RECOVERY OF FUNCTIONS FROM NIKOLSKY'S CLASSES IN UNIFORM METRIC BY UNEXACT INFORMATION

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ABSTRACT. For the classes of functions with bounded r difference (in $L^p(0,1)$), the problem of optimal recovery of functions (C(N)D-1 problem) was completely solved before for the multidimensional Nikol'skii-Besov classes. In the one-dimensional case, in continuation of the previously obtained results, in the article a complete C(N)D-study of the optimal recovery problem was carried out, as a result of which it was established that among all conceivable computational aggregates, Lagrangian splines along a uniform grid belong to the best ones, and in the sense of simplicity of the device - for the best. As it is known, the Lagrange interpolation polynomials themselves have poor approximative properties, which is the subject of an extensive literature.

At the same time, in the article are obtained the estimates of recovery by unexact informationins and determined the largest bound on unexact information that preserves the order of recovery from exact information.

1. INTRODUCTION

The Lagrange polynomials themselves do not have high computational qualities [1, p.192] "*Do not use algebraic interpolation polynomials with equally spaced nodes for a significant number of nodes in computational practice*", whereas in [2, p.377], let's call it *Lokutsievsky's observation*, they are given the chance for them "*In numerical practice Lagrangian spline mostly (however, without special justifications) is taken with respect to the system of equally spaced nodes*".

Here is arisen the problem of investigation of possibilities of computational aggregates constructed by Lagrangian interpolation polynomials in conditions "*When and how it is correct to use in computing mathematics Lagrange polynomials?*".

Key words and phrases. recovery problem under exact and unexact information, limiting error of optimal recovery under exact and unexact information, Lagrange interpolation polynomials, Lagrange spline interpolation.

The purpose of this article is to close this *"demonstration of mathematical intuitiveness"* with an exact theorem, *"If the Lagrange interpolation polynomials are used correctly, namely, in spline form, there are no better ones computational aggregates"*, and even with C(N)D-2 and C(N)D-3 research.

This study will be carried out in the scheme of Computational (Numerical) diameter as the basis of a new approach to Approximation Theory, Computational Mathematics and Numerical Analysis (a complete definition, discussion and comparison with known results and similar considerations, the history of the problem are given in [3]-[4]).

For the sake of completeness, we give the full formulation of the C(N)D-statement, consisting of three successive problems. Let's start with C(N)D-1.

Given T, F, Y, D_N (explanation is given below):

C(N)D-1: To find $\asymp \inf\limits_{\left(l^{(N)},\varphi_N\right)\in D_N} \sup\limits_{f\in F} \|Tf\left(\cdot\right) - \varphi_N\left(l_1(f),...,l_N(f);\cdot\right)\|_Y \equiv$

$\equiv \delta_N\left(0;T;F;D_N\right)_Y \equiv \delta_N\left(0;D_N\right)_Y$– *the informative power of a set of computational aggregates* $D_N \equiv D_N\left(F\right)_Y$;

The first problem C(N)D-1 is a concentrated collective definition of various implementations of the general approach to the approximation of a function by numerical information of a finite volume from it, which can be not only its values at points, but also the Fourier coefficients, with the transition to linear functionals of the function being approximated (not even necessarily linear). All of this in various variations are scattered in the vast literature, the C(N)D-1 approach is a synthesis of the known, which the authors of the article do not pretend to be, but with its own exact, as it seems to us, the name *"Informative power of a given set of computational aggregates"*. With an accompanying explanation, as the task of finding the correct order when recovery from accurate information, which is a preparation for already, with our claim to novelty, the continuation of the studies made in C(N)D-2 and C(N)D-3.

This problem in the case of $Tf = f$-recovery of functions in the of Nikol'skii-Besov classes $B^r_{p,\theta}$ and Lizorkin-Triebel classes scale, in the successive expansion of classes and with various relationships between their parameters, was the subject of research by many authors, the final result is given in the formulation from the article [5]:

Theorem (E.Novak, H.Tribel, 2005). *Let* $\Omega \subset R^s$ *be a bounded Lipschitz domain,* $0 < p \leq \infty, 0 < q \leq \infty, 1 < \theta \leq \infty, r > 0$. *Then*

(1)
$$\inf_{h_1,h_2,...,h_N\in L^q(\Omega)} sup\left\{\left\|f(x)-\sum_{i=1}^{n} f(\xi_i)h_i(x)\right\|_{L^q(\Omega)} : f\in B^r_{p,\theta}(\Omega)\right\}\asymp$$
$$\asymp N^{-\frac{r}{s}+max\left\{\frac{1}{p}-\frac{1}{q},0\right\}}.$$

In this connection, we note $\Omega=(0,1)^s, 2\le p\le q\le\infty$ that relation (1) was established in an earlier work [6]

(2)
$$\inf_{L_N\left(B^r_{p,\theta}(0,1)^s\right)\times\{\varphi_N\}_{L^q(0,1)^s}} \sup_{f\in B^r_{p,\theta}(0,1)^s} \|f\,(x)-\varphi_N\,(l_1(f),...,l_N(f);x)\|_{L^q(0,1)^s}\asymp$$
$$\asymp N^{-\left(\frac{r}{s}-\left(\frac{1}{p}-\frac{1}{q}\right)\right)},$$

where the optimal estimates are sharped on the de la Vallee-Poussin means of the trigonometric Fourier series.

The optimal computational aggregates in the C(N)D-1 problem may not be the only ones, and each of them is subjected to further research C(N)D-2 and C(N)D-3. In this vein, the formulation of this problem is of particular importance due to the mathematical beauty and historical content of the famous Lagrange polynomials.

Under these conditions, the purpose of the article is to prove that in relations (1) - (2), upper bounds in the one-dimensional case $\Omega=[0,1]$, along with Theorem A and the de la Valle-Poussin means in [6], are also sharps on computational aggregates $\bar\varphi\,(l_1(f),...,l_N(f);\cdot)$ - in the form of Lagrangian splines.

Thus, the main result of the article is that among all conceivable computational aggregates constructed by linear information from the functions being reconstructed, the best are Lagrangian splines on a uniform grid, which confirms Lokutsievsky's forecast. After that, the Lagrangian splines on a uniform grid are subjected to further research (N)D-2 and (N)D-3, the statements of which are as follows.

C(N)D-2: To construct a *computational* aggregate $\left(\bar l^{(N)},\bar\varphi_N\right)$ from $D_N\equiv D_N(F)_Y$, which suports an order $\asymp\delta_N\,(0;D_N)_Y$, and investigate the problems of existence and finding a sequence $\bar\varepsilon_N\equiv\bar\varepsilon_N\left(D_N;\left(\bar l^{(N)},\bar\varphi_N\right)\right)_Y$ C(N)D-2–limiting error corresponding to $\left(\bar l^{(N)},\bar\varphi_N\right)$ such that

$$\delta_N\,(0;Tf;F;D_N)_Y\asymp\delta_N\left(\bar\varepsilon_N;Tf;F;\left(\bar l^{(N)},\bar\varphi_N\right)\right)_Y\equiv$$
$$\equiv\sup\left\{\|Tf\,(\cdot)-\bar\varphi_N\,(z_1(f),...,z_N(f);\cdot)\|_Y:\ f\in F,\ \left|\bar l_\tau\,(f)-z_\tau(f)\right|\le\bar\varepsilon_N^{(\tau)}\,(\tau=1,...,N)\right\},$$

with fulfillment the condition

$$\forall \eta_N \uparrow +\infty (0 < \eta_N < \eta_{N+1},\, \eta_N \to +\infty) : \overline{\lim_{N\to\infty}} \frac{\delta_N\left(\eta_N \bar{\varepsilon}_N; Tf; F; (\bar{l}^{(N)}, \bar{\varphi}_N)\right)_Y}{\delta_N\left(0; Tf; F; D_N\right)_Y} = +\infty.$$

C(N)D-3: Establish massiveness of the limiting error $\bar{\varepsilon}_N\left(D_N, \left(\bar{l}^{(N)}, \bar{\varphi}_N\right)\right)$: find as large as possible set $D_N\left(\bar{l}^{(N)}, \bar{\varphi}_N\right)$ (as usual, connected with the structure of the initial $\left(\bar{l}^{(N)}, \bar{\varphi}_N\right)$) of computational aggregates $\left(l^{(N)}, \varphi_N\right) \equiv \varphi_N\left(l_1(f), ..., l_N(f); \cdot\right)$, constructed by all possible non necessarily linear functionals $l_1, ..., l_N$ such that $\forall \eta_N \uparrow +\infty (0 < \eta_N < \eta_{N+1}, \eta_N \to +\infty)$:

$$\overline{\lim_{N\to\infty}} \frac{\delta_N\left(\eta_N \bar{\varepsilon}_N; Tf; F; \left(l^{(N)}, \varphi_N\right)\right)_Y}{\delta_N\left(0; Tf; F; D_N\right)_Y} = +\infty,$$

here $\bar{\varepsilon}_N$ is a non-negative sequence. In the case $\bar{\varepsilon}_N \equiv 0$ we will talk about the task of recovery by accurate information, in the rest – by inaccurate.

Here we assume that F is a class of functions and $Y-$ normalized space (or $Y \equiv C$, where C here and below, the field of complex numbers) defined on Ω_F and Ω_Y respectively. A mathematical model is given by the operator $T : F \mapsto Y$. $l_1, ..., l_N$ is a set of functionals $l^{(N)}$ (not necessarily linear) by which each function f from F is associated with a finite sequence $l_1(f), ..., l_N(f)$ - *numerical information about f volume N. An algorithm of processing approximate information $z_1(f), ..., z_N(f)$ about f* obtained from functionals $l_1(f), ..., l_N(f)$ with accuracy ε_N is, by definition, a numerical function $\varphi_N\left(z_1, ..., z_N; \cdot\right)$, which for any fixed $(z_1, ..., z_N) \in C^N$ is a function belongs to Y of variable $(\cdot) \in \Omega_Y$ (the set, which consist all such φ_N is denoted by $\{\varphi_N\}_Y$). Then, after substituting $z_1 = z_1(f), ..., z_N = z_N(f)$, the function $\varphi_N\left(z_1(f), ..., z_N(f); \cdot\right)$ acquires the status of *a computation aggregates* for approximate calculation in the operator $Tf \equiv u(\cdot; f)$ metric Y. Thus, each computation aggregates $\left(l^{(N)}; \varphi_N\right) = (l_1, ..., l_N; \varphi_N)$ is defined by a pair (sometimes we will move from a complex record of a computation aggregates $\varphi_N\left(y_1, ..., y_N; \cdot\right)$ to a shortened record $\varphi_N\left(\{y_\tau\}_{\tau=1}^N; \cdot\right)$). We denote D_N a set of complexes $\left(l^{(N)}; \varphi_N\right) = (l_1, ..., l_N; \varphi_N)$.

In conclusion, we note that the practical application of each computation aggregates from C(N)D-2 and -3 is carried out through the construction of a physical device for measuring digital information $l_1, ..., l_N$

on the class F with accuracy $\bar{\varepsilon}_N$ (it is clear that the more ε_N, the device is easier in technical design and cheaper in operation) and through software $\bar{\varphi}_N$ - algorithms for computer calculations.

Before proceeding to the results of this article, we give a brief history of the issue.

The problem of creating the algebraic polynomial with the smallest degree passing through the set of points $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$ was solved by Joseph Louis Lagrange [7] in 1795 . This polynomial is given by

$$P_n(x) = \sum_{k=0}^{n} y_k \prod_{t=0, t\neq k}^{n} \frac{(x - x_k)}{x_t - x_k},$$

is named after Lagrange (this is usual in such cases, according to Pearson's book [8]; these polynomials were known to Waring (1779) and Euler (1783)).

Approximative possibilities of the Lagrange polynomials were investigated deeply by Faber, Bernstein, Martzinkiewich, Zigmund, Fikhtengoltz, Natanson, Babenko, Lokutziewsky and etc. (see, for example, [1, 2], [9]-[11], however, the initial information is available in any book on numerical analysis). It has been established that in some cases, such as those constructed from Chebyshev nodes, we emphasize that with irrational coordinates, they are *acceptable* (see, for example, [10]), in others they are *bad* (as reflected in the title of the chapter "Negative Results" in [11, pp. 511-537]) from the standpoint of the theory of approximations, more precisely, the constructive theory of functions.

As it was relatively recently found out, as a means of approximating functions by their values at points, Lagrange interpolation polynomials are among the best if and only if the number of nodes is equal to the order of differentiability of the interpolated function (see [12]):

When measuring the error of the data in the metric $\|\cdot\|_{l_n^p}$ *(where for* $\rho = (\rho_1, ..., \rho_n) \in R^n$ *the norm* $\|\rho\|_{l_n^p}$ *is equal to* $\left(\sum_{\tau=1}^{n} |\rho_\tau|^p\right)^{\frac{1}{p}}$ *or* $\max_{\tau=1,...,n} |\rho_\tau|$ *depending on* $1 \le p < \infty$ *or* $p = \infty$*) for the class* $W_\infty^{(n)}(M; a, b)$ *of functions defined on the interval* $[a, b]$ *and having a continuous derivative of order* n*, which by modulo bounded by a constant* M*, where* n *is the number of different points* $x_1, ..., x_n$ *from* $[a, b]$*, in which* ε*- approximate values of the reconstructed function are given with accuracy for every* $x, a \le x \le b$ *equality* $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$$\inf_{\varphi} \sup \left\{ |f(x) - \varphi(z_1, ..., z_n; x, \varepsilon)| : f \in W_\infty^{(n)}, \|\{z_\tau - f(x_\tau)\}_{\tau=1}^n\|_{l_n^p} \le \varepsilon \right\} =$$

$$= sup \left\{ \left| f(x) - \sum_{\tau=1}^{m} \left(\prod_{i \ne \tau} \frac{x - x_i}{x_\tau - x_i} \right) z_\tau \right| : f \in W_\infty^{(n)}, \|\{z_\tau - f(x_\tau)\}_{\tau=1}^n\|_{l_n^p} \le \varepsilon \right\} =$$

$$= \frac{M}{n!} \prod_{i=1}^{n} (x - x_i) + \varepsilon \left(\sum_{\tau=1}^{n} \left| \prod_{i \ne \tau} \frac{x - x_i}{x_\tau - x_i} \right|^q \right)^{\frac{1}{q}}.$$

In the problem of studying the approximative possibilities of Lagrange interpolation polynomials, when smoothness is specified in the $W_p^r(0,1)$ Sobolev class scale and the recovery error of functions is measured in the Lebesgue metric $L^q(0,1)$, for all $1 \le p, q \le \infty, rp > 1$ in the framework of C(N)D-1, the following picture emerges.

For $2 \le p \le q \le \infty$ and $1 \le q \le p \le \infty$, the Lagrange interpolation polynomials give the best (in $L^q(0,1)$) among all conceivable computational units (in order) recovery, moreover, interpolation, of functions with a bounded (in $L^p(0,1)$) r-th derivative on the interval $[0,1]$, at a rate of $<< N^{-r+\max\{0;1/p-1/q\}}$, unless they are used in a spline form $L_{N,r}(x)$ with $N = (r-1)k (k = 1, 2, ...)$, which allows, for a fixed r-smoothness, to carry out an approximation with arbitrary accuracy by choosing the number of nodes N.

In the remaining cases $1 \le p < q \le 2$ and $1 \le p < 2 \le q \le \infty$, the speed $N^{-r+\max\{0;\frac{1}{p}-\frac{1}{q}\}}$ of approximation by Lagrangian interpolation splines is worse than the highest possible speed restoration of functions from linear information[13] and the coding width of functions [14] , by a power factor equal to $N^{\frac{1}{p}-\frac{1}{2}}$ and $N^{\frac{1}{p}-\frac{1}{q}}$, respectively.

In this connection, we note that in [15] for the proof of upper bound is referred to [16]–[17]. However, a complete proof was not found in them, and, as can be seen from [15] , the connection between the order of smoothness of a class and the number of nodes was not established; therefore, for the sake of completeness, the necessary proof was given in [18]-[19] .

If you limit yourself to computing units $P_N(W_p^r(0,1)) \times \{\varphi_N\}_{L^q(0,1)}$, built in terms of values at the points of the approaching function, then in all cases $1 \le p, q \le \infty$ the lagranzhevs belong to The best (see (4.7)), i.e. There is no need to search for other computing units built in terms of points.

Thus, a complete study of the approximation capabilities of the interpolation polynomials of Lagrange was carried out, the boundaries of their inconsistency and effectiveness were clarified. As a result, *we come to a fundamentally new conclusion that the use of Lagrange polynomials as a basic splan in cases of* $2 \le p \le q \le \infty$ *and* $1 \le q \le p \le \infty$ *leads to the best of all conceivable approximation units according to linear information. Such a high assessment even in the most important cases* $p = q = 2, p = q = \infty$ *and* $p = 2, q = \infty$ *has never been used, which can be understood as the complete rehabilitation of this computing unit of 1795 of the creation. In the remaining incident, you have to turn to other proximity means.*

Part C(N)D-2 establishes the limiting error of reconstruction by Lagrangian interpolation splines. In part C(N)D-3, it is shown that any computational unit built on all possible sets of linear functionals cannot give a marginal error greater (in order) than Lagrangian interpolation splines. As it turned out, the limiting recovery errors in all cases of the effectiveness of Lagrangian spline interpolation are of the order of the informative power of the corresponding set of computing units $\asymp N^{-r+\max\left\{0,\frac{1}{p}-\frac{1}{q}\right\}}$.

This article explores the Nikolsky classes.

2. Necessary definitions and auxiliary statements

Recall the notation and definitions that we will use throughout the paper.

We start by defining the Lagrangian interpolation spline $L_{N,\rho}(f;x)$. For $\rho \ge 2, N = \rho k (k = 1, 2, ...)$ and for a set of numbers $z_0, z_1, ..., z_N$ we introduce the function $L_{N,\rho}(z_0, z_1, ..., z_N; x) \equiv L_{N,\rho}(f;x)$, defined on $[0,1]$, which can be represented on a segment $\frac{i\rho}{N} \le x \le \frac{(i+1)\rho}{N}, (i = 0, 1, ..., k-1)$ in the form of algebraic polynomial (Lagrange inerpolation polynomials with nodes $\left\{\frac{i\rho+\tau}{N}\right\}_{\tau=0}^{\rho}$)

$$L_{N,\rho}^{(i)}(x) = \sum_{\tau=0}^{\rho} z_{i\rho+\tau} \prod_{\substack{t=0 \\ t \ne \tau}}^{\rho} \frac{N}{\tau - t}\left(x - \frac{i\rho + t}{N}\right).$$

For $\rho = 1$, we will understand by $L_{N,1}^{(i)}(f;x) (i = 0, ..., N-1)$ the linear on every segment $\left[\frac{i}{N}, \frac{i+1}{N}\right]$ function coinciding with f at the endpoints $\frac{i}{N}$ and $\frac{i+1}{N}$.

And finally,

$$L^{(0)}_{N,N}(f;x) \equiv L_N(f;x) = \sum_{\tau=0}^{N} f(\frac{\tau}{N}) \prod_{\substack{t=0 \\ t\neq\tau}}^{N} \frac{N}{\tau - t}\left(x - \frac{t}{N}\right)$$

.

For completeness, we recall the definition of the Nikol'skii classes (see [20]).

The fact that $f(x)$ has a continuous on $[a,b]$ derivative of order ρ, we will write briefly as $f \in C^{\rho}[a,b]$. Similarly, $f \in AC[a,b]$ means that a function $f(x)$ is absolutely continuous on $[a,b]$.

For any function $g(x) \in C[\lambda,\mu]$ we denote by $\omega^{(2)}(g,t)$ $\left(0 \le t \le \frac{(\mu-\lambda)}{2}\right)$ 2-nd order module of continuity, i.e., the function

$$\omega^{(2)}(g,t) = \sup_{\lambda+t\le x\le\mu-t} |g(x-t) - 2g(x) + g(x+t)|\,.$$

If $g(x) \in L^q(0,1)\,(1 \le q < +\infty)$, then its 2-nd order module of continuity (in the metric of $L^q(0,1)$) is defined by

$$\omega^{(2)}(g,t)_q = \sup_{0\le h\le t}\left(\int_h^{1-h} |g(x-h) - 2g(x) + g(x+h)|^q\,dx\right)^{\frac{1}{q}}\ (0 \le t \le 1)\,.$$

To unify notation, as it was mentioned above, we consider the space $C[0,1]$ instead of $L^\infty(0,1)$, and for $g(x) \in C[0,1]$ we take $\omega^{(2)}(g,t)_\infty$ instead of $\omega^{(2)}(g,t)$.

Let r be a positive number and $1 \le q \le +\infty$. Then r can be represented uniquely in the form $r = \overline{r} + \alpha$, where $\overline{r} \in Z, 0 < \alpha \le 1$. The Nikol'skii space $H^r_q(0,1)$ is the set of functions $f(x)\,(0 \le x \le 1)$ such that $f^{(\overline{r}-1)} \in AC[0,1]$. $f^{(\overline{r})} \in L^q(0,1)$ and $\omega^{(2)}\left(f^{(\overline{r})},t\right)_q \le M\cdot|t|^\alpha$ (for some $M > 0$and all $t > 0$).

The norm in the Nikol'skii space is defined as

$$\|f\|_{H^r_q(0,1)} \equiv \|f\|_{L^q(0,1)} + \sup_{t>0}\frac{\omega^{(2)}\left(f^{(\overline{r})},t\right)_q}{|t|^\alpha}.$$

The Nikol'skii class is the unit ball in the Nikol'skii space, i.e., the set of all functions $f \in H^r_q(0,1)$ such that $\|f\|_{H^r_q(0,1)} \le 1$.

We also need to determine the Lagrange interpolation polynomial in a different notation.

Let $f(x) \in C^{\rho-1}[a, a+h\rho]$ and $L_{N,\rho}(f;x)$ be its Lagrange interpolating polynomial corresponding to the partition $a, a+h, ..., a+\rho h$, where $h = \frac{1}{N}$.

Lemma 1. Let function $g(x) \in C^1[\lambda, \mu]$ and there exists a point $c \in [\lambda, \mu]$ such that $g(c) = 0$. Then

$$\|g\|_{C[\lambda,\mu]} \leq \|g'\|_{C[\lambda,\mu]}(\mu - \lambda).$$

Proof. Due to the second Weierstrass theorem, there exists a point $\overline{x} \in [\lambda, \mu]$ such that $|g(x)|$ attains its maximal value at the point, i.e., $\|g\|_{C[\lambda,\mu]} = |g(\bar{x})|$.

Then, using the mean value theorem, we get that for some ξ from the segment with the endpoints $\overline{x}$ and c satisfies

$$\|g\|_{C[\lambda,\mu]} = |g(\bar{x})| = |g(\bar{x}) - g(c)| = |g'(\xi)|\,|\bar{x} - c| \leq |g'(\xi)|_{C[\lambda,\mu]}(\mu - \lambda)$$

Lemma 1 is proved.

Below we need the summation operators of trigonometric Fourier-Lebesgue series

$$\Lambda_N(f;x) = \sum_{m=-N}^{N} \lambda_m \hat{f}(m) e^{2\pi i m x}, \tag{3}$$

where $\{\lambda_m\}_{m=-N}^{N}, \lambda_m = \lambda_{-m}$ - valid value sequence.

In particular, for $N = 2^{n-1} + 1$

$$\lambda_m = \begin{cases} 1, \quad if \;\; |m| \leq 2^{n-2}; \\ \frac{2^{n-1}+1-|m|}{2^{n-2}}, \;\; if\, 2^{n-2} < |m| \leq 2^{n-1}; \\ 0, if \;\; |m| > 2^{n-1}. \end{cases} \tag{4}$$

we arrive at the Vallee-Poussin averages $V_N(f;x)$ (for details see, for example,[20, p.295]).

Lemma 2 (see [6]). Let s be a positive integer, $2 \leq p \leq q \leq \infty$ and an integer $r > 0$, such that $\frac{r}{s} - \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Then

$$\delta_N(0; Tf = f; H_p^r(0,1)^s; L_N(H_p^r(0,1)^s) \times \{\varphi_N\}_{L^q(0,1)^s})_{L^q(0,1)^s} \asymp N^{-\frac{r}{s}+\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

3. The main results

The paper is devoted to concretization of the general problem in the following case:$F = H_p^r(0,1)$ is the Nikol'skii class, where $2 \leq p < +\infty$, $r > 1 + \frac{1}{p}$, $Y = L^\infty \equiv C[0,1]$,

$$D_N \equiv L_N = L_N\left(H_p^r(0,1)\right) \times \{\varphi_N\}_{C[0,1]} = \{(l_0, l_1, ..., l_N) : l_\tau = l_\tau(f) -$$
$$all\,possible\,linear\,functional\,on\,the\,linear\,span\,F\} \times \{\varphi_N\}_{C[0,1]}.$$

Everywhere below $\rho - 1 = \bar{\sigma}$, where $1 < r - \frac{1}{p} = \sigma = \overline{\sigma} + \alpha, \bar{\sigma} \in Z$, $0 < \alpha \leq 1$.

At last, let $\varepsilon_N \downarrow 0\,(N \uparrow +\infty)\,, |l_\tau(f) - z_\tau| \leq \varepsilon_N(\tau = 0, 1, ..., N)$.

Under these conditions, the following theorem holds.

Theorem 1. *Let numbers* $2 \leq p < \infty$ *and* $r > 1 + \frac{1}{p}$ *,then* $(N = \rho k\ (k = 1, 2, ...))$

C(N)D-1. $\delta_N(0; L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \equiv$

$$\equiv \inf_{\left(l^{(N)},\varphi_N\right) \in L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}} \sup_{f \in H_p^r(0,1)} \|f(x) - \varphi_N(l_0(f), l_1(f), ..., l_N(f); x)\|_{C[0,1]} \asymp N^{-r+\frac{1}{p}}.$$

C(N)D-2. *For the Lagrange interpolation spline* $L_{N,\rho}(f;x)$ *the sequences* $\bar{\varepsilon}_N(L_{N,\rho}(f;x)) \equiv \bar{\varepsilon}_N(L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}; L_{N,\rho}(f;x)) = N^{-r+\frac{1}{p}}$ *is the limiting error sequences: at first,*

$$\delta_N(0; L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \asymp$$

(5)

$$\asymp \delta_N(\bar{\varepsilon}_N(L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}; L_{N,\rho}(f;x)) = N^{-r+\frac{1}{p}}; L_{N,\rho}(f;x))_{C[0,1]} \equiv$$

$$\equiv \sup\left\{\|f(x) - L_{N,\rho}(z_0, z_1, ..., z_N; x)\|_{C[0,1]} : f \in H_p^r(0,1), \left|f\left(\frac{\tau}{N}\right) - z_\tau\right| \leq \bar{\varepsilon}_N(L_{N,\rho}(f;x))\,(\tau = 0, 1, ..., N)\right\}$$

secondly, for every positive sequence $\{\eta_N\}_{N=1}^{\infty}$ *increasing to* $+\infty$ *the following equality holds*

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N \bar{\varepsilon}_N(L_{N,\rho}(f;x)) = \eta_N N^{-r+\frac{1}{p}}; Tf = f; H_p^r(0,1); L_{N,\rho}(f;x)\right)_{C[0,1]}}{\delta_N\left(0; Tf = f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}} = +\infty$$

C(N)D-3: *Every computational aggregate constructed by an arbitrary linear information can not have the limiting error greater, by order, than the limiting error for Lagrange interpolation splines: for every increasing to* $+\infty$ *positive sequence* $\{\eta_N\}_{N=1}^{\infty}$ *and every* $C \geq 1$, *for every set of admissible functional* $l^{(N)} \equiv \{l_0; l_1; ...; l_N\}$ *such that* $|l_\tau(1)| \leq C\,(\tau = 0, 1, ..., N)$, *and for the corresponding computing aggregates* $\left(l^{(N)}, \varphi_N\right)$ *from* $l^{(N)} \times \{\varphi_N\}_{L^q(0,1)}$ *we have*

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N N^{-r+\frac{1}{p}}; Tf = f; H_p^r(0,1); l^{(N)} \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}}{\delta_N\left(0; Tf = f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}} = +\infty.$$

Proof. The lower bound in C(N)D-1 was proved in Lemma 2 for $s = 1$, $q = \infty$, i.e.

(6)
$$\delta_N(0; Tf = f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} >> N^{-(r-\frac{1}{p})}.$$

To prove the upper bound, we made the notation $1 < r - \frac{1}{p} = \sigma = \overline{\sigma} + \alpha$, where $\bar{\sigma} \in Z$, $0 < \alpha \leq 1$.

Let denote $\rho - 1 = \bar{\sigma}, N = \rho k (k = 2, 3, ...), h = \frac{1}{N}$ and define the aggregate of approximation – the Lagrange interpolating spline

$$\bar{\varphi}_{N,\rho}(f; x) \equiv \bar{\varphi}_{N,\bar{\sigma}+1}(f; x) = L_{N,\rho}^{(i)}(f; x)\, if\ x \in \left[\frac{i\rho}{N}, \frac{(i+1)\rho}{N}\right] (i = 0, ..., k-1)\,.$$

For $1 \leq p < \infty$, $\quad r > 1 + \frac{1}{p}$ we prove

(7)
$$\sup_{f \in H_p^r(0,1)} \|f(x) - \bar{\varphi}_{N,\rho}(f; x)\|_{C(0,1)} << N^{-r+\frac{1}{p}}.$$

By the embedding theorems, from $f(x) \in H_p^r(0,1)$ it follows that $f(x) \in H_\infty^{r-\frac{1}{p}}(0,1)$. Since $H_\infty^{r-\frac{1}{p}} \subset C[0,1]$, for some $i (i = 0, 1..., k-1)$ we have

$$\|f(x) - \bar{\varphi}_{N,\rho}(f; x)\|_{C[0,1]} = \left\|f(x) - L_{N,\rho}^{(i)}(f; x)\right\|_{C\left[\frac{i\rho}{N}, \frac{(i+1)\rho}{N}\right]}.$$

Let us estimate $\left\|f(x) - L_{N,\rho}^{(i)}(f; x)\right\|_{C\left[\frac{i\rho}{N}, \frac{(i+1)\rho}{N}\right]}$ the difference for an integer $\rho \geq 2$ and a positive number h, by partition $\alpha, \alpha + h, ..., \alpha + \rho h$, that is, we prove that for any function $f(x) \in C^{\rho-1}[\alpha, \alpha + h\rho]$ the inequality holds

(8)
$$\|f(x) - L_{N,\rho}(f; x)\|_{C[a,a+h\rho]} \leq \rho^{\rho+1} \omega^{(2)}\left(f^{(\rho-1)}, h/2\right) h^{\rho-1},$$

where $L_{N,\rho}(f; x)$ is the Lagrange interpolating polynomial of f over the partition $a, a+h, ..., a+\rho h$.

If $f(x)$ is a polynomial of degree not greater than ρ, then, taking into account that $L_{N,\rho}(f; x)$ is also a polynomial of degree not greater than ρ coinciding with $f(x)$ at the partition points, we get $f(x) - L_{N,\rho}(f; x) \equiv 0$. On the other hand, $f^{(\rho-1)}(x)$ is a polynomial of degree not greater than 1, therefore, $\omega^{(2)}\left(f^{(\rho-1)}, t\right) \equiv 0$. Thus, condition (8) is trivially fulfilled.

Therefore, in the proof will we assume that $f(x)$is not a polynomial of degree not greater than ρ, i.e., $f(x) - L_{N,\rho}(f; x) \not\equiv 0$.

Let $u(x) = f(x) - L_{N,\rho}(f;x)$ and

$$\xi_1^{(0)} = a, \xi_2^{(0)} = a+h, ..., \xi_{\rho+1}^{(0)} = a+\rho h.$$

Then, by the definition of Lagrange polynomials, we get equalities

$$u\left(\xi_1^{(0)}\right) = u\left(\xi_2^{(0)}\right) = ... = u\left(\xi_{\rho+1}^{(0)}\right) = 0.$$

Now we choose $\xi_\tau^{(1)} \in \left[\xi_\tau^{(0)}, \xi_{\tau+1}^{(0)}\right]$ $(\tau = 1, 2, ..., \rho)$ such that

$$\left|u\left(\xi_\tau^{(1)}\right)\right| = \max_{x\in\left[\xi_\tau^{(0)},\xi_{\tau+1}^{(0)}\right]} |u(x)| = \|u\|_{C\left[\xi_\tau^{(0)},\xi_{\tau+1}^{(0)}\right]}.$$

We can assume that $\xi_\tau^{(1)} \in \left(\xi_\tau^{(0)}, \xi_{\tau+1}^{(0)}\right)$ $(\tau = 1, 2, ..., \rho)$.

Then the following relations holds:

(9) $$\|u\|_{C[a,a+\rho h]} = \|u\|_{C\left[\xi_1^{(0)},\xi_{\rho+1}^{(0)}\right]} = \|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]};$$

(10) $$\|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \le \|u'\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \cdot (\rho h);$$

(11) $$u'\left(\xi_1^{(1)}\right) = u'\left(\xi_2^{(1)}\right) = ... = u'\left(\xi_\rho^{(1)}\right) = 0.$$

Now we will prove the relations (9)-(11). We have

$$\xi_1^{(0)} < \xi_1^{(1)} < \xi_2^{(0)} \le \xi_\rho^{(0)} < \xi_\rho^{(1)} < \xi_{\rho+1}^{(0)}.$$

Sucth that

(12)
$$\|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \ge \left|u\left(\xi_1^{(1)}\right)\right| = \max_{x\in\left[\xi_1^{(0)},\xi_2^{(0)}\right]} |u(x)| \ge \max_{x\in\left[\xi_1^{(0)},\xi_1^{(1)}\right]} |u(x)| = \|u\|_{C\left[\xi_1^{(0)},\xi_1^{(1)}\right]},$$

and, similarly,

(13) $$\|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \ge \left|u\left(\xi_\rho^{(1)}\right)\right| = \|u\|_{C\left[\xi_\rho^{(0)},\xi_{\rho+1}^{(0)}\right]} \ge \|u\|_{C\left[\xi_\rho^{(1)},\xi_{\rho+1}^{(0)}\right]}.$$

Then, it is obviously that

$$\|u\|_{C\left[\xi_1^{(0)},\xi_{\rho+1}^{(0)}\right]} = \max\left\{ \|u\|_{C\left[\xi_1^{(0)},\xi_1^{(1)}\right]}; \|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]}; \|u\|_{C\left[\xi_\rho^{(1)},\xi_{\rho+1}^{(0)}\right]} \right\}.$$

From this and taking into account (12) and (13) we get (9).

Validity of (11) follows from the Fermat theorem (obout the necessary condition of an extremum).

Now we will prove (10). Since $\rho \geq 2$ and $\xi_1^{(1)} < \xi_2^{(0)} \leq \xi_\rho^{(0)} < \xi_\rho^{(1)}$ and $u\left(\xi_2^{(0)}\right) = 0$ applying Lemma 1, we obtain validity of (10):

$$\|u\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \leq \|u'\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \cdot \left(\xi_\rho^{(1)} - \xi_1^{(1)}\right) \leq \|u'\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \cdot (\rho h).$$

In a similar way, we choose $\xi_\tau^{(2)} \in \left(\xi_\tau^{(1)}, \xi_{\tau+1}^{(1)}\right)$ $(\tau = 1, 2, ..., \rho - 1)$, such that $\left|u'\left(\xi_\tau^{(2)}\right)\right| = \max_{x \in \left[\xi_\tau^{(1)},\xi_{\tau+1}^{(1)}\right]} |u'(x)|$.

Then, repeating almost word by word the proof of (9)-(11), we get

$$\|u'\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} = \|u'\|_{C\left[\xi_1^{(2)},\xi_{\rho-1}^{(2)}\right]}; \tag{14}$$

$$\|u'\|_{C\left[\xi_1^{(2)},\xi_{\rho-1}^{(2)}\right]} \leq \left\|u''\right\|_{C\left[\xi_1^{(2)},\xi_{\rho-1}^{(2)}\right]} \cdot (\rho h); \tag{15}$$

$$u''\left(\xi_1^{(2)}\right) = u''\left(\xi_2^{(2)}\right) = ... = u''\left(\xi_{\rho-1}^{(2)}\right) = 0.$$

Continuing by this way, we obtain a set of points

$$\left\{\xi_1^{(1)}, \xi_2^{(1)}, ..., \xi_\rho^{(1)}\right\}; \left\{\xi_1^{(2)}, \xi_2^{(2)}, ..., \xi_{\rho-1}^{(2)}\right\}; , ..., \left\{\xi_1^{(\rho-1)}, \xi_2^{(\rho-1)}\right\},$$

such that

$$\left\|u^{(\rho-2)}\right\|_{C\left[\xi_1^{(\rho-2)},\xi_3^{(\rho-2)}\right]} = \left\|u^{(\rho-2)}\right\|_{C\left[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}\right]}, \tag{16}$$

$$\left\|u^{(\rho-2)}\right\|_{C\left[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}\right]} \leq \left\|u^{(\rho-1)}\right\|_{C\left[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}\right]} \cdot (\rho h), \tag{17}$$

$$u^{(\rho-1)}\left(\xi_1^{(\rho-1)}\right) = u^{(\rho-1)}\left(\xi_2^{(\rho-1)}\right) = 0. \tag{18}$$

Thus, by (9), (10), (14)-(17) we obtain

$$\|f(x) - L_{N,\rho}(x,f)\|_{C[a,a+h\rho]} = \|f(x) - L_{N,\rho}(x,f)\|_{C\left[\xi_1^{(1)},\xi_\rho^{(1)}\right]} \leq$$

(19)

$$\leq \|u'\|_{C[\xi_1^{(1)},\xi_\rho^{(1)}]}(\rho h) \leq \left\|u''\right\|_{C[\xi_1^{(2)},\xi_{\rho-1}^{(2)}]}(\rho h)^2 \leq ... \leq \left\|u^{(\rho-1)}\right\|_{C[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}]}(\rho h)^{\rho-1}.$$

Now we choose a point $\bar{x} \in \left[\xi_1^{(\rho-1)}, \xi_2^{(\rho-1)}\right]$, such that $\left|u^{(\rho-1)}(\bar{x})\right| = \max\limits_{x\in\left[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}\right]} \left|u^{(\rho-1)}(x)\right|$. Here we can assume that $\bar{x} \in \left(\xi_1^{(\rho-1)}, \xi_2^{(\rho-1)}\right)$.

We put $\delta = \min\left(\bar{x} - \xi_1^{(\rho-1)}, \xi_2^{(\rho-1)} - \bar{x}\right) > 0$. Notice that

$$\delta \le \frac{\xi_2^{(\rho-1)} - \xi_1^{(\rho-1)}}{2} \le \frac{\rho h}{2}. \tag{20}$$

Without lost of generality, we may assume that $\delta = \bar{x} - \xi_1^{(\rho-1)}$. Then (see (18))

$$u^{(\rho-1)}(\bar{x} - \delta) = u^{(\rho-1)}(\xi_1^{(\rho-1)}) = 0$$

and

$$u^{(\rho-1)}(\bar{x}+\delta) = u^{(\rho-1)}(2\bar{x}-\xi_1^{(\rho-1)}) = u^{(\rho-1)}(2\frac{\left(\xi_1^{(\rho-1)} + \xi_2^{(\rho-1)}\right)}{2}-\xi_1^{(\rho-1)}) = u^{(\rho-1)}(\xi_2^{(\rho-1)}) = 0,$$

therefore,

$$\left|u^{(\rho-1)}(\bar{x} - \delta) - 2u^{(\rho-1)}(\bar{x}) + u^{(\rho-1)}(\bar{x} + \delta)\right| = 2\left|u^{(\rho-1)}(\bar{x})\right| \ge \left|u^{(\rho-1)}(\bar{x})\right|. \tag{21}$$

Further, taking into account the equality $u^{(\rho-1)}(x) = f^{(\rho-1)}(x) - L_{N,\rho}^{(\rho-1)}(x, f)$ and that the $L_{N,\rho}^{(\rho-1)}(x, f)$ is a polynomial of degree not greater than one, i.e.,

$$L_{N,\rho}^{(\rho-1)}(\bar{x} - \delta) - 2L_{N,\rho}^{(\rho-1)}(\bar{x}) + L_{N,\rho}^{(\rho-1)}(\bar{x} + \delta) = 0,$$

we get

$$u^{(\rho-1)}(\bar{x}-\delta)-2u^{(\rho-1)}(\bar{x})+u^{(\rho-1)}(\bar{x}+\delta) = f^{(\rho-1)}(\bar{x}-\delta)-2f^{(\rho-1)}(\bar{x})+f^{(\rho-1)}(\bar{x}+\delta). \tag{22}$$

Sequently, by applying (21), (22) and (20) we get the following inequality

$$\left\|u^{(\rho-1)}\right\|_{C\left[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}\right]} = \left|u^{(\rho-1)}(\bar{x})\right| \le \omega^{(2)}(u^{(\rho-1)}, \delta) =$$

$$= \omega^{(2)}(f^{(\rho-1)}, \delta) \le \omega^{(2)}(f^{(\rho-1)}, \frac{\rho h}{2}).$$

From this estimate and (19) it follows the required inequality (8):

$$\|f(x) - L_{N,\rho}(f; x)\|_{C[a,a+h\rho]} \le \left\|u^{(\rho-1)}\right\|_{C[\xi_1^{(\rho-1)},\xi_2^{(\rho-1)}]} (\rho h)^{\rho-1} \le$$

$$\leq \omega^{(2)}(f^{(\rho-1)}, \frac{\rho h}{2})(\rho h)^{\rho-1} \leq \rho^{\rho+1}\omega^{(2)}(f^{(\rho-1)}, \frac{h}{2})h^{\rho-1}.$$

Inequality (8) is completely proved.

Further, from (8) it follows the inequalities $\left(r - \frac{1}{p} = \sigma = \overline{\sigma} + \alpha\right)$

$$\left\| f(x) - L^{(i)}_{N,\rho}(f;x) \right\|_{C\left[\frac{i\rho}{N}, \frac{(i+1)\rho}{N}\right]} << \omega^{(2)}(f^{(\bar{\sigma})}, \frac{h}{2}) \cdot h^{\bar{\sigma}} <<$$

$$<< h^{\alpha} \cdot h^{\bar{\sigma}} = N^{-\left(r-\frac{1}{p}\right)}.$$

Therefore, for every $f \in H^r_p(0,1)$

$$\|f(x) - \bar{\varphi}_{N,\bar{\sigma}+1}(x)\|_{C[0,1]} << N^{-\left(r-\frac{1}{p}\right)},$$

which together with (6) gives a solution of the problem C(N)D-1.

And now we consider the problem of C(N)D-2. Firstly we consider the first part of C(N)D-2. *Upper estimate.* Let $r - \frac{1}{p} > 1$. We recall that everywhere below $\bar{\varepsilon}_N(L_{N,\rho}(f;x)) = N^{-r+\frac{1}{p}}$.

In notation we put $N = \rho k$ $(i = 0, ..., k-1)$

$$\bar{\varphi}_{N,\rho}\left(z_0, z_1, ..., z_N; x\right) = \sum_{\tau=0}^{\rho} z_{i\rho+\tau} \prod_{\substack{t=0 \\ t\neq\tau}}^{\rho} \frac{x - x_{i\rho+t}}{x_{i\rho+\tau} - x_{i\rho+t}} \; if \; x \in \left[\frac{i\rho}{N}, \frac{(i+1)\rho}{N}\right],$$

here and everywhere bellow $x_l = \frac{l}{N}(l = 0, 1, ..., N)$.

For every function $f(x) \in H^r_p(0,1)$, by the inequality (7) we have

$$\left\| f(x) - \bar{\varphi}_{N,\rho}(z_0, z_1, ..., z_N; x) \right\|_{C[0,1]} \leq$$

(23)

$$\leq \|f(x) - \bar{\varphi}_{N,\rho}(f;x)\|_{C[0,1]} + \|\bar{\varphi}_{N,\rho}(f;x) - \bar{\varphi}_{N,\rho}(z_0, z_1, ..., z_N; x)\|_{C[0,1]} \leq$$

$$\leq \; C \cdot N^{-(r-\frac{1}{p})} + \|\bar{\varphi}_{N,\rho}(f;x) - \bar{\varphi}_{N,\rho}(z_0, z_1, ..., z_N; x)\|_{C[0,1]} \cdot$$

Now we will estimate the second term in the last sum. Let assume that $0 \leq x \leq 1$ and put $i = \min\left\{j = 0, 1, ..., k-1 : x \in \left[\frac{j\rho}{N}, \frac{(j+1)\rho}{N}\right]\right\}$. Then, by the definition of $\bar{\varphi}_{N,\rho}(f;x)$,

$$\bar{\varphi}_{N,\rho}(f;x) = L^{(i)}_{N,\rho}(f,x) = \sum_{\tau=1}^{\rho} f(x_{i\rho+\tau}) \cdot \prod_{\substack{t=0 \\ t\neq\tau}}^{\rho} \frac{x - x_{i\rho+t}}{x_{i\rho+\tau} - x_{i\rho+t}},$$

from the inequalities $|f(x_\tau) - z_\tau| \le \bar{\varepsilon}_N(L_{N,\rho}(f;x))$ $(\tau = 0,1,...,N)$ we obtain

$$\left|\bar{\varphi}_{N,\rho}(f(0), f(\frac{1}{N}), ...,, f(1);x) - \bar{\varphi}_{N,\rho}(z_0, z_1, ..., z_N;x)\right| =$$

$$= \sum_{\tau=0}^{\rho} |f(x_{i\rho+\tau}) - z_{i\rho+\tau}| \cdot \left| \prod_{\substack{t=0 \\ t\ne\tau}}^{\rho} \frac{(x - x_{i\rho+t})}{(x_{i\rho+\tau} - x_{i\rho+t})} \right| << \bar{\varepsilon}_N(L_{N,\rho}(f;x)).$$

From here and inequality (23) follows the upper estimate in (5).

Now we turn to the problem of finding limiting error. The proofs for C(N)D-2 and C(N)D-3 are similar.

We will proove that the error $\bar{\varepsilon}_N(L_{N,\rho}(f;x)) = N^{-\left(r-\frac{1}{p}\right)}$ is limiting for all functionals bounded on the function $f(x) \equiv 1$.

Let such kind of functionals $l_0, l_1, ..., l_N(|l_\tau(1)| \le c,\ c \ge 1\,(\tau = 0,1,...,N))$ and the function $\varphi_N\left(z_0, z_1, ..., z_N;x\right)$, which belongs to $C(0,1)$ for every fixed $z_0, z_1, ..., z_N$.

For a given sequence $\eta_N \uparrow +\infty (N \to +\infty)$, we put

$$g_N(x) \equiv \frac{1}{c}\eta_N^*\bar{\varepsilon}_N(L_{N,\rho}(f;x)),$$

where $\eta_N^* = \min\left\{\eta_N; \ln(N+1)\right\}$.

It is easy to see that $g_N(x) \in H_p^r(0,1)$ and

$$|l_\tau(g_N)| = \frac{1}{c}\eta_N^*\bar{\varepsilon}_N(L_{N,\rho}(f;x))\,|l_\tau(1)| \le$$
$$\le \eta_N^*\bar{\varepsilon}_N(L_{N,\rho}(f;x)) \le \eta_N\bar{\varepsilon}_N(L_{N,\rho}(f;x))\,(\tau = 1,...,N)\,,$$

therefore, for $\bar{\gamma}_N^{(\tau)} = -\frac{l_\tau(g_N)}{\eta_N\bar{\varepsilon}_N}$ $(\bar{\varepsilon}_N \equiv \bar{\varepsilon}_N(L_{N,\rho}(f;x)))$ satisfies

$$\left\| g_N(x) - \varphi_N\left(l_1(g_N) + \bar{\gamma}_N^{(1)}\eta_N\bar{\varepsilon}_N, ..., l_N(g_N) + \bar{\gamma}_N^{(N)}\eta_N\bar{\varepsilon}_N; x\right)\right\|_{C[0,1]} =$$

$$= \|g_N\|_{C[0,1]} = \frac{1}{c}\eta_N^*\bar{\varepsilon}_N(L_{N,\rho}(f;x)) >>$$

$$>> \delta_N\left(0; f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}\overline{\eta}_N$$

where

$$\overline{\eta}_N = \frac{1}{c}\eta_N^* \to +\infty \, (N \to +\infty),$$

that proves simultaneous fulfillment of C(N)D-2 and C(N)D-3.

Theorem 1 is proved.

The following theorem is devoted to the restoration of functions from their values at points, i.e. according to the information obtained from P_N - functionals of the form $l_\tau(f) = f(\xi_\tau) \, (\xi_\tau \in [0,1]\,; \tau = 0,1,..,N)$ and, in contrast to Theorem 1, here$1 \leq p < \infty$.

Theorem 2. *Let* $1 \leq p < \infty$ *and* $r (r = 1, 2, \dots)$, *such that* $r > 1 + \frac{1}{p}$. *Then satisfies* $(N = \rho k \, (k = 1, 2, ...))$

C(N)D-1: $\delta_N(0; P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \equiv$

$$\equiv \inf_{\xi_0,\xi_1,\dots,\xi_N;\varphi_N} \sup_{f \in H_p^r(0,1)} \|f(x) - \varphi_N(f(\xi_0), f(\xi_1), ..., f(\xi_N); x)\|_{C[0,1]} \asymp N^{-r+\frac{1}{p}},$$

C(N)D-2: *For the Lagrange interpolation spline* $L_{N,\rho}(f;x)$ *the sequence* $\bar{\varepsilon}_N(P_N; L_{N,\rho}(f;x)) \equiv \bar{\varepsilon}_N\left(P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}; L_{N,\rho}(f;x)\right) = N^{-r+\frac{1}{p}}$ *is the limiting error sequence: at first,*

$$\delta_N(0; P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \asymp$$

$$\asymp \delta_N(\bar{\varepsilon}_N\left(P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}; L_{N,\rho}(f;x)\right) = N^{-r+\frac{1}{p}}; L_{N,\rho}(f;x))_{C[0,1]} \equiv$$

$$\equiv \sup\left\{\|f(x) - L_{N,\rho}(z_0(f), z_1(f), ..., z_N(f); x)\|_{C[0,1]} :\right.$$

$$\left. f \in H_p^r(0,1), \left|f\left(\frac{\tau}{N}\right) - z_\tau\right| \leq \bar{\varepsilon}_N\left(P_N; L_{N,\rho}(f;x)\right)(\tau = 0,1,...,N)\right\},$$

secondly, for every positive increasing to $+\infty$ *sequence* $\{\eta_N\}_{N=1}^\infty$ *the following equality holds*

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N \bar{\varepsilon}_N\left(P_N; L_{N,\rho}(f;x)\right) = \eta_N N^{-r+\frac{1}{p}}; Tf = f; H_p^r(0,1); L_{N,\rho}(f;x)\right)_{C[0,1]}}{\delta_N(0; Tf = f; H_p^r(0,1); P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]}} = +\infty.$$

C(N)D-3: Any computational aggregates constructed by values at points cannot have a limiting error large (in order) than the limiting error of Lagrangian interpolation splines: for any increasing to $+\infty$ a positive sequence, the equality

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N N^{-r+\frac{1}{p}}; Tf = f; H_p^r(0,1); P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}}{\delta_N\left(0; Tf = f; H_p^r(0,1); P_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}} = +\infty.$$

The proof of C(N)D-1 repeats the corresponding proof from Theorem 1. The proofs of C(N)D-2 and C(N)D-3 repeat corresponding proofs from Theorem 1 for $c = 1$.

Corollary. *For all* p, q, r *such that* $1 \le p < q = \infty, r \ge 1$ *and* $rp > 1$ *we have*

$$\sup_{f \in H_p^r(0,1)} \|f(x) - L_{N,\rho}(f; x)\|_{C[0,1]} \asymp N^{-r+\frac{1}{p}}.$$

In the context of the Computational (Numerical) diameter, the problems of recovery of functions (in $C[0,1]$) from the classes $H_p^r(0,1)$ have the final solution in the sense that any possible computational aggregates (within the framework of this problem statement) give approximation not better than the most simple computiational aggregate – Lanrange interpolation splines.

We should note that earlier in the recovery theory (including [5]-[6],[15]-[17],[21]-[22]) the problem C(N)D in its parts C(N)D-2 and C(N)D-3 did not investigated.

We pass to the final part of the article. As indicated in the definition of C(N)D, each optimal computing unit in C(N)D-1 undergoes further C(N)D-2 and -3 studies.

It is established below (in Theorem 3) that the computational aggregates confirming the lower bound obtained in Theorem 1 include the Vallee-Poussin averages constructed from the trigonometric Fourier coefficients and the corresponding limiting error is determined, which is worse for $\frac{1}{N}$ than the limiting error of the Lagrange interpolation polynomial in the form splines.

Theorem 3. *Let* $2 \le p < \infty$ *and* $r > 1 + \frac{1}{p}$. *Then*

C(N)D-1: $\delta_N(0; L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \equiv$

(24)

$$\equiv \inf_{(l^{(N)}, \varphi_N) \in L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}} \sup_{f \in H_p^r(0,1)} \|f(x) - \varphi_N(l_1(f), ..., l_N(f); x)\|_{C[0,1]} \asymp N^{-r+\frac{1}{p}},$$

C(N)D-2: *For the Valle-Poussin averages* (3)-(4) *of the trigonometric Fourier series* $\bar{\varphi}\left(\left\{\hat{f}(m)\right\}_{m=-2^{n-1}}^{2^{n-1}}; x\right) \equiv V_N(f; x)$ *sequence* $\bar{\varepsilon}_N(V_N(f; x)) \equiv \bar{\varepsilon}_N\left(L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}; V_N(f; x)\right) = N^{-r-\left(1-\frac{1}{p}\right)}$ *is the limiting error sequence: at first, for* $V_N(z; x) = \sum_{m=-2^{n-1}}^{2^{n-1}} z_m \lambda_m e^{2\pi i m x}$, *where* λ_m *from* (4)

$$\delta_N(0; L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \asymp \delta_N(\bar{\varepsilon}_N(V_N(f; x)) = N^{-r-\left(1-\frac{1}{p}\right)}; V_N(f; x))_{C[0,1]} \equiv$$

$$= \sup\Big\{\|f(x) - V_N\left(z_{-2^{n-1}}, ..., z_{2^{n-1}}; x\right)\|_{C[0,1]} :$$

$$f \in H_p^r(0,1)^s, \left|\hat{f}(m^{(\tau)}) - z_\tau\right| < \bar{\varepsilon}_N\left(V_N(f;x)\right)\left(\tau = -2^{n-1}, ..., 2^{n-1}\right)\Big\} \asymp N^{-r+\frac{1}{p}},$$

secondly, for every increasing to $+\infty$ *positive sequence* $\{\eta_N\}_{N=1}^{\infty}$ *the following equality holds*

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N\bar{\varepsilon}_N\left(V_N(f;x)\right) = \eta_N N^{-r-\left(1-\frac{1}{p}\right)}; Tf = f; H_p^r(0,1); V_N\left(\left\{f(m^{(\tau)})\right\}_{|\tau|\le 2^{n-1}}; x\right)\right)_{C[0,1]}}{\delta_N\left(0; Tf = f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}} = \infty.$$

C(N)D-3:*Let integer number* $N \ge 1$. *Then for any computational aggregates*

$$\varphi_N^{(\lambda)}\left(\left\{f\left(m^{(\tau)}\right)\right\}_{|\tau|\le N}; \ x\right) = \sum_{|\tau|\le N} \lambda_N\left(\tau\right)\hat{f}(m^{(\tau)})e^{2\pi i m^{(\tau)}x}, \tag{25}$$

constructed by trigonometric Fourier-Lebesgue series with arbitrary spectrum with $2N+1$ *harmonics and sequence* $\left\{\lambda_N^{(\tau)}\right\}_{|\tau|\le N}$, *where* $\lambda(x) \equiv \lambda_N(x)$ *is given on the segment* $[-\left(N+1\right), N+1]$ *even, positive function whith condition* $\lambda(0) = 1$ *and* $\lambda(N+1) = 0$, *does not have limiting error grater (by order) than limiting error* $\bar{\varepsilon}_N\left(V_N(f;x)\right) = N^{-r-\left(1-\frac{1}{p}\right)}$ *of Valle-Poussen operator* $V_N\left(f;x\right)$*: for any ncreasing goes to* $+\infty$ *positive sequences* $\{\eta_N\}$ *satisfies*

$$\lim_{N\to\infty} \frac{\delta_N\left(\eta_N N^{-r-\left(1-\frac{1}{p}\right)}; Tf = f; H_p^r(0,1); \varphi_N^{(\lambda)}\left(\left\{m^{(\tau)}\right\}_{\tau=1}^{N}; x\right)\right)_{C[0,1]}}{\delta_N\left(0; Tf = f; H_p^r(0,1); L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]}\right)_{C[0,1]}} = +\infty. \tag{26}$$

It is assumed that $\lambda(x)$ *either convex up on* $[0, N+1]$, *either convex down and differentiable on* $[0, N+1]$, *and for some* $0 < x_N < N$ *satisfies the inequality*

$$\frac{\left(\lambda'(x_N)x_N - \lambda(x_N)\right)^2}{|\lambda'(x_N)|} \ge cN,$$

where c *does not depend on* N.

Proof. The upper bound in (24) by accurate information follows from Lemma 2, which with (6) gives a proof of C(N)D-1.

We turn to the problem C(N)D-2. Firstly, we prove the first part of problem C(N)D-2.

The upper estimate. Let $r-\frac{1}{p}>1$. Recall that according to the statement of Theorem 3 everywhere below $\bar{\varepsilon}_N\left(V_N(f;x)\right)=N^{-r-\left(1-\frac{1}{p}\right)}$. Let number N. Without loss of generality, we can assume that $N=2^{n-1}+1$.

Let denote

$$\varphi\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)=V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right).$$

For any function $f(x)\in H_p^r(0,1)$ and functionals $l_1(f)=\hat{f}(m^{(-2^{n-1})}),...,l_N(f)=\hat{f}(m^{(2^{n-1})})$ where $\left\{m^{(-2^{n-1})},...,m^{(2^{n-1})}\right\}$ there is some ordering set. Again, according to the Lemma 2, we have

$$\left\|f(x)-V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]}\le$$

$$\le\left\|f(x)-V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]}+$$

$$+\left\|V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)\right\}_{|\tau|\le 2^{n-1}};x\right)-V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]}\le$$

$$\le C\cdot N^{-(r-\frac{1}{p})}+\left\|V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)\right\}_{|\tau|\le 2^{n-1}};x\right)-V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]}.$$

We estimate the second term in the last sum:

$$\left\|V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)\right\}_{|\tau|\le 2^{n-1}};x\right)-V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]}=$$

$$=\left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}}\lambda_\tau\hat{f}(m^{(\tau)})e^{2\pi i\tau x}-\sum_{\tau=-2^{n-1}}^{2^{n-1}}\lambda_\tau\left(\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right)e^{2\pi i m^{(\tau)}x}\right\|_{C[0,1]}=$$

$$=\left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}}\lambda_\tau\left(\hat{f}(m^{(\tau)})-\hat{f}\left(m^{(\tau)}\right)-\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right)e^{2\pi i m^{(\tau)}x}\right\|_{C[0,1]}=$$

$$=\left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}}\lambda_\tau\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}e^{2\pi i m^{(\tau)}x}\right\|_{C[0,1]},$$

whence from inequalities $\left|\gamma_N^{(\tau)}\right|\le 1$ $(\tau=0,1,...,N)$ and from

$$\|V_N(f;x)\|_{C[0,1]}=\left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}}\lambda_\tau e^{2\pi i m^{(\tau)}x}\right\|_{C[0,1]}\asymp N$$

we get

$$= \left\| V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)\right\}_{|\tau|\le 2^{n-1}};x\right) - V_N\left(\left\{\hat{f}\left(m^{(\tau)}\right)+\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right\}_{|\tau|\le 2^{n-1}};x\right)\right\|_{C[0,1]} =$$

$$= \left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}} \lambda_\tau \bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)} e^{2\pi i m^{(\tau)} x}\right\|_{C[0,1]} =$$

$$= \bar{\varepsilon}_N\left(V_N(f;x)\right)\left\|\sum_{\tau=-2^{n-1}}^{2^{n-1}} \lambda_\tau \gamma_N^{(\tau)} e^{2\pi i m^{(\tau)} x}\right\|_{C[0,1]} \le \bar{\varepsilon}_N\left(V_N(f;x)\right)N.$$

Next, we define $\bar{\varepsilon}_N\left(V_N(f;x)\right)$ from the equality

$$N^{-r+\frac{1}{p}} = \bar{\varepsilon}_N\left(V_N(f;x)\right)N,$$

such that

$$\bar{\varepsilon}_N\left(V_N(f;x)\right) = N^{-r-\left(1-\frac{1}{p}\right)},$$

which together with inequality

$$N^{-r+\frac{1}{p}} \le \delta_N\left(0; L_N(H_p^r(0,1))\times\{\varphi_N\}_{C[0,1]}\right)_{C[0,1]} \asymp$$

$$\asymp \delta_N\left(\bar{\varepsilon}_N\left(V_N(f;x)\right)=N^{-r-(1-\frac{1}{p})}; L_N(H_p^r(0,1))\times\{\varphi_N\}_{C[0,1]}\right)_{C[0,1]},$$

proves the first part of C(N)D-2.

Now we turn to the problem of finding the limiting error. Since the lower bound in C(N)D-2 is contained in C(N)D-3, we immediately turn to this case.

Let us prove that the quantity $\bar{\varepsilon}_N\left(V_N(f;x)\right) = N^{-r-\left(1-\frac{1}{p}\right)}$ is the limiting error with respect to all computational aggregates of the type (25), constructed using trigonometric Fourier coefficients with an arbitrary spectrum.

Let an increasing to $+\infty$ positive sequence η_N.

For an arbitrary set $\left\{m^{(\tau)}\right\}_{|\tau|\le N}$ and function $\bar{f}(x)\equiv 0 \in H_p^r(0,1)$ we have$(\lambda_\tau = \lambda(\tau))$

(27)

$$\sup_{\substack{f\in H_p^r(0,1)\\ \left|\gamma_N^{(\tau)}\right|\le 1\ (|\tau|\le N)}} \left\| f(x) - \sum_{|\tau|\le N}\lambda_\tau\left(\hat{f}(m^{(\tau)}) + \eta_N\bar{\varepsilon}_N\left(V_N(f;x)\right)\gamma_N^{(\tau)}\right)e^{2\pi i\tau x}\right\|_{C[0,1]} \ge$$

$$\geq \left\| \bar{f}(x) - \sum_{|\tau|\leq N} \lambda_\tau \left(\hat{\bar{f}}(m^{(\tau)}) + \eta_N \bar{\varepsilon}_N \left(V_N(f;x) \right) \right) e^{2\pi i \tau x} \right\|_{C[0,1]} \geq$$

$$\geq \sum_{|\tau|\leq N} \eta_N \lambda_\tau \bar{\varepsilon}_N \left(V_N(f;x) \right) = N^{-r-\left(1-\frac{1}{p}\right)} \eta_N \sum_{|\tau|\leq N} \lambda_\tau = \frac{1}{N} \delta_N(0) \eta_N \sum_{|\tau|\leq N} \lambda_\tau .$$

Let estimate from below $\sum\limits_{\tau=0}^{N} \lambda_\tau$. By the conditions of the theorem the function $\lambda(x)$ is convex up or convex down.

Let first $\lambda(x)$ is the convex upward on the segment $[0, N+1]$, then $(\lambda(0) = 1)$

If $\lambda(x)$ convex up on $[0, N+1]$, then for the line $l(x)$ connecting the points $(0,1)$ and $(N+1, 0)$ satisfies

$$\sum_{\tau=0}^{N} \lambda(\tau) \geq \sum_{\tau=0}^{N} l(\tau) = \sum_{\tau=0}^{N} \left(1 - \frac{\tau}{N+1} \right) = \frac{N+1}{2} \geq \frac{N}{2}. \tag{28}$$

For the case of a convex downward function $\lambda(x)$, denoting the tangent to it at a point x_N , by conditions of the theorem, we have

$$\sum_{\tau=0}^{N} \lambda(\tau) = \sum_{\tau=0}^{N+1} \lambda(\tau) \geq \sum_{\tau=0}^{[b_N]} \left(\lambda'(x_N)(\tau - x_N) + \lambda(x_N) \right) =$$

(29)

$$= \left(\left[\frac{\lambda'(x_N)x_N - \lambda(x_N)}{\lambda'(x_N)} \right] + 1 \right) \left(\lambda(x_N) - \lambda'(x_N)x_N + \frac{1}{2}\lambda'(x_N) \left[\frac{\lambda'(x_N)x_N - \lambda(x_N)}{\lambda'(x_N)} \right] \right) \geq$$

$$\geq \frac{1}{2} \left(\frac{\lambda'(x_N)x_N - \lambda(x_N)}{\lambda'(x_N)} \right) \left(\lambda(x_N) - \lambda'(x_N)x_N \right) = \frac{1}{2} \frac{\left(\lambda'(x_N)x_N - \lambda(x_N) \right)^2}{|\lambda'(x_N)|} \geq cN,$$

where $[b_N]$ is the integer part of b_N.

From (27) - (29) it follows

$$\delta_N \left(\eta_N N^{-r-\left(1-\frac{1}{p}\right)}; \varphi_N^{(\lambda)} \left(\left\{ f\left(m^{(\tau)} \right) \right\}_{|m|\leq N}; x \right) \right)_{C[0,1]} >>$$

$$>> \delta_N(0; L_N(H_p^r(0,1)) \times \{\varphi_N\}_{C[0,1]})_{C[0,1]} \eta_N .$$

therefore and holds (26).

Note that the coefficients (4) of the Vallee-Poussin averages are convex upward and therefore satisfy the condition C(N)D-3, which implies the fulfillment of C(N)D-2. The theorem is completely proved.

In this connection, we note that C(N)D-3 contains some summation methods from the criterion of S. M. Nikolsky [23] (here, as a convex function, we can take $\lambda_N(x) = 1 - \frac{x^2}{(N+1)^2}$, $x_N = \frac{N}{2}$).

4. Numerical experiments

Here we present results of numerical experiments. Let us take the values $(M \geq N)$

$$\Delta_{lagr.}(f;N;M) = \max_{x=x_0,x_1,...,x_M} |f(x) - L_N(x;f)|$$

and

$$\Delta_{spl.}(f;\rho;k;N;M) = \max_{i=0,...,k-1} \max_{\substack{x = x_\tau \in \left[\frac{\rho i}{N}, \frac{\rho(i+1)}{N}\right] \\ \tau = 0,...,M}} \left| f(x) - L^{(i)}_{N,\rho}(x;f) \right|,$$

as the accuracy of recovery a function f for a given number $(N+1)$ of nodes of the uniform grid on the segment $[0,1]$. Here $M \geq N$, positive integers ρ and k are such that $\rho \cdot k = N$, the polynomials $L_N(x;f)$ and $L^{(i)}_{N,\rho}(x;f)$ are constructed by values of function $f(x)$ at the nodes of the uniform grid $\{\frac{l}{N}\}_{l=0}^{N}$, and $\{x_\tau\}_{\tau=0}^{M}$ is a given set of points from [0,1].

Thus, the error of approximate aggregate for recovery function is evaluated at the points of a sufficiently dense grid $\{x_\tau\}_{\tau=0}^{M}$ on $[0,1]$.

The results of computing experiment are presented in the following table:

Function: $f(x) = sinx$			
The order of uniform splitting $N = \rho k = 100$		*Errors* Δ *in* M *points* $M = 1000, x_\tau = \frac{\tau}{M}(\tau = 0,1,...,M)$	
ρ	k	$\Delta_{spl.}$	$\Delta_{lag.}$
2	50	4,38E-7	2,06E+13
4	25	2,88E-12	
5	20	5,08E-13	
10	10	7,72E-14	
20	5	5,22E-11	
25	4	1,51E-9	
50	2	2,89E-2	

Here the errors Δ are calculated by two ways, via Visual Basic and using Maple, with the same results, under the accuracy $\varepsilon = 10^{-16}$.

Let us comment the dynamics of the Lagrangian spline-interpolation: in all cases it is better by $10^{15} - 10^{27}$ times than the interpolation by Lagrange polynomials, with decrease of the accuracy when we pass along from the left end of the segment [0,1] to the right one (it is showed in a Fig. 1).

Acknowledgments. The work was supported by the Ministry of Science and Higher Education of the Republic of Kazakhstan, project number AP19680525.

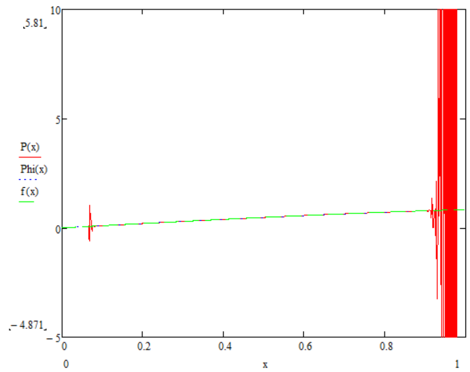

Figure 1. Dynamics Lagrangian spline interpolation

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