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ON EXACT SOLUTIONS OF DELAY PDEs USED IN MODELING OF DISTRIBUTED FORMATIONS

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Abstract: In this paper we study a multidimensional nonlinear differential equation of parabolic type with delay, considered as a mathematical model for changing the density of a distributed formation of autonomous robots. The problem of constructing exact solutions is considered. The variant of reduction method, allowing to separate search of dependence of the solution on time and on spatial variables, is offered. The dependence of solutions on spatial variables is found from a system of algebraic equations, and the time dependence is found from a system of ordinary differential equations with delay. A number of examples are given for the construction of exact solutions, which are explicitly expressed in terms of elementary and Lambert functions.

Keywords: equations of parabolic type; delay PDEs; exact solutions; formation modeling; Lambert function.

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1 Introduction and problem statement

One of the main trends in the development of modern robotics and multiagent systems is the miniaturization of autonomous vehicles, combining them into «warms» and «flocks» for interaction in order to increase the efficiency and success of solving problems and completing missions [8]. This trend inevitably leads to the need to use distributed characteristics in mathematical modeling, such as densities, concentrations, etc. and application of partial differential equations. Everything here is similar to how macroparameters (temperature, pressure, etc.) are successfully used for description in fluid and gas mechanics. Partial differential equations began to be used in modeling the movement of formations of moving objects with distributed characteristics. Thus, in [31], to model a formation with control based on the principle of deviation from the leader's movement, a parabolic type equation of the following form was used

$$u_t = F\left(u, u_x, u_{xx}\right), \quad t \geq 0, \quad x \in \mathbb{R}. \tag{1}$$

In [13], a distributed model of exploration of three-dimensional space by two types of robots that propagate into the surrounding space from some common base and interact with each other was proposed. The model is described by a system of two partial derivative equations of parabolic type. In some cases, a certain time is spent on the exchange of information between robots and the leader and the formation of a feedback signal, and it becomes necessary to take into account the delay in the equations of motion. For example, in [2], taking into account the delay, the problem of equidistant distribution of autonomous robots on a given segment of a straight line was considered. Problems of this kind with the need to take into account the delay in the general case lead to the study of equations of the form

$$u_t(t,x) = F\left(u(t-\tau,x), u(t,x), u_x(t,x), u_{xx}(t,x)\right), \tag{2}$$

$$t \geq 0, \quad \tau > 0, \quad x \in \mathbb{R}.$$

Note that many models and problems that arise in biology, chemistry, economics, and other fields also lead to equations of the form (2), where the rate of a process may depend not only on the current, but also on previous states. The presence of delay in the nonlinear equation (2) significantly complicates the study and construction of exact solutions. A number of results in this direction were obtained in [3, 17, 20, 21, 22, 23, 24, 25], the most studied are equations with one space variable. A detailed summary of the results and an extensive bibliographic list of more than 600 papers on the construction of exact solutions for partial differential equations with delay are given in the monograph [27]. We also note a detailed (219 sources cited) survey [29] on exact solutions and methods of numerical integration of a delayed PDEs. The existence of traveling waves and various types of their stability (asymptotic, exponential, global) for reaction-diffusion systems with delay were studied in papers [1, 9, 14, 15, 16, 19, 28, 32].

In this article, we will consider a multidimensional special case of the equation (2) of the following form

$$u_t = \nabla \cdot \left(u^\lambda \nabla u \right) + \alpha \bar{u}, \tag{3}$$

where $u = u(\mathbf{x}, t)$, $\bar{u} = u(\mathbf{x}, \bar{t})$, $\bar{t} = t - \tau$, $\mathbf{x} \in \mathbb{R}^n$; $\tau > 0$; ∇ — gradient operator relative to $\mathbf{x} \in \mathbb{R}^n$; $\lambda \neq 0$ — real parameter of medium nonlinearity; $\alpha \neq 0$ — some constant. In the equation (3), $u(\mathbf{x}, t)$ represents the formation density at time t at a point in space $\mathbf{x} \in \mathbb{R}^n$, therefore, only non-negative solutions $u(\mathbf{x}, t) \geq 0$ are of interest. The left side in (3) is the rate of formation density change over time, the first term on the right side describes the nonlinear diffusion process of formation propagation in space, and the second term is a distributed control action formed with some informational delay $\tau > 0$ according to the principle of linear feedback. Wherein, in the case of negative feedback $\alpha < 0$, when zero density is reached, the control signal is switched off, i.e. $\bar{u} = 0$ for $u(\mathbf{x}, t) = 0$.

Note that the above interpretation of the equation (3) is certainly not the only one. This equation can also be considered as an equation of nonlinear heat conduction with a linear source (sink), in which the process of heat release (or absorption) occurs with some delay $\tau > 0$. An example of such a process is heat transfer in a nuclear reactor [7, 10]. The equation (3) can also be considered as a mathematical model of the distribution of a biological species over the habitat [18], while the second term on the right side (3) takes into account reproduction, and the value $\tau > 0$ represents breeding time. Other interpretations of the equation (3) are also possible.

Thus, for many applications it is of interest to study the equation (3), construct its exact solutions, and study their properties. The main purpose of this work is to construct exact multidimensional solutions of the equation (3) in the form of a generalized separation of variables. To do this, we transform (3) by a simple replacement $u^\lambda = v$ to the equation for the new desired function v

$$v_t = v\Delta v + \nu |\nabla v|^2 + \alpha\lambda v^{1-\nu} \bar{v}^\nu, \tag{4}$$

where $\nu = 1/\lambda$, $v = v(\mathbf{x}, t)$, $\bar{v} = v(\mathbf{x}, \bar{t})$, Δ — the Laplace operator in $\mathbb{R}^n$. Exact multidimensional solutions of the equation (4) will be sought in the form of a multiplicative separation of variables [26]

$$v(\mathbf{x}, t) = \psi(t)\xi(\mathbf{x}), \tag{5}$$

with multidimensional function $\xi(\mathbf{x})$ the following kind

$$\xi(\mathbf{x}) = \frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + C, \tag{6}$$

where is the function $\psi(t) \in C^1([0, +\infty))$, non-zero numeric symmetric matrix A of size $n \times n$, the constant vector $\mathbf{B} \in \mathbb{R}^n$ and the constant $C \in \mathbb{R}$ are to be determined. In addition, in the case of $\lambda = 1$, multidimensional solutions of the equation (3) will be sought in the form of an additive separation of variables [26]

$$u(\mathbf{x}, t) = \varphi(t) + \xi(\mathbf{x}), \tag{7}$$

where the function $\varphi(t) \in C^1([0, +\infty))$ and the function $\xi(\mathbf{x})$ has the form (6).

2 Main results

After substituting the function (5) into the equation (4) and elementary transformations, we arrive at the expression

$$\psi'\xi = \psi^2\xi\Delta\xi + \frac{1}{\lambda}\psi^2|\nabla\xi|^2 + \alpha\lambda\psi^{1-\frac{1}{\lambda}}\bar{\psi}^{\frac{1}{\lambda}}\xi.$$

Here and below $\psi' = \dfrac{d\psi(t)}{dt}$, $\bar{\psi} = \psi(\bar{t}) \equiv \psi(t-\tau)$. Since $\xi(\mathbf{x})$ is defined by the formula (6), by direct calculation we find

$$|\nabla\xi(\mathbf{x})|^2 = (A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2,$$
$$\Delta\xi(\mathbf{x}) = \operatorname{tr} A. \tag{8}$$

Here $\operatorname{tr} A$ denotes the trace of matrix A. Taking into account the relations (8), we rewrite the last equality as

$$\left(\psi' - \operatorname{tr} A\psi^2 - \alpha\lambda\psi^{1-\frac{1}{\lambda}}\bar{\psi}^{\frac{1}{\lambda}}\right)\left(\frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + C\right) =$$
$$= \frac{1}{\lambda}\psi^2\left((A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2\right). \tag{9}$$

Lemma 1. *If the numerical symmetric matrix A, the vector $\mathbf{B} \in \mathbb{R}^n$ and the constant $C \in \mathbb{R}$ satisfy the system of algebraic equations (SAE)*

$$A = 2\sigma A^2, \quad \mathbf{B} = 2\sigma A\mathbf{B}, \quad C = \sigma|\mathbf{B}|^2, \tag{10}$$

where $\sigma \neq 0$ — separation constant, then the equality (9) reduces to an ordinary differential equation (ODE) with a delay of the following form:

$$\psi'(t) = \left(\operatorname{tr} A + \frac{1}{\lambda\sigma}\right)\psi^2(t) + \alpha\lambda\psi^{1-\frac{1}{\lambda}}(t)\psi^{\frac{1}{\lambda}}(t-\tau). \tag{11}$$

Proof. If the numeric symmetric matrix A, the vector $\mathbf{B} \in \mathbb{R}^n$ and the constant $C \in \mathbb{R}$ satisfy SAE (10), then (9) can be rewritten as

$$\left[\psi'(t) - \left(\operatorname{tr} A + \frac{1}{\lambda\sigma}\right)\psi^2(t) - \alpha\lambda\psi^{1-\frac{1}{\lambda}}(t)\psi^{\frac{1}{\lambda}}(t-\tau)\right] \times$$
$$\times\left[(A^2\mathbf{x}, \mathbf{x}) + 2(A\mathbf{B}, \mathbf{x}) + |\mathbf{B}|^2\right] = 0.$$

This equality turns into an identity when the expression in the first square bracket, which is the ODE (11), vanishes. The lemma is proven. □

Thus, the construction of exact multidimensional solutions of the nonlinear heat equation with a linear delayed source (sink) (3) using the ansatz (5) has been reduced to integrating an ODE with delay (11) and the solvability of the system algebraic equations SAE (10).

We note that the SAE (10) was previously studied by the authors in [11, 12] in the study and construction of exact multidimensional solutions of the nonlinear system of reaction-diffusion equations using the construction (6). Therefore, we will not repeat its complete study, but only present the solution of this system. The matrix

$$A = \frac{1}{2\sigma}\, S E_m S^T, \tag{12}$$

is a solution to the matrix equation (10). Here E_m — is a diagonal matrix with $m \in \{1, 2, \dots, n\}$ ones and $n - m$ zeros on the diagonal, S – is an arbitrary orthogonal matrix. At the same time, we have

$$\operatorname{tr} A = \frac{m}{2\sigma}, \quad m \le n, \quad n \in \mathbb{N}, \quad n \ge 2. \tag{13}$$

The second equation (10) is a system of n linear homogeneous algebraic equations with respect to the components $b_1, \dots, b_n$ of the required vector $\mathbf{B}$. For any fixed matrix A of the form (12) with $\operatorname{rank} A = m < n$, there always exists a non-trivial solution of the linear homogeneous system, with m components of the vector $\mathbf{B}$ can be chosen arbitrarily from m — dimensional linear manifold. In the case when $\operatorname{rank} A = m = n$, i.e. for $E_m \equiv E$, the linear homogeneous system of equations has a solution — an arbitrary vector $\mathbf{B} \in \mathbb{R}^n$. After successively finding solutions to the matrix and vector equations, the constant C is uniquely determined from the scalar equation (10).

Thus, we have arrived at the validity of the following main result of this paper.

Theorem 1. *If the matrix A is given by the formula (12), the vector $\mathbf{B}$ is determined from the linear SAE $\mathbf{B} = 2\sigma\, A\mathbf{B}$, and the function $\psi(t) \in C^1([0, +\infty))$ satisfies ODE with delay (11), then the following function*

$$u(\mathbf{x}, t) = \left[\psi(t)\left(\frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + \sigma|\mathbf{B}|^2\right)\right]^{\frac{1}{\lambda}}, \tag{14}$$

is an exact multidimensional solution of the equation (3).

Note that the spatial structure of solutions (14) is determined by the rank of the matrix A of the form (12). If $\operatorname{rank} A = m = 1$, then we have «pseudo-multidimensional» exact solutions, that is, solutions with a linear combination of space variables. If $1 < \operatorname{rank} A = m < n$, then we obtain exact solutions of the equation (3) anisotropic in space variables. Finally, if $\operatorname{rank} A = m = n$, then we have exact solutions that are radially symmetric in space variables. In what follows, we will consider $m > 1$, $m \in \mathbb{Z}$.

Now let $\lambda = 1$ and the solution is constructed as an additive separation of variables (7). In this case, the assertion is true.

State 1. *Nonlinear equation with delay*

$$u_t = u\Delta u + |\nabla u|^2 + \alpha \bar{u}, \quad \bar{u} = u(\mathbf{x}, t-\tau), \tag{15}$$

has an exact solution of the form (7), in which the function $\xi(\mathbf{x})$ is given by the formula (6), a nonzero numerical symmetric matrix A of size $n \times n$, the constant vector $\mathbf{B} \in \mathbb{R}^n$, and the constant $C \in \mathbb{R}$ satisfy SAE

$$A(\operatorname{tr} A + \alpha) + 2A^2 = 0, \quad \mathbf{B}(\operatorname{tr} A + \alpha) + 2A\mathbf{B} = 0, \tag{16}$$
$$C(\operatorname{tr} A + \alpha) + |\mathbf{B}|^2 = 0,$$

and the function $\varphi(t) \in C^1([0, +\infty))$ satisfies a linear ODE with delay

$$\varphi'(t) = \operatorname{tr} A\, \varphi(t) + \alpha \varphi(t-\tau), \tag{17}$$

where $\tau > 0$ — numeric lag parameter.

Note that the exact solutions of the equation (15) in the one-dimensional case, in the form of a multiplicative and additive separation of variables, were constructed in [3].

Consider the solvability of the matrix equation (16). Obviously, it always has a trivial solution $A = 0$. Therefore, in what follows, we will consider only nontrivial solutions of this equation, assuming that $\operatorname{tr} A + \alpha \neq 0$. Otherwise, the matrix equation $A^2 = 0$, in the class of symmetric matrices, has a solution of the real matrix $A = 0$.

State 2. *Let E_m – be a diagonal matrix with $m \in \{1, 2, \dots, n\}$ ones and $n-m$ zeros placed on the diagonal arbitrarily. Then the matrix*

$$A = \nu S E_m S^T, \quad \nu = -\frac{\alpha}{2+m}, \tag{18}$$

where S — arbitrary orthogonal matrix, is a solution to the matrix equation (16).

Proof. The trace of the matrix A defined by the formula (18) has the form $\operatorname{tr} A = m\nu \equiv -\dfrac{\alpha m}{2+m}$. Taking this relation into account, the matrix equation (16) will be rewritten as $\nu A = A^2$. It is easy to check that the solution to this matrix equation is $A = \nu P$, where P is an arbitrary idempotent matrix, i.e. matrix satisfying the equality $P^2 = P$. It is known [6] that any idempotent matrix P can be written as $P = M E_m M^{-1}$, where M — arbitrary nonsingular matrix with order n, E_m — a diagonal matrix with $m \in \{1, \dots, n\}$ units and $n-m$ zeros on the diagonal; E_m is also idempotent: $E_m^2 = E_m$. Since we are only interested in symmetric matrices A, we must also take idempotent matrices P as symmetric ones, i.e. $P = S E_m S^T$, where S — is an arbitrary orthogonal matrix. From here we obtain the final form of the matrix A defined by the formula (18). The assertion has been proven. □

Second equations (16) are systems of n linear homogeneous algebraic equations with respect to the components $b_1, \dots, b_n$ of the required vector $\mathbf{B}$. For any fixed matrix (18) with $\operatorname{rank} A = m < n$, there always exists a nontrivial solution to the linear homogeneous system, and the components

$b_1, \dots, b_m$ of the vector $\mathbf{B}$ can be chosen arbitrarily from m – dimensional linear manifold. In the case when $\operatorname{rank} A = m \equiv n$, i.e. for $E_m \equiv E$, the linear homogeneous system of equations has a solution – an arbitrary vector $\mathbf{B} \in \mathbb{R}^n$. After successively finding solutions to the matrix and vector equations, the constants C is uniquely determined from the scalar equation (16).

3 Exact Solutions to ODEs with Delay

First, consider a linear ODE with delay (17), which, taking into account the formula (18), can be written as

$$\varphi'(t) = -\frac{\alpha m}{2+m}\,\varphi(t) + \alpha\varphi(t-\tau), \tag{19}$$

where $m \in \{1, 2, \dots, n\}$. The solution of this equation will be sought by the Euler method

$$\varphi(t) = \varphi_0 e^{\mu t}, \tag{20}$$

where $\varphi_0 \neq 0$ — arbitrary constant. In particular, we can put $\varphi_0 = \varphi(t)|_{t=0}$. Substituting (20) into (19) we obtain the transcendental characteristic equation for determining μ

$$\mu = -\frac{\alpha m}{2+m} + \alpha e^{-\mu\tau}.$$

The solution of this transcendental equation can be expressed in terms of the special Lambert W function [4, 30]. The Lambert $W(z)$ function is defined as the solution of the equation $z = W\exp(W)$. The derivative of the function $W(z)$ has the form

$$\frac{dW(z)}{dz} = \frac{W(z)}{z\,(W(z)+1)}.$$

This equality can be viewed as an ordinary differential equation that is satisfied by the Lambert $W(z)$ function. The Lambert $W(z)$ function is defined on the interval $z \in \left(-e^{-1}, +\infty\right)$ and takes values from the interval $(-\infty, +\infty)$. For negative values of the argument, the Lambert function is bivalent. In the following, we will use the Lambert function to represent solutions of ordinary differential equations with delay. Thus, the solution of the equation (19) is expressed by the following formula

$$\varphi(t) = \varphi_0 \exp\left(\frac{(m+2)\, W\left(\alpha\tau\exp\left(\dfrac{\alpha\tau m}{m+2}\right)\right) - \alpha\tau m}{\tau(m+2)}\, t\right),$$

where $W(\cdot)$ is the Lambert W function, $\varphi_0 \neq 0$ — arbitrary constant.

Example 1. Let $n=4$, $m=3$, then the equation (15) in four-dimensional coordinate space based on state 1 has an exact, spatially anisotropic, solution

$$u(x_1,x_2,x_3,x_4,t)=\varphi_0\exp\left(\frac{5\,W\left(\alpha\tau\exp\left(\frac{3}{5}\alpha\tau\right)\right)-3\alpha\tau}{5\tau}\,t\right)+$$

$$+\frac{\alpha}{160}\left(-13x_1^2-7x_2^2-12x_3^2-16x_4^2+6\sqrt{3}\,x_1x_2-4\sqrt{3}\,x_1x_3-12x_2x_3\right)+$$

$$+b_1x_1+\left(-\frac{1}{3}\sqrt{3}\,b_1+\frac{2}{3}\,b_2\right)x_2+b_2x_3-\frac{5}{18\alpha}\left(12b_1^2-4\sqrt{3}\,b_1b_2+13b_2^2\right),$$

where $W(\cdot)$ is the Lambert W function, $\alpha\neq 0$, $\psi_0\neq 0$, b_i, $i=1,2$ — arbitrary constants.

Now let's move on to constructing exact solutions to ODEs with delay (11). Here we have to consider two cases $\operatorname{tr}A+\dfrac{1}{\lambda\sigma}=0$ and $\operatorname{tr}A+\dfrac{1}{\lambda\sigma}\neq 0$.

1. Let the elements of the main diagonal of the desired matrix A be such that its trace $\operatorname{tr}A$ satisfies the equality

$$\operatorname{tr}A+\frac{1}{\lambda\sigma}=0, \tag{21}$$

in this case, we must look for the solution of the matrix equation (10) loaded with the condition (21). Taking into account the formula (13), the additional condition (21) on the trace of the matrix A is equivalent to the equality $\lambda=-2/m$. Then the delay ODE (11) becomes simpler and takes the form

$$\psi'(t)=-\frac{2\alpha}{m}\,\psi^{1+\frac{m}{2}}(t)\psi^{-\frac{m}{2}}(t-\tau), \tag{22}$$

where $m\in\{1,2,\ldots,n\}$, $m\leq n$, $n\in\mathbb{N}$, $n\geq 2$. The equation (22) can be rewritten as $\left(\psi^{-\frac{m}{2}}(t)\right)'=\alpha\psi^{-\frac{m}{2}}(t-\tau)$ or

$$z'(t)=\alpha z(t-\tau), \tag{23}$$

that is, it becomes linear with respect to the new function $z(t)=\psi^{-\frac{m}{2}}(t)$. The linear delay ODE (23) has a general solution $z(t)=z_0e^{\mu t}$, $z_0=\mathrm{const}\neq 0$, where the parameter μ is from the transcendental equation $\mu e^{\mu\tau}=\alpha$, whose solution is expressed in terms of the special Lambert W function. Finally, we obtain the exact solution of the equation (22) of the following form

$$\psi(t)=\psi_0\exp\left(-\frac{2\,W(\alpha\tau)}{m\tau}\,t\right),$$

where $W(\alpha\tau)$ is the Lambert W function, $\psi_0\neq 0$ — is an arbitrary constant, in particular, $\psi_0=\psi(t)|_{t=0}$ — is an arbitrary constant. Thus, in this case, taking into account the theorem 1, the assertion is true.

State 3. *Nonlinear equation with delay*

$$u_t = \nabla \cdot \left(u^{-\frac{2}{m}} \nabla u \right) + \alpha \bar{u}, \quad \bar{u} = u(\mathbf{x}, t - \tau),$$

where $m \in \{1, 2, \ldots, n\}$, $m \leq n$, $n \in \mathbb{N}$, $n \geq 2$ *has the exact solution*

$$u(\mathbf{x}, t) = \left[l_0 \exp\left(-\frac{2\, W(\alpha\tau)}{m\tau}\, t \right) \left(\frac{1}{2}(A\mathbf{x}, \mathbf{x}) + (\mathbf{B}, \mathbf{x}) + \sigma |\mathbf{B}|^2 \right) \right]^{-\frac{m}{2}},$$

where the numerical symmetric matrix A *has the form* (12), *and the vector* $\mathbf{B}$ *is determined from the linear* SAE $\mathbf{B} = 2\sigma\, A\mathbf{B}$, $W(\alpha\tau)$ *is the Lambert* W *function,* $l_0 \neq 0$ *— arbitrary constant.*

Example 2. Let $m = 2$, then by state 3 for $\lambda = -1$ the nonlinear equation with delay

$$u_t = \Delta \ln u + \alpha \bar{u}, \quad \bar{u} = u(x, y, z, t - \tau),$$

has a particular exact solution anisotropic in space variables of the following form

$$u(x, y, z, t) = \left[\psi_0 \exp\left(-\frac{W(\alpha\tau)}{\tau}\, t \right) \xi(x, y, z) \right]^{-1},$$

where

$$\xi(x, y, z) = \frac{1}{256\sigma} \left(55\, x^2 + 39\, y^2 + 34\, z^2 - 30\, xy - 6\sqrt{30}\, xz - 10\sqrt{30}\, yz \right) -$$
$$-\frac{1}{3}(5\, l_1 + \sqrt{30}\, l_2)\, x + l_1 y + l_2 z + \frac{\sigma}{9} \left(34\, l_1^2 + 10\sqrt{30}\, l_1 l_2 + 39\, l_2^2 \right),$$

$W(\alpha\tau)$ is the Lambert W function, $\alpha \neq 0$, $\psi_0 \neq 0$, $\sigma \neq 0$, l_1, l_2, l_3 — arbitrary parameters.

Example 3. Let $m = 3$, then by state 3 $\lambda = -2/3$ the nonlinear equation with a linear delayed source (sink)

$$u_t = \nabla \cdot \left(u^{-\frac{2}{3}} \nabla u \right) + \alpha \bar{u}, \quad \bar{u} = u(x, y, z, t - \tau),$$

in three-dimensional coordinate space has a particular exact radially symmetric solution

$$u(x, y, z, t) = \left[\psi_0 \exp\left(-\frac{2\, W(\alpha\tau)}{3\tau}\, t \right) \xi_0(x, y, z) \right]^{-\frac{3}{2}},$$

where

$$\xi_0(x, y, z) = \frac{1}{4\sigma} \left(x^2 + y^2 + z^2 \right) + l_1 x + l_2 y + l_3 z + \sigma \left(l_1^2 + l_2^2 + l_3^2 \right),$$

$W(\alpha\tau)$ is the Lambert W function, $\alpha \neq 0$, $\psi_0 \neq 0$, $\sigma \neq 0$, l_1, l_2, l_3 — arbitrary parameters.

If the delay parameter τ is chosen as a fixed positive number, then the general solution of the linear ODE with delay (23) can be easily written as a linear combination of sine and cosine. Let $\tau = -\dfrac{(4k+1)\pi}{2\alpha}$, $\alpha < 0$, k – be any non-negative integer, then the general solution of the equation (22) has the form

$$\psi(t) = [C_1 \cos(\alpha t) + C_2 \sin(\alpha t)]^{-\frac{2}{m}},$$

where C_1, C_2 — are arbitrary constants such that $C_1 C_2 \neq 0$.

2. Let us start constructing exact solutions to ODEs with delay (11) in the general case, assuming that the inequality $\operatorname{tr} A + \dfrac{1}{\lambda\sigma} \neq 0$ holds, or, taking into account the formula (13), condition $\lambda \neq -2/m$. By a simple change $\psi(t) = \theta^{\lambda}(t)$ we rewrite the equation (11) as a relation for the new function $\theta(t)$

$$\theta'(t) = \frac{a}{\lambda}\,\theta^{\lambda+1}(t) + \alpha\theta(t-\tau), \tag{24}$$

where $a \equiv \operatorname{tr} A + \dfrac{1}{\lambda\sigma} = \dfrac{1}{\sigma}\left(\dfrac{m}{2} + \dfrac{1}{\lambda}\right) \neq 0$ — constant. In the case of $\lambda = -1$ ($m \neq 2$), the equation (24) becomes linear and its general solution has the form

$$\theta(t) = \theta_0 \exp\left(\frac{W(\alpha\tau)}{\tau}\,t\right) + \frac{1}{\alpha\sigma}\left(\frac{m}{2} - 1\right),$$

where $W(\alpha\tau)$ is the Lambert W function, $\theta_0 \neq 0$ — arbitrary constant.

Example 4. Let $\lambda = -1$, $m = 3$, then the nonlinear equation with a linear delayed source (sink) in a three-dimensional coordinate space of the following form

$$u_t = \nabla \cdot \left(u^{-1}\nabla u\right) + \alpha\bar{u}, \quad \bar{u} = u(x, y, z, t-\tau),$$

by Theorem 1 has a particular exact radially symmetric solution

$$u(x, y, z, t) = \frac{\theta_0 \exp\left(\dfrac{W(\alpha\tau)}{\tau}\,t\right) + \dfrac{1}{2\alpha\sigma}}{\xi_0(x, y, z)},$$

where

$$\xi_0(x, y, z) = \frac{1}{4\sigma}\left(x^2 + y^2 + z^2\right) + l_1 x + l_2 y + l_3 z + \sigma\left(l_1^2 + l_2^2 + l_3^2\right),$$

$W(\alpha\tau)$ is the Lambert W function, $\alpha \neq 0$, $\theta_0 \neq 0$, $\sigma \neq 0$, l_1, l_2, l_3 — arbitrary parameters.

4 Conclusion

In conclusion, we briefly formulate the main results. The article studies the problem of constructing exact solutions to a nonlinear equation of parabolic type with delay, considered as a mathematical model for changing the density of a distributed formation of autonomous robots. A variant of the reduction method is proposed, which makes it possible to separate the search for the dependence of the solution on time and on spatial variables. The dependence on spatial variables is found from a system of algebraic equations, and the dependence on time from a system of ordinary differential equations with delay. A number of examples are given of constructing exact solutions that are explicitly expressed in terms of elementary functions and the Lambert function. As noted in the review [5], most recently partial differential and delayed equations have been used for modelling the movement of formations of interacting robots in swarm robotics, while the main difficulty is that «analytical solutions of PDEs are available only for a small number of special cases». Therefore, the exact solutions of delay PDEs found in the article will be useful in the study of swarm robotics models.

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