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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 20, No. 2, pp. 144–145 (2023) УДК 510.67
https://doi.org/10.33048/semi.2023.20.??? MSC 03C64

TYPE-PRESERVING FORMULAS IN WEAKLY O-MINIMAL THEORIES

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Communicated by P.P. Petrov

Abstract: In the present paper we study properties of p-preserving convex-to-right (left) formulas in weakly o-minimal theories, where p is a non-algebraic 1-type. It is proved that in the case of an existence of a p-preserving convex-to-right (left) formula that is not equivalence-generating, there exists a p-preserving convex-to-left (right) formula that is also not equivalence-generating; it was shown how it is built from the original formula.

Keywords: weak o-minimality, type-preserving formula, convex-to-right (left) formula, equivalence relation, convexity rank.

1 Preliminaries

Let L be a countable first-order language. Throughout this paper we consider L–structures and suppose that L contains a binary relation symbol $<$ which is interpreted as a linear order in these structures. This paper concerns the notion of *weak o-minimality* which was initially deeply studied by H.D. Macpherson, D. Marker and C. Steinhorn in [1]. A subset A of a linearly ordered structure M is *convex* if for all $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters)

Kulpeshov, B.Sh., Type-preserving formulas in weakly o-minimal theories.

This work was supported by Science Committee of Ministry of Science and Higher Education of the Republic of Kazakhstan, Grant No. BR20281002.

Received January, 1, 2023, Published December, 31, 2023.

subset of M is a union of finitely many convex sets in M. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal structures.

Let A and B be arbitrary subsets of a linearly ordered structure M. Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$, and $A < b$ means that $A < \{b\}$. For an arbitrary subset A of M we introduce the following notations: $A^+ := \{b \in M \mid A < b\}$ and $A^- := \{b \in M \mid b < A\}$. For an arbitrary one-type p we denote by $p(M)$ the set of realizations of p in M. If $B \subseteq M$ and E is an equivalence relation on M then we denote by B/E the set of equivalence classes (E–classes) which have representatives in B. If f is a function on M then we denote by $Dom(f)$ the domain of f. A theory T is said to be *binary* if every formula of the theory T is equivalent in T to a boolean combination of formulas with at most two free variables.

Further throughout the paper we consider an arbitrary complete theory T (if unless otherwise stated), where M is a sufficiently saturated model of T.

Definition 1. [2] Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic.

(1) An L_A-formula $F(x, y)$ is said to be *p-preserving* if there exist α, γ_1, $\gamma_2 \in p(M)$ such that

$$[F(M, \alpha) \setminus \{\alpha\}] \cap p(M) \neq \emptyset \text{ and } \gamma_1 < F(M, \alpha) \cap p(M) < \gamma_2.$$

(2) A p-preserving formula $F(x, y)$ is said to be *convex-to-right (left)* if there exists $\alpha \in p(M)$ such that $F(M, \alpha) \cap p(M)$ is convex, α is the left (right) endpoint of the set $F(M, \alpha) \cap p(M)$ and $\alpha \in F(M, \alpha)$.

Definition 2. [3] Let $F(x, y)$ be a p–preserving convex-to-right (left) formula. We say that $F(x, y)$ is said to be *equivalence-generating* if for any $\alpha, \beta \in p(M)$ such that $M \models F(\beta, \alpha)$ the following holds:

$$\sup[F(M, \alpha) \cap p(M)] = \sup[F(M, \beta) \cap p(M)]$$
$$(\text{resp. } \inf[F(M, \alpha) \cap p(M)] = \inf[F(M, \beta) \cap p(M)]).$$

Let us recall some notions originally introduced in [1]. Let $Y \subset M^{n+1}$ be $\emptyset$-definable, let $\pi : M^{n+1} \to M^n$ be a projection that drops the last coordinate, and let $Z := \pi(Y)$. For each $\bar{a} \in Z$, let $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose that for every $\bar{a} \in Z$, the set $Y_{\bar{a}}$ is bounded from above but has no supremum in M. Let $\sim$ be an $\emptyset$-definable equivalence relation on M^n, defined as follows:

$$\bar{a} \sim \bar{b} \text{ for all } \bar{a}, \bar{b} \in M^n \setminus Z, \text{ and } \bar{a} \sim \bar{b} \Leftrightarrow \sup Y_{\bar{a}} = \sup Y_{\bar{b}}, \text{ if } \bar{a}, \bar{b} \in Z.$$

Let $\overline{Z} := Z/\sim$, and for each tuple $\bar{a} \in Z$, we denote by $[\bar{a}] \sim$ the class of tuple $\bar{a}$. There is a natural $\emptyset$–definable linear ordering on $M \cup \overline{Z}$, defined as follows.

Let $\bar{a} \in Z$ and $c \in M$. Then $[\bar{a}] < c$ if and only if $w < c$ for all $w \in Y_{\bar{a}}$. If $\bar{a} \not\sim \bar{b}$, then there exists some $x \in M$ such that $[\bar{a}] < x < [\bar{b}]$ or $[\bar{b}] < x < [\bar{a}]$, and therefore $<$ induces a linear order on $M \cup \overline{Z}$. We call such a set $\overline{Z}$ *a sort* (in this case, an $\emptyset$–definable sort) in $\overline{M}$, where $\overline{M}$ is the Dedekind completion of the structure M, and we view $\overline{Z}$ as naturally embedded in $\overline{M}$. Similarly, we can obtain a sort in $\overline{M}$ by considering infima instead of suprema.

Thus, we will consider definable functions from M in its Dedekind completion $\overline{M}$, more precisely into definable sorts of the structure $\overline{M}$, representing infima or suprema of definable sets.

Let $A, D \subseteq M$, D be infinite, $Z \subseteq \overline{M}$ be an A–definable sort and $f : D \to Z$ be an A-definable function. We say that f is *locally increasing* (*locally decreasing, locally constant*) on D if for any $a \in D$ there exists an infinite interval $J \subseteq D$ containing $\{a\}$ such that f is strictly increasing (strictly decreasing, constant) on J; we also say that f is *locally monotonic* on D if it is locally increasing or locally decreasing on D.

Let f be an A-definable function on $D \subseteq M$, E be an A-definable equivalence relation on D. We say that f is *strictly increasing (decreasing) on D/E* if for any $a, b \in D$ with conditions $a < b$ and $\neg E(a, b)$ we have $f(a) < f(b)$ $(f(a) > f(b))$.

Theorem 1. [4, 5] *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. Suppose that there exists an A-definable function f the domain of which contains the set $p(M)$, and f is not a constant on $p(M)$. Then f is locally monotonic or locally constant on $p(M)$ and there exists an A-definable equivalence relation $E(x, y)$ partitioning $p(M)$ into infinitely many convex classes such that f is strictly monotonic on $p(M)/E$.*

Definition 3. [6, 7] Let M be a weakly o-minimal structure, $B, D \subseteq M$, $A \subseteq \overline{M}$ be a B-definable sort and $f : D \to A$ be a B-definable function that is locally increasing (decreasing) on D. We say that f *has depth* n on D if there are B-definable equivalence relations $E_1(x, y), \dots, E_n(x, y)$, partitioning D into infinitely many infinite convex classes, so that for any $2 \le i \le n$ every E_i-class is partitioned into infinitely many infinite convex E_{i-1}-subclasses and the following holds:

- f is strictly increasing (decreasing) on each E_1-class,
- f is locally decreasing (increasing) on D/E_k for any odd $k \le n$ (or the same, f is strictly decreasing (increasing) on every $E_{k+1}(a, M)/E_k$ for any $a \in D$),
- f is locally increasing (decreasing) on D/E_k for for any even $k \le n$,
- f is strictly monotonic on D/E_n.

In this case, the function f is called *locally increasing (decreasing) of depth n*.

Obviously, a strictly increasing (decreasing) function is *locally increasing (decreasing) of depth* 0.

Theorem 2. [7] *Let T be a weakly o-minimal theory. Then any definable function into a definable sort has finite depth.*

In [8], Definition 3 was extended by introducing the notion of *a locally constant function of depth* n, i.e., in Definition 3 the function f is a constant on every E_1-class. Note that in this case the function f can be either locally increasing or locally decreasing on D/E_1.

Definition 4. [9] Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$. *The rank of convexity of the set* A ($RC(A)$) is defined as follows:

1) $RC(A) = -1$ if $A = \emptyset$.
2) $RC(A) = 0$ if A is finite and non-empty.
3) $RC(A) \geq 1$ if A is infinite.
4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:
   - For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
   - For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.
5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α, we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there are no definable (with parameters) equivalence relations with infinitely many infinite convex classes. Clearly, each o-minimal theory has convexity rank 1.

In [3] properties of p-preserving convex-to-right (left) formulas were studied. In particular, it was established that some of these formulas generate an equivalence relation with infinite convex classes. These results were used at studying the number of pairwise non-isomorphic countable models for a weakly o-minimal theory in series of papers [10]–[12]. Here we continue studying properties of p-preserving convex-to-right (left) formulas. In particular, we prove the following: if there exists a p-preserving convex-to-right (left) formula that is not equivalence-generating then there exists a p-preserving convex-to-left (right) formula that is also not equivalence-generating (Theorem 3). We show how such a formula is constructed from an existing one.

2 Properties of p-preserving formulas

Fact 1. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. Then either $p(M')$ is densely ordered or $p(M')$ is discretely ordered for any $M' \succeq M$.*

Example 1. Let $M := \langle \mathbb{Q}, <, f^1 \rangle$ be a linearly ordered structure, where $\mathbb{Q}$ is the set of rational numbers, and $f(x) = x + 1$. Obviously, f is strictly increasing on M. It can be proved that M is an o-minimal 1-transitive structure. Let $p(x) := \{x = x\}$. Obviously, $p \in S_1(\emptyset)$, p is non-algebraic and $p(M)$ is densely ordered. Consider the following formula:

$$F(x, y) := y \leq x \leq f(y).$$

Obviously, $F(x, y)$ is a p-preserving convex-to-right formula that is not equivalence-generating.

Let $G(x, y) := F(y, x)$. Since $y \leq f(x)$ iff $f^{-1}(y) \leq x$, $G(x, y) \equiv f^{-1}(y) \leq x \leq y$ and $G(x, y)$ is a p-preserving convex-to-left formula that is also not equivalence-generating.

If $M = \langle \mathbb{Z}, < f^1 \rangle$, where $\mathbb{Z}$ is the set of integers, then M also is an o-minimal 1-transitive structure. Obviously, $p(x) := \{x = x\} \in S_1(\emptyset)$ and $p(M)$ is discretely ordered. Consider the following formulas:

$$F_1(x, y) := y \leq x < f(y), \qquad F_2(x, y) := y \leq x < f^2(y).$$

Since $F_1(M, \alpha) \setminus \{\alpha\} = \emptyset$, $F_1(x, y)$ is not p-preserving. But $F_2(x, y)$ is a p-preserving convex-to-right formula.

Proposition 1. *Let T be a weakly o-minimal theory of convexity rank 1, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic, $F(x, y)$ be a p-preserving convex-to-right (left) formula. Then for any $\alpha \in p(M)$ the following holds:*

$$M \models \forall x[F(x, \alpha) \land x \neq \alpha \to \exists z(F(z, x) \land \neg F(z, \alpha))].$$

Proof. Without loss of generality, suppose that $F(x, y)$ is convex-to-right. Since T has convexity rank 1 and $p(M)$ is 1-indiscernible over A, the function $f(y) := \sup F(M, y)$ is strictly monotonic on $p(M)$. Prove that f is strictly increasing on $p(M)$. Assume the contrary: f is strictly decreasing on $p(M)$. Then consider the following formula:

$$\phi(x) := \forall y[y < x \to F(x, y)].$$

Obviously, we have $\phi(M) \cap p(M) \neq \emptyset$ and $\neg\phi(M) \cap p(M) \neq \emptyset$, whence $p(M)$ is not 1-indiscernible over A. Consequently, f is strictly increasing on $p(M)$.

Take an arbitrary $\alpha \in p(M)$ and consider $F(M, \alpha)$. Take an arbitrary $\beta \in F(M, \alpha)$ such that $\beta \neq \alpha$. Obviously, $\alpha < \beta$. Since f is strictly increasing on $p(M)$, $f(\alpha) < f(\beta)$, whence there exists $\gamma \in F(M, \beta) \setminus F(M, \alpha)$. □

Proposition 2. *Let T be a weakly o-minimal theory of convexity rank 1, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic, $F(x, y)$ be a p-preserving convex-to-right (left) formula. Then $G(x, y) := F(y, x)$ is a p-preserving convex-to-left (right) formula.*

Proof of Proposition 2. Without loss of generality, suppose that $F(x, y)$ is convex-to-right. Take an arbitrary $\alpha \in p(M)$. Since $f(y) := \sup F(M, y)$ is strictly increasing on $p(M)$, $F(\alpha, M)$ is convex, $\alpha \in F(\alpha, M)$ and there exists $\gamma_1 \in p(M)$ such that $\gamma_1 < F(\alpha, M)$. And also for any $\beta \in p(M)$ with $\beta > \alpha$ we have $f(\beta) > \alpha$, i.e. $\alpha \notin F(M, \beta)$, whence α is the right endpoint of $F(\alpha, M)$. Thus, $G(x, y)$ is p-preserving convex-to-left.

Example 2. Let $M := \langle M, <, E^2, f^1 \rangle$ be a linearly ordered structure, where $M = \mathbb{Q} \times \mathbb{Q}$ is ordered lexicographically.

We define E as follows: for any $a = (a_1, a_2), b = (b_1, b_2) \in M$ we have $E(a, b)$ iff $a_1 = b_1$. Obviously, $E(x, y)$ is an equivalence relation partitioning M into infinitely many infinite convex classes so that the induced ordering on E-classes is dense.

We define f as follows: for any $a = (a_1, a_2) \in M$ we have $f((a_1, a_2)) = (a_1 + 1, -a_2)$. Then f is strictly decreasing on each E-class and f is strictly increasing on M/E.

It can be proved that $Th(M)$ is a weakly o-minimal theory, M is 1-transitive and $Th(M)$ has 2^ω countable models.

Let $p(x) := \{x = x\}$. Obviously, $p \in S_1(\emptyset)$, p is non-algebraic and $p(M) = M$. Consider the following formula:

$$F(x, y) := y \leq x \leq f(y).$$

Obviously, $F(x, y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Observe that for any $\alpha \in p(M)$ there exists $\beta \in p(M)$ such that $\alpha < \beta$ and

$$M \models F(\beta, \alpha) \wedge \exists x[F(x, \alpha) \wedge \neg F(x, \beta)].$$

Let $G(x, y) := F(y, x)$. Since f is strictly increasing on M/E, $G(x, y)$ is p-preserving. Observe also that $F(\gamma, M)$ consists of two convex sets for any $\gamma \in p(M)$. Then $G(M, \gamma)$ is not convex, whence $G(x, y)$ is not convex-to-left. Let

$$G'(x, y) := \exists t[F(y, t) \wedge t \leq x \leq y].$$

It can be checked that $G'(x, y)$ is a p-preserving convex-to-left formula that is not equivalence-generating.

Here $F(M, \alpha)$ has the right endpoint $f(\alpha)$ in M, but $G'(M, \alpha)$ has no a left endpoint in M.

If we define f as follows: $f((a_1, a_2)) = (a_1 + 1, -a_2 + \sqrt{2})$, then $F(M, \alpha)$ has no a right endpoint in M.

Example 3. Let $M := \langle M, <, E_1^2, E_2^2, f^1 \rangle$ be a linearly ordered structure, where $M = \mathbb{Q}^3$ is ordered lexicographically.

We define E_1 and E_2 as follows: for any $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in M$ we have $E_1(a, b)$ iff $a_1 = b_1 \wedge a_2 = b_2$, and $E_2(a, b)$ iff $a_1 = b_1$. Obviously, $E_1(x, y)$ and $E_2(x, y)$ are equivalence relations partitioning M into infinitely many infinite convex classes so that $E_1(M, a) \subset E_2(M, a)$ for any $a \in M$.

We define f as follows: for any $a = (a_1, a_2, a_3) \in M$ we have

$$f((a_1, a_2, a_3)) = (a_1 + 1, -a_2, a_3).$$

Then f is strictly increasing on each E_1-class, f is strictly decreasing on $E_2(M, a)/E_1$ for each $a \in M$ and f is strictly increasing on M/E_2.

It can be proved that $Th(M)$ is a weakly o-minimal theory. Let $p(x) := \{x = x\}$. Obviously, $p \in S_1(\emptyset)$ and p is non-algebraic. Consider the following formula:

$$F(x,y) := y \leq x \leq f(y).$$

Obviously, $F(x,y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Here $F(\gamma, M)$ consists of three convex sets for any $\gamma \in p(M)$.

Example 4. Let $M := \langle M, <, E_1^2, E_2^2, \ldots, E_{n-1}^2, f^1 \rangle$ be a linearly ordered structure, where $M = \mathbb{Q}^n$ is ordered lexicographically.

We define E_k for each $1 \leq k \leq n-1$ as follows: for any $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in M$ we have

$$E_k(a,b) \text{ iff } a_1 = b_1 \wedge a_2 = b_2 \wedge \ldots \wedge a_{n-k} = b_{n-k}.$$

Obviously, $E_k(x,y)$ is an equivalence relation partitioning M into infinitely many infinite convex classes for each $1 \leq k \leq n-1$ so that $E_1(M,a) \subset E_2(M,a) \subset \ldots \subset E_{n-1}(M,a)$ for any $a \in M$.

We define f as follows: for any $a = (a_1, a_2, \ldots, a_n) \in M$ we have

$$f((a_1, a_2, \ldots, a_n)) = (a_1 + 1, -a_2, a_3, \ldots, (-1)^{i+1} a_i, \ldots, (-1)^{n+1} a_n).$$

Then f is strictly increasing on M/E_{n-1}. If n is even then f is strictly decreasing on each E_1-class, and f is strictly increasing (decreasing) on $E_{k+1}(M,a)/E_k$ for each $a \in M$, where $1 \leq k \leq n-2$ and k is odd (even). If n is odd then f is strictly increasing on each E_1-class, f is strictly decreasing (increasing) on $E_{k+1}(M,a)/E_k$ for each $a \in M$, where $1 \leq k \leq n-2$ and k is odd (even).

It can be proved that $Th(M)$ is a weakly o-minimal theory. Let $p(x) := \{x = x\}$. Obviously, $p \in S_1(\emptyset)$ and p is non-algebraic. Consider the following formula:

$$F(x,y) := y \leq x \leq f(y).$$

Obviously, $F(x,y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Here $F(\gamma, M)$ consists of n convex sets for any $\gamma \in p(M)$.

Thus, we established the following:

Proposition 3. *For each natural $n \geq 1$ there exist a weakly o-minimal theory T, $M \models T$, non-algebraic $p \in S_1(\emptyset)$ and a p-preserving convex-to-right (left) formula $F(x,y)$ such that for any $\gamma \in p(M)$ the set $F(\gamma, M)$ consists of n convex sets in $p(M)$.*

Notation 1. Let $E(x,y)$ be an equivalence relation with convex classes on M, and $f : M \to \overline{M}$ be a function to a definable sort. Take arbitrary $a, b \in M$ and suppose that $f(b) \in \overline{M} \setminus M$. We write $E^*(a, f(b))$ if $a < f(b)$ and there exists $c \in M$ such that $f(b) < c$ and $E(a,c)$. We also write $E^*(f(b), a)$ if $f(b) < a$ and there exists $c \in M$ such that $c < f(b)$ and $E(c,a)$.

Proposition 4. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ non-algebraic, $F(x,y)$ a p-preserving convex-to-right (left) formula.*

Suppose that the function $f(y) := \sup F(M, y)$ is locally monotonic of depth n on $p(M)$ for some $n \in \omega$. Then

(1) n is even (odd) iff f is locally increasing (decreasing) on $p(M)$;

(2) $F(\gamma, M)$ consists of $n+1$ convex sets in $p(M)$.

Proof. By the hypothesis $f(y) := \sup F(M, y)$ is locally monotonic of depth n on $p(M)$ for some $n \in \omega$. Consequently, there exist A-definable equivalence relations $E_1(x, y), \dots, E_n(x, y)$ partitioning $p(M)$ into infinitely many infinite convex classes so that $E_1(M, a) \subset \dots \subset E_n(M, a)$ for any $a \in p(M)$.

Prove (1). If n is even then we assert that f is locally increasing on $p(M)$. Assume the contrary: f is locally decreasing on $p(M)$. Then f is strictly decreasing on each E_1-class, f is locally increasing on $p(M)/E_k$ for every odd $k \leq n$, f is locally decreasing on $p(M)/E_k$ for every even $k \leq n$. Whence we obtain that f is strictly decreasing on $p(M)/E_n$. Then we have a contradiction with the 1-indiscernibility of $p(M)$.

Similarly, we can establish that if n is odd then f is locally decreasing on $p(M)$.

Let's prove (2). If f is locally monotonic of depth 1 then by (1) f is locally decreasing on $p(M)$, i.e. f is strictly decreasing on each E_1-class and f is strictly increasing on $p(M)/E_1$. Take an arbitrary $\gamma \in p(M)$ and consider $E_1(M, \gamma)$. Then there exist $a_1, a_2 \in p(M)$ such that

$$a_1 < a_2, E_1(a_1, a_2), f(a_2) < \gamma < f(a_1), E_1^*(\gamma, f(a_1)) \text{ and } E_1^*(f(a_2), \gamma).$$

Then we assert that the following convex sets are contained in $F(\gamma, M)$: the set definable by $\neg E_1(x, a_2) \wedge a_2 < x \leq \gamma$ and a nonempty proper convex subset of the set definable by $E_1(x, a_2) \wedge x < a_2$. These convex sets are separable by the formula $E_1(x, a_2) \wedge x > a_2$. Thus, $F(\gamma, M)$ consists of two convex sets in $p(M)$.

Case A. n is even.

Then by (1) f is locally increasing on $p(M)$. Take an arbitrary $\gamma \in p(M)$ and consider $E_1(M, \gamma)$. Then there exist $a_1, a_2 \in p(M)$ such that

$$a_1 < a_2, E_1(a_1, a_2), f(a_1) < \gamma < f(a_2), E_1^*(f(a_1), \gamma) \text{ and } E_1^*(\gamma, f(a_2)).$$

Then we assert that the following convex sets are contained in $F(\gamma, M)$: the sets definable by $\neg E_n(x, a_2) \wedge a_2 < x \leq \gamma$, $E_k(x, a_2) \wedge \neg E_{k-1}(x, a_2) \wedge x < a_2$ for every even $1 < k \leq n$, $E_k(x, a_2) \wedge \neg E_{k-1}(x, a_2) \wedge x > a_2$ for every odd $2 < k \leq n$, and a nonempty proper convex subset of the set definable by $E_1(x, a_2) \wedge x > a_2$. Observe that these convex sets are definably separable. For example, $\neg E_n(x, a_2) \wedge a_2 < x \leq \gamma$ and $E_n(x, a_2) \wedge \neg E_{n-1}(x, a_2) \wedge x < a_2$ are separable by the formula

$$E_n(x, a_2) \wedge \neg E_{n-1}(x, a_2) \wedge x \geq a_2.$$

Thus, $F(\gamma, M)$ consists of $n+1$ convex sets in $p(M)$.

Case B. n is odd. This case considered similarly. □

Example 5. Let $M := \langle M, <, E^2, R^2 \rangle$ be a linearly ordered structure, where $M = \mathbb{Q} \times \mathbb{Q}$ is ordered lexicographically.

We define E as follows: for any $a=(a_1,a_2), b=(b_1,b_2)\in M$ we have $E(a,b)$ iff $a_1=b_1$. Obviously, $E(x,y)$ is an equivalence relation partitioning M into infinitely many infinite convex classes so that the induced ordering on E-classes is dense.

We define R as follows: for any $a=(a_1,a_2), b=(b_1,b_2)\in M$ we have

$$R(b,a) \text{ iff } a_1\le b_1<a_1+\sqrt{2}.$$

Then $r(y):=\sup R(M,y)$ is locally constant on M, and more exactly, r is constant on each E-class and r is strictly increasing on M/E (i.e. r is locally constant of depth 1 on M).

It can be proved that $Th(M)$ is a weakly o-minimal theory. Let $p(x):=\{x=x\}$. Obviously, $p\in S_1(\emptyset)$, p is non-algebraic and $p(M)=M$. Also, $R(x,y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Observe that $R(\gamma,M)$ is convex for any $\gamma\in p(M)$.

Example 6. Let $M:=\langle M,<,E_1^2,E_2^2,R^2\rangle$ be a linearly ordered structure, where $M=\mathbb{Q}^3$ is ordered lexicographically.

We define E_1 and E_2 as follows: for any $a=(a_1,a_2,a_3), b=(b_1,b_2,b_3)\in M$ we have $E_1(a,b)$ iff $a_1=b_1\wedge a_2=b_2$, and $E_2(a,b)$ iff $a_1=b_1$. Obviously, $E_1(x,y)$ and $E_2(x,y)$ are equivalence relations partitioning M into infinitely many infinite convex classes so that $E_1(M,a)\subset E_2(M,a)$ for any $a\in M$.

We define R as follows: for any $a=(a_1,a_2,a_3), b=(b_1,b_2,b_3)\in M$ we have

$$R(b,a) \text{ iff } b_1=a_1+1 \text{ and } b_2<-a_2+\sqrt{2}.$$

Then $r(y):=\sup R(M,y)$ is locally constant on M, and more exactly, r is constant on each E_1-class, r is strictly decreasing on $E_2(M,a)/E_1$ for each $a\in M$ and r is strictly increasing on M/E_2 (i.e. r is locally constant of depth 2 on M).

It can be proved that $Th(M)$ is a weakly o-minimal theory. Let $p(x):=\{x=x\}$. Obviously, $p\in S_1(\emptyset)$, p is non-algebraic and $p(M)=M$. Also, $R(x,y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Here $R(\gamma,M)$ consists of two convex sets for any $\gamma\in p(M)$.

Example 7. Let $M:=\langle M,<,E_1^2,E_2^2,\ldots,E_n^2,f^1\rangle$ be a linearly ordered structure, where $M=\mathbb{Q}^{n+1}$ is ordered lexicographically.

We define E_k for each $1\le k\le n$ as follows: for any $a=(a_1,a_2,\ldots,a_n, a_{n+1})$, $b=(b_1,b_2,\ldots,b_n,b_{n+1})\in M$ we have

$$E_k(a,b) \text{ iff } a_1=b_1\wedge a_2=b_2\wedge\ldots\wedge a_{n-k+1}=b_{n-k+1}.$$

Obviously, $E_k(x,y)$ is an equivalence relation partitioning M into infinitely many infinite convex classes for each $1\le k\le n$ so that $E_1(M,a)\subset E_2(M,a)\subset\ldots\subset E_n(M,a)$ for any $a\in M$.

We define R as follows: for any $a=(a_1,\ldots,a_n,a_{n+1})$, $b=(b_1,\ldots,b_n, b_{n+1})\in M$ we have

$$R(b,a) \text{ iff } b_1=a_1+1, b_2=-a_2,\ldots,b_i=(-1)^{i+1}a_i,\ldots,b_{n-1}=(-1)^n a_{n-1},$$

and $b_n < (-1)^{n+1}a_n + \sqrt{2}$. Then $r(y) := \sup R(M, y)$ is locally constant on M, r is constant on each E_1-class and r is strictly increasing on M/E_n. If n is even then r is strictly decreasing (increasing) on $E_{k+1}(M, a)/E_k$ for each $a \in M$, where $1 \leq k \leq n-1$ and k is odd (even). If n is odd then r is strictly increasing (decreasing) on $E_{k+1}(M, a)/E_k$ for each $a \in M$, where $1 \leq k \leq n-1$ and k is odd (even). Here r is locally constant of depth n on M.

It can be proved that $Th(M)$ is a weakly o-minimal theory. Let $p(x) := \{x = x\}$. Obviously, $p \in S_1(\emptyset)$, p is non-algebraic and $p(M) = M$. Also, $R(x, y)$ is a p-preserving convex-to-right formula that is not equivalence-generating. Here $R(\gamma, M)$ consists of n convex sets for any $\gamma \in p(M)$.

Proposition 5. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ non-algebraic, $F(x, y)$ a p-preserving convex-to-right (left) formula. Suppose that the function $f(y) := \sup F(M, y)$ is locally constant of depth n on $p(M)$ for some natural $n \geq 1$. Then*

(1) there exist A-definable equivalence relations $E_1(x, y), \ldots, E_n(x, y)$ partitioning $p(M)$ into infinitely many infinite convex classes so that $E_1(M, a) \subset \ldots E_n(M, a)$ for any $a \in p(M)$, f is strictly increasing on $p(M)/E_n$ and n is even (odd) iff f is locally decreasing (increasing) on $p(M)/E_k$ for every odd $k \leq n$ and f is locally increasing (decreasing) on $p(M)/E_k$ for every even $k \leq n$;

(2) $F(\gamma, M)$ consists of n convex sets in $p(M)$.

Proof. Prove (1). By the hypothesis $f(y) := \sup F(M, y)$ is locally constant of depth n on $p(M)$ for some natural $n \geq 1$. Consequently, there exist A-definable equivalence relations $E_1(x, y), \ldots, E_n(x, y)$ partitioning $p(M)$ into infinitely many infinite convex classes so that $E_1(M, a) \subset \ldots \subset E_n(M, a)$ for any $a \in p(M)$, and f is constant on each E_1-class.

Without loss of generality suppose n is even. Since f has depth n, f is locally increasing or locally decreasing on $p(M)/E_k$ for every odd $k \leq n$ and f is locally decreasing or locally increasing on $p(M)/E_k$ for every even $k \leq n$. If f is locally increasing on $p(M)/E_1$ then f is strictly decreasing on $p(M)/E_n$, whence we have a contradiction with the 1-indiscernibility of $p(M)$. Consequently, f is locally decreasing on $p(M)/E_k$ on every odd $k \leq n$ and f is locally increasing on $p(M)/E_k$ on every even $k \leq n$. Similar reasons for case when n is odd.

Let's prove (2). If f is locally constant of depth 1 then by (1) f is constant on each E_1-class and f is strictly increasing on $p(M)/E_1$. Take an arbitrary $\gamma \in p(M)$. Then there exist $a_1, a_2 \in p(M)$ such that $a_1 < a_2$, $\neg E_1(a_1, a_2)$ and $f(a_1) < \gamma < f(a_2)$. Since f is strictly increasing on $p(M)/E_1$, there exist a nonempty proper convex subset of the set definable by the formula $\neg E_1(x, a_1) \wedge x > a_1$ that defines $F(\gamma, M)$. Thus, $F(\gamma, M)$ is convex in $p(M)$.

Case A. n is even.

Then by (1) f is locally decreasing on $p(M)/E_k$ for every odd $k \leq n$ and f is locally increasing on $p(M)/E_k$ for every even $k \leq n$.. Take an arbitrary

$\gamma \in p(M)$. Then there exist $a_1, a_2 \in p(M)$ such that $a_1 < a_2$, $E_2(a_1, a_2)$, $\neg E_1(a_1, a_2)$, $f(a_2) < \gamma < f(a_1)$. Then we assert that the following convex sets are contained in $F(\gamma, M)$: the sets definable by $\neg E_n(x, a_2) \wedge a_2 < x \leq \gamma$, $E_k(x, a_2) \wedge \neg E_{k-1}(x, a_2) \wedge x < a_2$ for every even $1 < k \leq n$, $E_k(x, a_2) \wedge \neg E_{k-1}(x, a_2) \wedge x > a_2$ for every odd $2 < k \leq n$, and a nonempty proper convex subset of the set definable by $E_2(x, a_2) \wedge \neg E_1(x, a_2) \wedge x < a_2$. Observe that these convex sets are definably separable. For example,

$$\neg E_n(x, a_2) \wedge a_2 < x \leq \gamma \text{ and } E_n(x, a_2) \wedge \neg E_{n-1}(x, a_2) \wedge x < a_2$$

are separable by the formula $E_n(x, a_2) \wedge \neg E_{n-1}(x, a_2) \wedge x \geq a_2$. Thus, $F(\gamma, M)$ consists of n convex sets in $p(M)$.

Case B. n is odd. This case considered similarly. □

Theorem 3. *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ non-algebraic, $F(x, y)$ a p-preserving convex-to-right formula that is not equivalence-generating. Then*

$$G(x, y) := \exists t[F(y, t) \wedge t \leq x \leq y]$$

is a p-preserving convex-to-left formula that is also not equivalence-generating.

Proof. Consider $f(y) := \sup F(M, y)$. By Theorems 1 and 2 either f is locally monotonic of depth n for some $n \in \omega$ or f is locally constant of depth n for some natural $n \geq 1$.

Case 1. f is locally monotonic of depth n for some $n \in \omega$.

By Proposition 4 $F(\gamma, M)$ consists of $n+1$ convex sets in $p(M)$ being to the left from γ and including the element γ. Then obviously $G(M, \gamma) \cap p(M)$ is convex, γ is the right endpoint of $G(M, \gamma) \cap p(M)$ and $\gamma \in G(M, \gamma)$, whence we obtain that $G(x, y)$ is a p-preserving convex-to-left formula. Since f is strictly increasing on $p(M)/E_n$, we obatin that $G(x, y)$ is not equivalence-generating.

Case 2. f is locally constant of depth n for some natural $n \geq 1$.

By Proposition 5 $F(\gamma, M)$ consists of n convex sets in $p(M)$ being to the left from γ and including the element γ. The remaining reasons are similar as in Case 1. □

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