R$ is σ-un-compact. Let (e_n) the standard basis of ℓ^1, thus $TR(e_n)$ has a norm unbounded convergent subsequence in F. Clearly $TR(e_n) = T(x_n)$ and hence $T(x_n)$ has a subsequence norm unbounded convergent in F.
4) $\Rightarrow$ 1) Let (x_n) be a norm bounded of E. Since E and E' are order continuous norm (because E is reflexive), implies that from [2, Theorem 4.25] (x_n) has a weak Cauchy subsequence $(x_{\varphi(n)})$ in E. By assumption we have $(T(x_{\varphi(n)}))$ has a subsequence norm unbounded convergent in F. It follows that T is σ-un-compact operator. □

Let us recall that an operator T from a Banach space X into a Banach lattice F is (sequentially) uaw-compact if $T(B_X)$ is relatively (sequentially) uaw-compact in F. Equivalently, for every bounded net (x_α) (respectively, every bounded sequence (x_n)), its image has a subnet (respectively, subsequence), which is uaw-convergent [10]. Next main result establishes a necessary and sufficient condition for each σ-un-Dunford-Pettis operator $T : E \longrightarrow E$ to be σ-un-compact (res, σ-uaw-compact, weakly compact,) and for T^2 to be σ-un-compact.

Proposition 2. *Let E be a Banach lattice such that E and its dual have order continuous norms, then the following statements are equivalent:*

(1) *Every σ-un-Dunford-Pettis operator $T : E \longrightarrow E$ is σ-un-compact.*
(2) *Every σ-un-Dunford-Pettis operator $T : E \longrightarrow E$ is σ-uaw-compact.*
(3) *Every σ-un-Dunford-Pettis operator $T : E \longrightarrow E$ is weakly compact.*
(4) *Every σ-un-Dunford-Pettis operator $T : E \longrightarrow E$, T^2 is σ-un-compact.*
(5) *E has property (b).*

Proof. 1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Let (x_n) be a norm bounded sequence in E. By hypothesis, T is σ-uaw-compact, we infer that the sequence (Tx_n) has a subsequence unbounded absolute weakly convergent to y in E. Since the norm of E' is order continuous, then from [13, Theorem 7], this subsequence of (Tx_n) is weakly convergent to y in E. It follows that T is weakly compact.
3) $\Rightarrow$ 4) see Theorem 1.
4) $\Rightarrow$ 5) Assume that E does not have property (b), which implies that it is not a KB-space. From the proof of Theorem 2 of Wnuk [11], it follows that E contain sublattice isomorphic to c_0, and there exists a positive projection $P : E \longrightarrow c_0$. It is clear that P is σ-un-Dunford-Pettis, but $P^2 = P$ is not σ-un-compact. If were false, the restriction P to the Banach lattice c_0 is mapping linear Id_{c_0} will be σ-un-compact, this is a contradiction.
5) $\Rightarrow$ 1) Let $T : E \to F$ be a σ-un-Dunford-Pettis operator. Since E has property (b) and an order continuous norm, then E is KB-space. Furthermore,

by assumption, E' is KB-space, which implies that E is reflexive. From [1, Corollary 3.2], we conclude that T is σ-un-compact. □

The results below present the necessary and sufficient conditions for each σ-un-Dunford-Pettis operator T between two Banach lattices to be weakly compact.

Theorem 2. *Let E and F be two Banach lattices such that E is KB-space. The following assertions are equivalent:*

(1) *Each σ-un-Dunford-Pettis operator T from E into F is weakly compact.*
(2) *One of the following conditions is valid:*
   (a) *The norm of E' is order continuous.*
   (b) *F is a reflexive.*

Proof. 1) $\Rightarrow$ 2) Assume that the norm of E' is not order continuous and F is not reflexive. we need to construct a σ-un-Dunford-Pettis operator T from E into F that is not weakly compact. Since E' does not have an order continuous norm, it follows from [7, Theorem 2.4.14] that there exists a sublattice of E which is isomorphic to ℓ^1 and a positive projection $P : E \longrightarrow \ell^1$. As F is not reflexive, then there exists a sequence $(y_n) \in B_F$ without any weakly convergent subsequences, where B_F is the closed unit ball of F.
Now, we consider the operator $Q : \ell^1 \longrightarrow F$ defined by

$$Q(x_1, x_2, ...) = \sum_{n=1}^{\infty} x_n y_n$$

We have $Q \circ P$ is σ-un-Dunford-Pettis (because $Q \circ P$ is Dunford-Pettis operator), but is not weakly compact. In fact, let the standard basis $(e_n) \in \ell^1$. We have the sequence (e_n) is norm bounded and $TP(e_n) = y_n$.
$2)a) \Rightarrow 1)$ Since E and E' are KB-space, then by [7, Theorem 2.4.15] E is reflexive. It follows that T is weakly compact.
$2)b) \Rightarrow 1)$ Since F is a reflexive, then T is weakly compact. □

Example 3. *Let $E = F = \ell^1$.*
The space ℓ^1 is KB-space and identity operator $Id : \ell^1 \to \ell^1$ is a σ-un-Dunford-Pettis operator but not weakly compact. However, ℓ^1 is not reflexive, and its dual does not have an order continuous norm.

As consequence:

Corollary 2. *Let E be a Banach lattice such that E is KB-space.*

(1) *Each σ-un-Dunford-Pettis operator $T : E \to E$ is weakly compact.*
(2) *E is a reflexive.*

Example 4. *We note that there exist an operator T which is σ-un-Dunford-Pettis from $C[0,1]$ into the AM-space σ-Dedekind complete F, but modulus $|T|$ exist and is not σ-un-Dunford-Pettis. in fact, we consider the operators which are mentioned in the* [8, Example 3.4]

$$\begin{array}{lll} T_n : & C[0,1] & \to F_n \\ & f & \to (T_n(f))(p) = 2^n \int_{B_n} f(t).sin(2\pi pt)dt (p \in \mathbb{N}). \end{array}$$

$$\begin{array}{lll} T : & C[0,1] & \to F \\ & f & \to T(f) = (T_1 f, \frac{1}{2} T_2 f, ...) \end{array}$$

Where F_n is the vector space ℓ^∞ with the norm $\|.\|_n$ defined by

$$\|(\lambda_1, \lambda_2, ...)\|_n = \|(\lambda_1, \lambda_2, ...)\|_\infty + n. \limsup_{k \in \mathbb{N}} |\lambda_k|$$

for each $(\lambda_n) \in \ell^\infty$.

$$B_n = [2^{-n}, 2^{-n+1}] \subset [0,1] \text{ for each } n \in \mathbb{N} \text{ and}$$

$$F = \{(x_n) : x_n \in F_n \text{ for each } n \text{ and the sequence } (\|x_n\|_n) \text{ is bounded}\}$$

, with the norm $\|(x_n)\| = sup\{\|x_n\|_n : n \in \mathbb{N}\}$.
The operator T_n is weakly compact. As $C[0,1]$ admits σ-un-DPP (Definition 1), then it's σ-un-Dunford-Pettis and its modulus $|T_n|$ exists defined by :

$$(|T_n|(f))(p) = 2^n \int_{B_n} f(t).|sin(2\pi pt)|dt (p \in \mathbb{N}; f \in C[0,1]).$$

The operator T is weakly compact. Since $C[0,1]$ admits σ-un-DPP, then it's σ-un-Dunford-Pettis and its modulus $|T|$ exists and is given by

$$|T|(f) = (\frac{1}{n}|T_n|(f))_{n \leq 1} \text{ (for } f \in C[0,1])$$

, but $|T|$ is not σ-un-Dunford-Pettis. Indeed, If were false, by [12, Corollary 3.20] *$|T|$ be weakly compact, and this is a contradiction.*

Now, we give a sufficient conditions under which T is a σ-un-Dunford-Pettis operator if and only if $|T|$ is a σ-un-Dunford-Pettis operator.

Proposition 3. *Let $T : E \longrightarrow F$ be an order bounded disjointness-preserving operator between two Banach lattices. T is a σ-un-Dunford-Pettis iff $|T|$ is a σ-un-Dunford-Pettis.*

Proof. Let (x_n) be a weakly null sequence in E. From [2, Theorem 2.40], we have

$$|Tx_n| = |T(|x_n|)| = |T|(|x_n|)$$

Since $|T|$ is also disjointness-preserving operator, then $||T|x_n| = ||T|||x_n| = |T||x_n| = |Tx_n|$. So for each $u \in E^+$,

$$||T|x_n| \wedge u = |Tx_n| \wedge u.$$

And hence T is σ-un-Dunford-Pettis operator if and only if $|T|$ is σ-un-Dunford-Pettis. ∎

We note that a weakly compact (resp, order weakly compact) operator is not necessarily σ-un-Dunford-Pettis and conversely is not true in general. In fact, the identity operator of $L_2[0,1]$ is weakly compact (resp, order weakly compact), but it's not σ-un-Dunford-Pettis (see [12, Proposition 3.1]). However the operator mentioned in [1, Remark 3.4] is σ-un-Dunford-Pettis, but it is not weakly compact and not order weakly compact.

In [2], a Banach space X is said to have Dunford-Pettis property (abb; DPP), if each weakly compact operator $T : X \longrightarrow Y$ is Dunford-Pettis for each Banach space Y. On the other hand, a Banach space X is said to have the reciprocal Dunford-Pettis property (abb; RDPP), if each Dunford-Pettis operator T from X to any Banach space Y is weakly compact.

Now, using the concept of unbounded norm convergence in Banach spaces, we introduce a new property that generalizes the Dunford-Pettis property, which we call the σ-un-Dunford-Pettis property.

Definition 1.

(1) *A Banach space X is said to have the σ-un-Dunford-Pettis property (abb, σ-unDPP), if each weakly compact operator T from X into F is σ-un-Dunford-Pettis for each Banach lattice F.*
(2) *A Banach space X is said to have the reciprocal σ-un-Dunford-Pettis property (abb, R-σ-unDPP), if each σ-un-Dunford-Pettis operator T from X into F is weakly compact for each Banach lattice F.*
(3) *A Banach lattice E is said to have the reciprocal o-σ-un-Dunford-Pettis property (abb, R-o-σ-unDPP), if each σ-un-Dunford-Pettis operator T from E into F is order weakly compact for each Banach lattice F.*

Observe that DPP is included in the σ-unDPP. In fact, let operator T from Banach space E from arbitrary Banach lattice F weakly compact. Since E has the Dunford-Pettis property, then T is a Dunford-Pettis, which implies that T is σ-un-Dunford-Pettis.

The following table presents examples of spaces that satisfy the σ-unDPP, R-σ-unDPP, and o-σ-unDPP, along with their justifications.

Space	**Justification**
ℓ^1 and c_0 have σ-unDPP	ℓ^1 is AL-space and c_0 is AM-space (see [2, Theorem 5.85])
$L^2[0,1]$ doesn't have σ-unDPP	$Id_{L^2[0,1]}$ is weakly compact but not σ-un-Dunford-Pettis
c_0 doesn't have R-σ-unDPP	Id_{c_0} σ-un-Dunford-Pettis but not weakly compact
ℓ^1 has the o-σ-unDPP	Every operator $T : \ell^1 \to F$ is order weakly compact and σ-un-Dunford-Pettis
$L^2[0,1]$ doesn't have o-σ-unDPP	$Id_{L^2[0,1]}$ is order weakly compact but not σ-un-Dunford-Pettis

It's clearly if E has the o-σ-unDPP, then also has the σ-unDPP.

As consequence of Proposition 1, we have the following corollary

Corollary 3.
Let E be a Banach lattice has σ-unDPP. If $T : E \longrightarrow E$ is a weakly compact operator, then T^2 is σ-un-compact.

Recall from [2] that an operator T from a Banach lattice E into a Banach lattice F is called M-weakly compact if $||T(x_n)|| \longrightarrow 0$ for every norm bounded disjoint sequence (x_n) of E. Clearly every M-weakly compact operator is u-M-weakly compact.

Proposition 4. (1) *If E atomic order continuous and E' has the positive Schur property, then E has the σ-unDPP.*

(2) *If the norm of E is order continuous has o-σ-unDPP, then E is weakly sequentially continuous.*

Proof. (1) Let $T : E \to F$ be a weakly compact. Since E' has the positive Schur property, by [5, Theorem 3.3] T is M-weakly-compact, which implies that T is u-M-weakly-compact. As E atomic order continuous, from Proposition 9, we infer that T is σ-un-Dunford-Pettis.
(2) Obvious.

□

As AL- and AM-spaces has the σ-un-Dunford-Pettis property (see [2, Theorem 5.85]), we have the following proposition:

Proposition 5. (1) *Let E and F be pair Banach lattices. Then, the adjoint of each σ-un-Dunford-Pettis operator T from E into F is σ-un-Dunford-Pettis, if one of the following conditions is valid:*

(a) *F is an AL-space or AM-space and E has the R-σ-unDPP.*

(b) *E has the R-σ-unDPP and E' atomic*

(2) *Let E be an AL-space or AM-space and F' a Banach lattice has R-σ-unDPP. Then each operator $T : E \longrightarrow F$ is σ-un-Dunford-Pettis whenever its adjoint operator $T' : F' \longrightarrow E'$ is σ-un-Dunford-Pettis.*

The following result is a generalization of [2, Theorem 5.83].

Theorem 5. *Let E be a reflexive Banach lattice. If E has the σ-un-Dunford-Pettis property, then E is finite dimensional.*

Proof. Let E be a reflexive Banach lattice. Then Id_E is weakly compact. Assume that E has the σ-un-Dunford-Pettis property and not finite dimensional. By [6, Proposition 6.2] E is not un-complete. Let (x_n) be a un-Cauchy, from [13, Theorem 10] the sequence (x_n) is weakly convergent. Since Id_E is a σ-un-Dunford-Pettis, it follows that (x_n) is unbounded norm convergent, which is impossible, thus E is finite dimensional. □

As a consequence, we obtain the following characterization:

Corollary 4. *Let E be a Banach lattice, then the following assertion are equivalent:*

(1) *E is reflexive with σ-un-Dunford-Pettis property.*
(2) *E is finite dimensional.*
(3) *E is reflexive with Dunford-Pettis property.*

Proposition 6. *Let X be a Banach space. then X has the σ-un-Dunford-Pettis property whenever the dual of X has the σ-un-Dunford-Pettis property.*

Proof. Let Q and T be two weakly compact operators such that $\ell^1 \xrightarrow{Q} X \xrightarrow{T} F$, where X is a Banach space and F is a Banach lattice. This implies that Q' and T' are weakly compact and we have $F' \xrightarrow{T'} X' \xrightarrow{Q'} \ell^\infty$. As X' has the σ-un-Dunford-Pettis property, then Q' is σ-un-Dunford-Pettis operator. Consequently, $Q'T' = (TQ)'$ is a σ-un-compact operator (Theorem 1). Since ℓ^∞ has order unit, Thus $(TQ)'$ is compact operator and hence QT is compact, which implies that QT is σ-un-compact. We infer that from Theorem 1 the operator T is σ-un-Dunford-Pettis operator. it follows that the Banach lattice X has the σ-un-Dunford-Pettis property. □

Recall that from [2] an operator $T : E \to X$ from a Banach lattice to a Banach space is order weakly compact whenever $T[0, x]$ is a relatively weakly compact subset of X for each $x \in E^+$. Now, we will present another condition under which every positive σ-un-Dunford-Pettis operator it dominates is also a σ-un-Dunford-Pettis operator.

Proposition 7. *Let E and F be two Banach lattices such that E has the o-σ-unDPP and S, T be operators defined from E to F with $0 \le S \le T$. If T is σ-un-Dunford-Pettis, then S is σ-un-Dunford-Pettis.*

Proof. Let T and S be two operators from E into F such that $0 \le S \le T$ and T is σ-un-Dunford-Pettis, then T is order weakly compact, which implies that S is also order weakly compact. Since E has the o-σ-unDPP, it follows that S is σ-un-Dunford-Pettis. □

3 The relationships between σ-un-Dunford-Pettis and some other operators.

In this section commences our examination of the connections between the σ-un-Dunford-Pettis and u-M-weakly compact operators.

We note that a σ-un-Dunford-Pettis operator is not necessarily u-M-weakly compact. However a u-M-weakly compact is not σ-un-Dunford-Pettis in general. Indeed, the canonical injection $i : l^1 \longrightarrow l^\infty$ is σ-un-Dunford-Pettis operator, but it's not u-M-weakly compact. In fact, consider the standard basis $(e_n) \in l^1$, we have $\|i(e_n)\| = \|e_n\| = 1$, since (e_n) is order bounded in l^∞, then by [6, Proposition 2.3] (e_n) is not un-convergent to 0. Additionally, the identity operator of $L^2[0,1]$ is u-M-weakly compact. In fact, let (x_n) norm bounded disjoint sequence, implies $x_n \xrightarrow{uaw} 0$ in $L^2[0,1]$, and we have the norm of $L^2[0,1]$ is order continuous, then from [13, Theorem 4] implies $id(x_n) = x_n \xrightarrow{un} 0$. However, $id_{L^2[0,1]}$ is not σ-un-Dunford-Pettis operator (see [12, Proposition 3.1]).

In terms of sequentially un-compact sets, we give a new result about u-M-weakly compact operators as follows:

Lemma 1. *Let E and F be two Banach lattices. If $T : E \longrightarrow F$ is an u-M-weakly compact operator, then $T(A)$ is relatively sequentially un-compact set in F for each relatively sequentially un-compact set A in E.*

Proof. Let A be a relatively sequentially un-compact set in E, then (x_n) has a unbounded norm convergent subsequence (x_{n_k}) in E. From [6, Theorem 1.1] there exist a subsequence $(x_{n_{k'}})$ of (x_{n_k}) and a bounded disjoint sequence $(d_{k'})$ in E such that $x_{n_{k'}} - d_{k'} \xrightarrow{\|\cdot\|} 0$, implies $T(x_{n_{k'}}) - T(d_{K'}) \xrightarrow{un} 0$. Since T is u-M-weakly-compact, we have $T(d_{k'}) \xrightarrow{un} 0$, from the following inequality: for $u \in F^+$

$$|T(x_{n_{k'}})| \wedge u \quad \leq \quad |T(x_{n_{k'}}) - T(d_{k'})| \wedge u + |T(d_{k'})| \wedge u$$

we infer that $T(x_{n_{k'}}) \xrightarrow{un} 0$. So $T(A)$ is a relatively sequentially un-compact set of F.

□

Proposition 8. *Let E and F be a Banach lattices such that E is an atomic KB-space. if $T : E \longrightarrow F$ is an u-M-weakly compact operator then T is sequentially un-compact.*

Proof. Let B_E be the closed unit ball of E. Since E is atomic KB-space, from [6, Theorem 7.5] B_X is sequentially un-compact. As T is u-M-weakly compact, by Lemma 1 $T(B_E)$ is sequentially un-compact, it follows that T is sequentially un-compact. □

The next result, gives a sufficient condition under which the σ-un-Dunford-Pettis operator is u-M-weakly-compact, and conversely.

Proposition 9. *Let E and F be two Banach lattices.*

1. *If the norm of E' is order continuous, then each σ-un-Dunford-Pettis operator T from E into F is u-M-weakly-compact.*
2. *If the norm of E is order continuous and atomic, then each u-M-weakly-compact operator T from E into F is σ-un-Dunford-Pettis.*

Proof.
1. let (x_n) be a norm bounded disjoint sequence in E. Since the norm of E' is order continuous, from [13, Theorem 7], which implies that $x_n \xrightarrow{w} 0$, by assumption T is σ-un-Dunford-Pettis operator, implies $Tx_n \xrightarrow{un} 0$. Therefore, we conclude that T is u-M-weakly-compact.
2. let A be a weakly compact set of E. As E is order continuous and atomic, by [6, Proposition 4.16] A is σ-un-compact set of E, as T is u-M-weakly compact, from Lemma 1 $T(A)$ is relatively σ-un-compact set of F, it follows that by Theorem 1 T is σ-un-Dunford-Pettis.

□

Proposition 10. *Let X be a Banach space and E, F two Banach lattices. If $S : X \longrightarrow E$ is an operator σ-un-DP and $T : E \longrightarrow F$ is an u-M-weakly-compact, then the product operator $T \circ S$ is σ-un-Dunford-Pettis.*

Proof. Let $S : X \longrightarrow E$ be an operator σ-un-Dunford-Pettis and A a relatively weakly compact set in X, then from Theorem 1 that $S(A)$ is relatively sequentially un-compact set in E. As T is u-M-weakly-compact, now, using Lemma 1 that $T \circ S$ is a relatively σ-un-compact set in F, we infer that $T \circ S$ is a σ-un-Dunford-Pettis operator. □

Remark 6. *The condition " $T : E \longrightarrow F$ is u-M-weakly-compact operator" is essential in Proposition 10. Indeed, we consider the following operators which are mentioned in the* [12, Remark 1]*, add F has order unit.*

$$\begin{array}{rcl} S : E & \to & c_0 \\ x & \mapsto & (g_n(x)) \end{array} \qquad \begin{array}{rcl} R : c_0 & \to & F \\ (\lambda_n)_n & \mapsto & \sum_{n=1}^{\infty} \lambda_n y_n. \end{array}$$

Note that S is σ-un-Dunford-Pettis and R is not u-M-weakly-compact, but the product operator $T = R \circ S$ which is not σ-un-Dunford-Pettis.

As consequence of the preceding results we have the following.

Corollary 5. *Let E be a Banach lattice such that E' is order continuous. If $T : E \longrightarrow E$ is an σ-un-Dunford-Pettis operator, then T^2 is σ-un-Dunford-Pettis.*

The following theorem characterize Banach lattices E and F for which every σ-un-Dunford-Pettis operator $T : E \longrightarrow F$ is u-M-weakly-compact.

Theorem 7. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

1. *Each σ-un-Dunford-Pettis operator from E into F is u-M-weakly-compact.*

(2) *Each σ-un-compact operator from E into F is u-M-weakly-compact.*
(3) *One of the following conditions is valid:*
   (a) *The norm of E' is order continuous.*
   (b) *$F = \{0\}$.*

Proof. 1) $\Rightarrow$2) Obvious.
2) $\Rightarrow$3) Assume by way of contradiction that the norm of E' is not order continuous and F $\neq \{0\}$. Since the norm of E' is not order continuous, then from [7, Theorem 2.4.14 and Proposition 2.3.11] that E contains a sublattice isomorphic to ℓ^1 and there exists a positive projection P from E into ℓ^1. Furthermore, $F \neq \{0\}$, then there exists $0 < y \in F$. Consider the operator Q from ℓ^1 into F define by:

$$Q(\lambda_n) = (\sum_{n=1}^{\infty} \lambda_n)y$$

for each $(\lambda_n) \in \ell^1$. The operator Q is well defined. Consider $T = Q \circ P : E \to \ell^1 \to F$. Clearly T is σ-un-compact (T is compact), but T is not u-M-weakly-compact operator. In fact, let (e_n) be the canonical basis of ℓ^1, we have (e_n) norm bounded disjoint and $\||T(e_n)| \wedge y\| = \||y| \wedge y\| = \|y\| > 0$.
3)a)$\Rightarrow$ 1) Proposition 9 and 3)b)$\Rightarrow$ 1) Obvious. □

Theorem 8. *Let E and F be two Banach lattices and $T : E \to F$ be an operator. If every operator T σ-un-Dunford-Pettis is u-M-weakly-compact, then one of the following assertions is valid:*

(1) *The norm of E' is order continuous.*
(2) *The norm of F is order continuous.*

Proof. Assume that neither E' nor F is order continuous. we need to construct an operator $T : E \to F$ is σ-un-Dunford-Pettis, but is not u-M-weakly-compact. Since the norm of E' is not order continuous, then from ([7, Theorem 2.4.14 and Proposition 2.3.11]) there exist a sublattice of E isomorphic to ℓ^1 and a positive projection

$$P : E \to \ell^1$$

Since the norm of F is not order continuous, then by ([7, Theorem 2.4.2]) there exist a positive order bounded disjoint sequence (y_n) of F such that $\|y_n\| = 1$ and $y_n \leq y$, some $y \in F^+$
Consider the operator $Q : \ell^1 \to F$ defined by

$$Q(x_n) = \sum_{n=1}^{\infty} x_n y_n$$

It's clear that $T = Q \circ P : E \to \ell^1 \to F$ is σ-un-Dunford-Pettis, but T is not u-M-weakly-comapct operator. in fact, let the standard basis (e_n) of ℓ^1, implies ([13, lemme 2]) $e_n \xrightarrow{uaw} 0$ and

$$\begin{aligned} \||T(e_n)| \wedge y\| &= \|y_n \wedge y\| \\ &= \|y_n\| \\ &= 1 \end{aligned}$$

□

Drawing inspiration from AM-compact operators and unbounded norm convergence, we introduce the following class of operators.

Definition 2. *An operator T from a Banach lattice E into a Banach lattice F is called* ***AM-σ-unbounded norm compact*** *(AM-σ-un-compact, for short), if $T[-x, x]$ is relatively sequentially un-compact in F for any $x \in E^+$*

In this position, we continue our discussion on relationships σ-un-Dunford-Pettis with AM-σ-un-compact.

Let us recall from [3] that an operator T from a Banach lattice E into a Banach space Y is called AM-compact, if $T[-x, x]$ is relatively compact in Y for any $x \in E^+$. It's clear that every AM-compact operator is AM-σ-un-compact, and if T is order bounded then every AM-σ-un-compact operator is AM-compact. Note that AM-σ-un-compact operators is not necessarily σ-un-Dunford-Pettis. in fact, consider the canonical injection $i : c_0 \to l^\infty$ is AM-σ-un-compact operator (see [12, Proposition 3.2]), but is not σ-un-Dunford-Pettis. And converse is not true in general. Indeed, we consider Banach lattices l^∞ as the norm of l^∞ is not order continuous and the Banach lattice $(l^\infty)'$ is not discrete, it follows from [3, Theorem 2.5] that there exist two positive operators S,T from l^∞ into l^∞ with $0 \le S \le T$ and T is AM-compact but S is not one, implies S is not un-σ-AM-compact because S is order bounded operator, and T is σ-un-Dunford-Pettis operator. Since l^∞ has o-σ-unDPP, it follows from Proposition 7 that S is σ-un-Dunford-Pettis.

Now, we give a sufficient condition under which σ-un-Dunford-Pettis is an AM-σ-un-compact operator.

Proposition 11. *Let E and F be two Banach lattices such that the norm of E is order continuous. Then each operator T σ-un-Dunford-Pettis from E into F is AM-σ-un-compact.*

Proof. Let $x \in E^+$. Since E is order continuous, it follows from [2, Theorem 4.9] that the order interval $[-x, x]$ is weakly compact. As T is a σ-un-Dunford-Pettis, Theorem 1 implies that $T[-x, x]$ relatively σ-un-compact in F. Consequently, T is an AM-σ-un-compact operator. □

Proposition 12. *Let E be a Banach lattice. Then each positive operator $T : E \longrightarrow E$ σ-un-Dunford-Pettis operator, T^2 is AM-σ-un-compact.*

Proof. Let $[-x, x]$ for each x in E^+. Since $T : E \longrightarrow F$ is σ-un-Dunford-Pettis positive operators, then T is order weakly compact, implies that

$T[-x,x]$ weakly relatively compact, it follows that by Theorem 1 $T^2[-x,x]$ is relatively sequentially un-compact. So T^2 is AM-un-compact. □

Theorem 9. *Let E and F be two Banach lattices . If each operator AM-σ-un-compact from E into F is σ-un-Dunford-Pettis then one of the following statements is valid:*

(1) *E is an order continuous norm.*
(2) *F is an order continuous norm.*

Proof. Assume that neither the norm of E nor F is order continuous. we have to construct an operator T from E into F is AM-σ-un-compact and is not σ-un-Dunford-Pettis. Since E is not order continuous, it follows from ([7, theorem 2.4.2]) that there exist a disjoint order bounded sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n. Hence, by [4, lemma 3.4] there exists a positive disjoint sequence (g_n) in E' with $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ if $n \neq m$. consider the positive operator $Q : E \to c_0$ defined by

$$Q(x) = (g_n(x))_1^\infty, \text{ for all } x \in E$$

The operator Q is well defined. On the other hand, since the norm of F is not order continuous, there exists a disjoint sequence (y_n) of F^+ and $y \in F^+$ such that $0 \le y_n \le y$ and $\|y_n\| = 1$ for each n.
Now, we consider the positive operator $S : c_0 \to F$ defined by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \text{ for all } (\lambda_n) \in c_0.$$

Also, we define the positive operator $T = S \circ Q : E \to c_0 \to F$. Clearly T is AM-compact then T is AM-σ-un-compact, but T is not σ-un-Dunford-Pettis. In fact, note that (x_n) is a weakly null sequence of E^+ and we chose y in F^+.

$$\begin{aligned} |T(x_n)| \wedge y &= |S \circ Q(x_n)| \wedge y \\ &= |S(g_n(x_n))| \wedge y \\ &= |\sum_{n=1}^{\infty} g_n(x_n) y_n| \wedge y \\ &= |y_n| \wedge y \\ &= y_n \end{aligned}$$

$\||T(x_n)| \wedge y\| = \|y_n\| = 1$ □

Finally, we propose a simple diagram illustrating the relationship between different class of operators.
The notations used are as follows:

- The collection of σ-un-compact operators will be denoted by σ-unK(E,F)
- The collection of σ-un-Dunford-Pettis operators will be denoted by σ-unDP(E,F).

- The collection of u-M-weakly compact operators will be denoted by uMWC(E,F).
- The collection of AM-σ-un-compact operators will be denoted by σ-$unAM_c(E,F)$.
- The collection of compact operators will be denoted by K(E,F)
- The collection of weakly compact operators will be denoted by WC(E,F).
- The collection of order weakly compact operators will be denoted by OWC(E,F).
- The collection of Dunford-Pettis operators will be denoted by DP(E,F).
- The collection of AM-compact operators will be denoted by $AM_c(E,F)$.

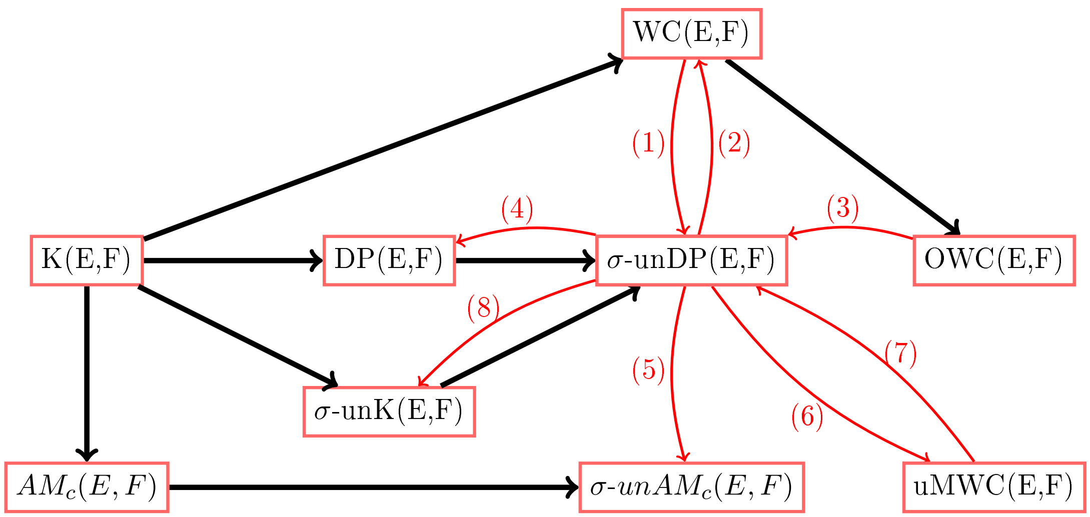

Figure 1: Connection between the classes of operators

- $\longrightarrow$: Inclusion without condition.
- $\longrightarrow$: Inclusion with condition.
- (1) : E has the unDP property.
- (2) : E has the R-unDP property..
- (3) : E has the o-unDP property..
- (4) : F has a order unit.
- (5) : E has an order norm continuous.
- (6) : E' has an order norm continuous.
- (7) : E is atomic and has an order norm continuous.
- (8) : E is a reflexive.

Acknowledgement. This research did not receive any specific grant from funding in the public, commercial, or not-for-profit sectors.

References

[1] N. Hafidi, J. H'michane, *Some results on the class of σ-unbounded Dunford-Pettis operators*, Commentationes Mathematicae Universitatis Carolinae, **62**:4 (2021), 431–443.

[2] C. D. Aliprantis, O. Burkinshaw, *Positive operators*, reprint of the 1985 original. Springer, Dordrecht, 2006.

[3] B. Aqzzouz, R. Nouira, L. Zroula, *Compactness of AM-compact operators on Banach lattices*, Annales des sciences mathematiques du Quebec. **31**:1, (2007), 1–11.

[4] B. Aqzzouz, A. Elbour, J. H'michane, *The duality problem for the class of b-weakly compact operators*, Positivity, **13**:4 (2009), 683–692.

[5] Z. L. Chen, A. W. Wickstead, *L-weakly and M-weakly compact operators*, Indagationes Mathematicae, **10**:3 (1999), 321–336.

[6] M. Kandić, M. A. A. Marabeh, V. G. Troitsky, *Unbounded norm topology in Banach lattices*, Journal of Mathematical Analysis and Applications, **451**:1 (2017), 259–279.

[7] P. Meyer-Nieberg, *Banach lattices*, Universitext. Springer-Verlag, Berlin, 1991.

[8] G. Groenewegen, V. A. Rooij, *The modulus of a weakly compact operator*, Mathematische Zeitschrift, 195 (1987), 473–480.

[9] Z. Niktab, K. H. Azar, R. Alavizadeh, S. S. Gavgani, *Some Properties of Unbounded M-Weakly and Unbounded L-Weakly Compact Operators*, Hindawi Journal of Mathematics, 2023

[10] N. E. Özcan, N. A. Gezer, O. Zabeti, *Unbounded absolutely weak Dunford-Pettis operators*, Turkish Journal of Mathematics, **43** (2019), 2731–2740.

[11] W. Wnuk, *Some characterizations of Banach lattices with the Schur property*, Revista matemática de la Universidad Complutense de Madrid, **2** (1989), 217–224

[12] L. Zraoula, I. ouba, J. H'michane, B. EL. Wahbi, *On the class of unbounded-U-Dunford-Pettis operators*, Filomat, **38**:25 (2024), 8701–8713.

[13] O. Zabeti, *Unbounded absolute weak convergence in Banach lattices*, Positivity, **22**:2 (2018), 501–505.

Issam Ouba
Department of Mathematics,
University of Ibn Tofail,
14000, Kenitra, Morocco
Email address: contact.issam.ouba@gmail.com

elmostafa bendib
Department of Mathematics and Computer Science,
University of Cadi Ayyad,
40000, Safi, Morocco
Email address: elmostafa.bendib@uca.ac.ma

Larbi Zraoula
Department of Mathematics and Computer Science,
CRMEF,
14000, Kenitre, Morocco
Email address: zraoularbi@yahoo.fr

Bouazza El Wahbi
Department of Mathematics,
University of Ibn Tofail,
14000, Kenitra, Morocco
Email address: bouazza.elwahbi@uit.ac.ma