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OPTIMIZATION OF INTERVAL FRACTIONAL FUNCTIONS OVER EFFICIENT SET IN STOCHASTIC INTEGER PROGRAMMING

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Abstract: This paper addresses the problem of multi-objective stochastic integer linear programming (MOSILP) with joint chance constraints. In real-world applications, decision-makers often face multiple conflicting objectives under uncertainty. To manage these uncertainties, we employ chance-constrained programming (CCP), which transforms probabilistic constraints into deterministic equivalents. We propose an approach to optimize a linear fractional function with interval coefficients over the efficient set of a MOSILP. The stochastic objectives and constraints are converted into deterministic functions, allowing us to avoid the impractical task of determining all efficient solutions. An implicit algorithm is introduced to find an optimal solution that minimizes the fractional function without fully enumerating the efficient set. The convergence and finiteness of the algorithm are proven, and a numerical example is presented to illustrate its application. The results demonstrate the effectiveness of the proposed method in solving complex multi-objective stochastic optimization problems..

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1 Introduction

Multiobjective stochastic optimization is one of the important fields of study in operations research. Many real-world problems involve multiple objectives with random parameters. Due to conflict between objectives, finding a feasible solution that simultaneously optimizes all objectives is usually impossible. Consequently, in practice, decision makers want to explore and understand the trade off between objectives before choosing a suitable solution.

The multiobjective stochastic program (MOSP) problem was studied by Teghem and al. [17] who have presented interactive methods in stochastic programming, PROTRADE of Goicoechea and al. [9], and PROMISE of Urli and Nadeau [18]. These methods have been successfully tested in real world contexts. Ben Abdelaziz and al. [3] proposed a compromise chance constrained approach to solve a MOSP portfolio selection problem. The chance constrained programming (CCP) technique is one which can be used to solve problems involving chance constraints, i.e.,constraints having finite probability of being violated. The CCP was originally developed by Charnes and Cooper [6] and has, in recent years, been generalized in several directions and has various applications.

We consider the Chance Constraints Multiple Objective Stochastic Integer Linear Programming problem (CCMOSILP) [12]:

$$(\mathbf{CCMOSILP})\begin{cases} \min\ Z^k = \sum_{i=1}^{n} C_i^k x_i \quad k = 1, ..., K & \text{(1a)} \\ s.t. x = (x_1, ..., x_n) \in D & \text{(1b)} \\ \quad \mathbb{P}(\sum_{i=1}^{n} T_{ij} x_i \leq h_j) \geq 1 - \alpha_j, \ \ j = 1, ..., J & \text{(1c)} \end{cases}$$

We assume throughout the paper that $D_{cs} = D \cap [\mathbb{P}(Tx \leq h) \geq \alpha] \neq \emptyset$, where $D = S \cap \mathbb{Z}$ which $S = \{x \in \mathbb{R}^n : \quad Ax = b, \quad x \geq 0\}$ is a nonempty bounded polyhedron. S is the set of deterministic constraints with A is $(m \times n)$ matrix, b is m vector and T is $(J \times n)$ matrix; C^k and h are random matrices of respective dimension $(1 \times n)$ and $(J \times 1)$. $\mathbb{P}$ is the partially known probability distribution α are specified probabilities. Let $\mathbf{E}_{cs}$ denote the set of efficient solutions, whose definition will be given in the next section.

In many situations, the decision maker faces a large number of different efficient solutions and the selection of his/her preferred solutions becomes a very hard task. A way of assessing some preferred solution is by optimizing a function (utility function written as a function of decision variables), particularly linear, optimization over the efficient set, an appropriate approach that has received increasing attention in recent years. In [14] Philip first studied the problem and suggested an algorithm based on moving to adjacent efficient vertices when the function is a linear function. Later, Isermann and Steuer [10] outlined a similar procedure for solving the problem of optimizing over the efficient set, where the objective function is one of the multiobjective linear programming. Abbas and Chaabane (2006) [2], proposed a method for the optimization over the efficient set of a multiple objective integer linear programming (MOILP), where different types of cuts are imposed in such a way that the improvement of the objective value at each iteration is guaranteed. Jorge [11], Chaabane and Pirlo [4], developed another approach that defines a sequence of progressively more constrained single-objective integer problems that successively eliminates undesirable points. Zerdani and Moulaï [21] developed an approach that optimizes an arbitrary linear function over an integer efficient set of multibjective Linear fractional programming problem (MOLFP) without explicitly having to enumerate all the efficient solutions. Drici et al [7] have optimized a linear fractional function over the integer efficient set of the (MOILP) problem using the Branch and Bound technique strengthened by efficient cuts and tests. Younsi and Moulai [20] have optimized a stochastic linear over the efficient set of the Multi-Objective Stochastic Integer Linear Programming problem, it is based on Jorge's approach [11] with the concepts L-shaped integer method, using an Augmented Weighted Tchebychev program to generate the set of nondominated objective vectors. Recent advancements in multi-objective stochastic fractional programming have developed methods to manage uncertainties using probabilistic constraints and interval-valued coefficients. A stochastic fractional problem involving an inequality type of constraints, where all quantities on the right side are log-normal random variables, and the objective function coefficients are fractional intervals is presented in the paper [19].

In Management Science, there are numerous decision marking problems where the objective functions are linear fractional functions with interval coefficients. This type of functions can be found in game theory, portfolio selection, agriculture based management systems, in which the coefficients are not certain when they are modeled mathematically. The basic problem P_E that we investigate is to minimize a main linear fractional function with interval coefficients F over the set $\mathbf{E}_{cs}$:

$$(P_E)\begin{cases} \min \ F(x) = \dfrac{\delta + \sum_{i=1}^{n} P_i x_i}{\beta + \sum_{i=1}^{n} Q_i x_i} & \text{(2a)} \\ s.t. x \in \mathbf{E}_{cs} & \text{(2b)} \end{cases}$$

where δ, P_i, β and $Q_i \in I^+$ are an intervals which represents the uncertain coefficients of the objective function, $\forall i = 1, ..., n$, with $\delta = [\delta^1, \delta^2]$, $\beta = [\beta^1, \beta^2]$, $P_i = [p_i^1, p_i^2]$ and $Q_i = [q_i^1, q_i^2]$.

The main difficulty of the problem arises from the nonconvexity of the efficient set $\mathbf{E}_{cs}$, which is the union of several faces of D_{cs} (the problem is to be solve without solving (CCMOSILP)).

Associated with P_E, the relaxed problem is

$$\begin{cases} \min \quad F(x) = \dfrac{[\delta^1, \delta^2] + \sum_{i=1}^{n} [p_i^1, p_i^2] x_i}{[\beta^1, \beta^2] + \sum_{i=1}^{n} [q_i^1, q_i^2] x_i} \\ s.t. \quad x \in D_{cs} \end{cases} \tag{3}$$

It has also been assumed that $[\beta^1, \beta^2] + \sum_{i=1}^{n} [q_i^1, q_i^2] x_i > 0$ for all $x \in D_{cs}$.

In this paper, we focus on the problem of optimizing a linear fractional function with interval coefficients F, over the efficient set of a MOSILP with a joint chance constraint. We address the general case where F is reduced into a deterministic linear fractional function. Then, the stochastic objective function is converted into deterministic function. We also transform the chance constraint into a deterministic constraints by using known inverse distribution function. A direct approach could consist of finding all efficient solutions of the CCMOSILP problem and then finding the best value of F on that set. This approach is not appropriate for practical purposes, because of the difficulty of determining the set of all efficient solutions. We thus propose an implicit technique that avoids searching for all efficient solutions but guarantees finding one that minimize F.

The structure of the paper is organized as follows: Section 2 presents the formulation of chance constrained and describes the process of transforming CCMOSILP problem into an equivalent deterministic and compiles the basic result used throughout the manuscript. Section 3 presents new approach for reducing the fractional function with interval coefficients into an deterministic function. A proposition is provided to support this approach. Section 4 is devoted to proposed the different steps of present method and the algorithm. Two propositions are provided to justify finiteness and convergence of the algorithm. An extensive numerical example is solved in Section 5 to show the optimum of proposed problem P_E, and finally Section 6 presents the concluding remarks.

2 Chance constrained and efficiency testing

2.1. Deterministic Equivalents of Probabilistic Constraints. Use of chance constrained programming (CCP) introduces a new requirement upon decision makers. This approach was first introduced by Charnes et al. (see, [5]), where the objective is often an expectational functional as we used earlier (the E-model), or the V-model minimizes the generalized mean square of the objective functions, or the P-model maximizes the probability of aspiration levels of the objective functions. Another variation includes an objective that is a quantile of a random function [13]. Ben Abdelaziz et al. [3] proposed a compromise chance constrained approach to solve multiobjective stochastic programming a portfolio selection problem.

We assume that C_i^k are independent normal distributed random variables that follows $N(\eta_{c_i^k}, \delta^2_{c_i^k})$, and assuming that all coefficients of $\eta_{c_i^k}$ are integers. The mean of Z^k is given by

$$\widetilde{Z}^k = Esp(Z^k) = \sum_{i=1}^{n} Esp[C_i^k]x_i = \sum_{i=1}^{n} \eta_{c_i^k} x_i, \quad k = 1, ..., K,$$

It is assumed that h_j, $j = 1, ..., J$ are independent normal variables $N(\eta_{h_j}, \delta^2_{h_j})$. Where η_{h_j} is the j^{th} mean and $\delta^2_{h_j}$ is the jth variance. The $\mathbb{P}(\sum_{i=1}^{n} T_{ij}x_i \leq h_j) \geq 1 - \alpha_j$ constraint can be expressed as:

$$\mathbb{P}\Big(\frac{\sum_{i=1}^{n} T_{ij}x_i - \eta_{h_j}}{\delta_{h_j}} \leq \frac{h_j - \eta_{h_j}}{\delta_{h_j}}\Big) \geq 1 - \alpha_j, \quad j = 1, ..., J$$

On rearranging, we get

$$\mathbb{P}\Big(\frac{\sum_{i=1}^{n} T_{ij}x_i - \eta_{h_j}}{\delta_{h_j}} \geq \frac{h_j - \eta_{h_j}}{\delta_{h_j}}\Big) \leq \alpha_j, \quad j = 1, ..., J$$

But $\frac{h_j-\eta_{h_j}}{\delta_{h_j}}$ is standard normal distribution with zero mean and unit variance.
Here $\Phi(.)$ represents the cumulative density function of the standard normal random variable, and $(-\mathbf{K}_{\alpha_j})$ denotes the value of the standard normal variable, then we have $\alpha_j = \phi(-\mathbf{K}_{\alpha_j})$.

Therefore, the constraint can be stated as:

$$\Phi\Big(\frac{\sum_{i=1}^{n} T_{ij}x_i - \eta_{h_j}}{\delta_{h_j}}\Big) \leq \Phi(-\mathbf{K}_{\alpha_j}), \quad j = 1, ..., J$$

The inequality will be satisfied only if

$$\frac{\sum_{i=1}^{n} T_{ij}x_i - \eta_{h_j}}{\delta_{h_j}} \leq -\mathbf{K}_{\alpha_j}, \quad j = 1, ..., J$$

Finally, the probabilistic constraint can be transformed into deterministic constraint as follows:

$$\sum_{i=1}^{n} T_{ij}x_i \leq \eta_{h_j} - \mathbf{K}_{\alpha_j}\delta_{h_j}, \quad j = 1, ..., J$$

The equivalent deterministic model for the chance constrained programming technique of the multi-objective stochastic problem form representing as:

$$(MCP)\begin{cases} \min & \widetilde{Z}^k = \sum_{i=1}^{n} \eta_{c_i^k} x_i, \quad k = 1, ..., K \\ s.t. & x \in D \\ & \sum_{i=1}^{n} T_{ij}x_i \leq \eta_{h_j} - \mathbf{K}_{\alpha_j}\delta_{h_j}, \quad j = 1, ..., J \end{cases} \tag{4}$$

We assume throughout the paper that
$D_{cs} = \{x \in \mathbb{R}^n : \ Ax = b, \ \sum_{i=1}^{n} T_{ij}x_i \leq \eta_{h_j} - \mathbf{K}_{\alpha_j}\delta_{h_j}, \ j = 1, ..., J, \ X \geq 0, \ integer\}$ is not empty.

In the sense of MCP programming, K objective are usually simultaneously each other in nature and concept of optimal solution gives place to concept of Pareto optimal (efficient, non dominated), for which the improvement of one objective function is attained only by sacrificing another objective function.

The solution to the problem (MCP) is to find all solutions that are efficient in the sense of the following definition:

Definition 1. *A point $x^* \in D_{cs}$ is said to be efficient solution for (4) if and only if there does not exist another point $x^1 \in D_{cs}$ such that $\widetilde{Z}^k(x^1) \leq \widetilde{Z}^k(x^*)$, $k \in 1, ..., K$ and $\widetilde{Z}^k(x^1) < \widetilde{Z}^k(x^*)$ for at least one $k \in 1, ..., K$.*

In this subsection we pay attention to some basic result which can help the reader to understand the algorithm of Section 4.

2.2. Efficiency testing . The following result (see[8]) is used in various steps of the algorithm to test the efficiency of a given feasible solution of problem (MCP).

Let x^0 be an arbitrary element of the region D_{cs}; $x^0 \in \mathbf{E}_{cs}$ if and only if the optimal value of the objective function Θ is null in the following integer linear programming problem:

$$(P(X^0)) \begin{cases} \min \quad \Theta = -\sum_{k=1}^{K} \Psi_k \\ s.t. \quad \begin{cases} x \in D_{cs} \\ \eta_{c_i^k} x + \Psi_k = \eta_{c_i^k} x^0 \\ \Psi^k \geq 0, \ \ integer \end{cases} \end{cases} \tag{5}$$

As is well-known, if the optimal value $\Theta = 0$, then $x^0 \in \mathbf{E}_{cs}$. Otherwise, any optimal solution $\hat{x}^0$ of $(P(x^0))$ is proved to be an efficient solution of (MCP) and its criterion vector dominates $\eta_c x^0$.

3 Construction of deterministic fractional function

3.1. Interval-Valued Number. Throughout this paper, assume that "I"denotes the set of all closed intervals in $\mathcal{R}$. For $A, B \in I$ such that $A = [a_L, a_R] = \{a : a_L < \ a < \ a_R, a \in \mathcal{R}\}$, where a_L is the left limit and a_R is the right limit of A.

Definition 2. *Let $* \in \{+, -, \times, \div\}$ be the arithmetic operations on the set of real numbers. If $A = [a_L, a_R]$ and $B = [b_L, b_R]$ are two intervals, then $A * B = \{a * b : a \in A, b \in B\}$*

In this paper, the intervals are considered to be bounded and closed. It is always assumed that $0 \notin B$, in the case of division. The used operations on intervals are as follows:

(1) $A + B = [a_L + b_L, a_R + b_R]$,
(2) $A - B = [a_L - b_L, a_R - b_R]$,
(3) $A \times B = [min\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}, max\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}]$,
(4) $\lambda A = \begin{cases} [\lambda a_L, \lambda a_R]; & \text{if} \quad \lambda \geq 0. \\ [\lambda a_R, \lambda a_L]; & \text{if} \quad \lambda < 0. \end{cases}$ whereλ is a real number.

(5) $A \div B = [a_L, a_R] \div [\frac{1}{b_L}, \frac{1}{b_R}]$, for $0 \notin B$
(6) If $\quad A, B \in I^+ \subset \mathcal{R}^+ \qquad$ then $\quad A \div B = [\frac{a_L}{b_R}, \frac{a_R}{b_L}]$.

3.2. Transforming the objectives fractional function. The objectives fractional function can be formulated as:

$$F(X) = \frac{\sum_{i=1}^{n}\Big[p_{L_i}^k, p_{R_i}\Big]x_i + \Big[\alpha_L, \alpha_R\Big]}{\sum_{i=1}^{n}\Big[q_{L_i}, q_{R_i}\Big]x_i + \Big[\beta_L, \beta_R\Big]} = \frac{\Big[\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L, \sum_{i=1}^{n} p_{R_i}x_i + \alpha_R\Big]}{\Big[\sum_{i=1}^{n} q_{L_i}x_i + \beta_L, \sum_{i=1}^{n} q_{R_i}x_i + \beta_R\Big]}. \tag{6}$$

Since $p_i; q_i; \beta; \alpha \in I^+$, the objectives can be reformulated in form of interval valued functions using the concept of interval analysis.

$$F(x) = \Bigg[\frac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R}, \frac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L}\Bigg] = \Big[F_{L_i}(x), F_{R_i}(x)\Big]. \tag{7}$$

Consider the following two mathematical programming problems (R_{cs1}) and(R_{cs2}).

$$(R_{cs1})\begin{cases} \min \quad F(x) = \Bigg[\dfrac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R}, \dfrac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L}\Bigg] \\ s.t. \quad x \in D_{cs} \end{cases} \tag{8}$$

$$(R_{cs2})\begin{cases} \min \quad G(x) = \Bigg[\dfrac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R} + \dfrac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L}\Bigg] \\ s.t. \quad x \in D_{cs} \end{cases} \tag{9}$$

To solve R_{cs1} (8),we used the following theorem.

Theorem 1. [22]
If x^* is an optimal solution of problem (R_{cs2}), then x^* is a non-dominated solution of problem (R_{cs1}).

4 Description of the method

The initial solution optimal (x^1) obtained by solving the relaxed deterministic problem (R_{cs2}) is tested of efficiency by solving the problem ($P(x^1)$ in order to obtain an initial efficient solution ($\hat{x}$). Seeking for equivalent efficient non-dominated there exist, and calculated valued of F_L, F_R by solving the following problem

$$(T^1)\begin{cases} \min \quad G(x) = \left[\dfrac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R} + \dfrac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L}\right] \\ s.t. \quad x \in D_{cs} \\ \qquad \eta_c x = \eta_c \hat{x}^1 \\ \qquad F_L = \dfrac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R}, \\ \qquad F_R = \dfrac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L} \end{cases} \tag{10}$$

Let ($\ddot{x}$) be an optimal solution of (T^1). At an iteration l,the feasible set D_{cs} is reduced gradually by eliminating all dominated solutions by $\eta_c\hat{x}^{l-1}$ (see Sylva and Crema,[15]) with the cut $G(x) \leq G_{opt}$ that insure that the new optimal solution x^l of the problem (R^l_{cs}) improves the optimum value. The resolution of the following problem (11) enables us to perform this elimination assuming that all coefficients of η_c are integers.

$$(R^l_{cs})\begin{cases} \min \quad G(x) = \left[\dfrac{\sum_{i=1}^{n} p_{L_i}x_i + \alpha_L}{\sum_{i=1}^{n} q_{R_i}x_i + \beta_R} + \dfrac{\sum_{i=1}^{n} p_{R_i}x_i + \alpha_R}{\sum_{i=1}^{n} q_{L_i}x_i + \beta_L}\right] \\ s.t. \quad x \in H_{cs}; \\ \qquad G(x) \leq G_{opt}; \end{cases} \tag{11}$$

where $H_{cs} = D_{cs} - \bigcup_{s=1}^{l} D_s$ and $D_s = \{x \in \mathbb{Z}^n | \eta_c x \geq \eta_c x^s\}$.

$\left\{\widetilde{C}X^s\right\}_{s=1}^{l}$ is a subset of non-dominated criteria vectors for problem MCP, with $\{X^s, s = 1, ..., l-1\}$ are solutions of (MCP) obtained at iterations $1, 2, ..;, l-1$ respectively.

$$H_{cs} = \begin{cases} \eta_{c_i^k} x \leq (\eta_{c_i^k} x^l - 1) y_l^k + M^k (1 - y_l^k). \\ \sum_{k=1}^{K} y_l^k \geq 1, \\ y_l^k \in \{0, 1\} \\ x \in D_{cs} \\ for\ \ all\ \ k = 1, 2, ...K, \ \ ; \end{cases}$$

where M^k is an upper bound for the k^{th} objective function of problem MCP. In practice, M_k can be taken, e.g., as the optimal value of the linear problem $\max\{\eta_{c_i^k} x |\ x \in D_{cs}\}$. Note that when $y_l^k = 0$, the constraint is not restrictive and when $y_l^k = 1$, a strict improvement is forced in the k^{th} objective function evaluated at $\ddot{x}^l$ or $\hat{x}^l$.

4.1. Algorithm.

step1: Transform the objectives $Z^k(x)$ [i.e. objectives (1a)] into deterministic objective function using the E-model.
step2: Convert stochastic constraints [i.e. constraints (1c)] into deterministic ones through the CCP by giving a significance level (α_j) for constraint j.
step3: Transform the objective $F(x)$ of (P_E) [i.e. objective (2a)] into the interval-valued form $[f_L(x); f_R(x)]$ using interval arithmetic.
step4: Let $G_{inf} = -\infty$, $G_{sup} = +\infty$, l=1
step5: Solve the relaxed problem (R_{cs2}),
- If its is unfeasible,**Terminated**, (P_E) is unfeasible.
- Otherwise, let x^l be an optimal solution of (R^l_{cs2})

step6: efficiency test Solve $P(x^l)$
- If $\Theta = 0$,**Terminated**, $x_{opt} = x^l$ is an non-dominated solution of (P_E). • Otherwise, $x_{inf} = G(x^l)$, let $\hat{x}^l$ an optimal of $P(x^l)$ and go to step 7.

step7: The solution obtained ,denote $\ddot{x}^l$, by solving the problem (T_l) is equivalent efficient solutions of $\hat{x}^l$.
- If $G(\ddot{x}^l) < G_{sup}$, set $G_{sup} = G(\ddot{x}^l)$ and $x_{opt} = \ddot{x}^l$
- If $G_{sup} = G_{inf}$, **Terminated**, x_{opt} is an non-dominated solution of (P_E).

step 8: Solve (R^l_{cs2})

• If its is unfeasible,**Terminated**, x_{opt} is an non-dominated solution of ((P_E).
• Otherwise, let x^{l+1} an optimal solution of (R^l_{cs2}).
• If $G(x^{l+1}) \leq G_{sup}$, **Terminated**. x_{opt} is an non-dominated solution of ((P_E).
• Otherwise, set $l = l + 1$ and go to step 6.

Theorem 2. *The algorithm converges to an optimal solution of the program P_E in a finite number of iterations, if such a solution exists.*

Proof. The algorithm converges in a finite number of iterations because it transforms probabilistic constraints into deterministic ones at each iteration and applies an efficiency test to verify whether a solution is non-dominated, as stated in Proposition 2.1. If the solution is not optimal, a cut is applied to eliminate dominated solutions, ensuring strict improvement of the objective function at each iteration (Proposition 3.2). Since the feasible set is finite and discrete, the algorithm progressively reduces this set until it reaches an optimal solution or determines that no solution exists, thereby ensuring finiteness and convergence. □

5 Example illustrative

To illustrate the proposed method, we consider the following example of the multi-objective stochastic coefficients in the objective functions and the parameters of the chance constraints follow normal distribution having known mean and standard deviation.

$$(\textbf{CCMOSILP})\begin{cases} \min & Z^1 = C^1_1 x_1 + C^1_2 x_2 \\ \min & Z^2 = C^2_1 x_1 + C^2_2 x_2 \\ \text{s.t} & \\ & x_1 + x_2 \leq h_1, \\ & x_2 \leq h_2, \\ & x_1 \leq 5, \\ & x_1, x_2 \geq 0, integer \end{cases}$$

Now, we assume that the means and the variances of normal random variables for $K = 1, 2$ are represented in Tabel-1.

Таблица 1. Means and variances.

	$K=1$		$K=2$			
	C_1	C_2	C_1	C_2	h_1	h_2
Mean	-2	1	1	-2	12	10
Variance	5	10	8	6	3	5

For specified probability levels $\alpha_1 = 0.04$ and $\alpha_{21} = 0.06$. Problem (5) be transformed into the following equivalent problems

$$(\mathbf{CCMOSILP})\begin{cases} \min & Z^1 = -2x_1 + x_2 \\ \min & Z^2 = x_1 - 2x_2 \\ \text{s.t} & \\ & x_1 + x_2 \leq 10,452, \\ & x_2 \leq 7,3805, \\ & x_1 \leq 5, \\ & x_1, x_2 \geq 0, integer \end{cases}$$

The main problem interval valued fractional is:

$$(P_E)\begin{cases} \min & F(x) = \dfrac{[2,10]x_1 + [5,6]x_2 + [\frac{7}{2},2]}{[\frac{1}{2},5]x_1 + [\frac{3}{4},2]x_2 + [\frac{1}{2},1]} \\ \text{s.t} & \\ & x_1, x_2 \in \mathbf{E}_{cs} \end{cases}$$

The relaxed problem (R_{cs2}) given by.

$$(R_{cs2})\begin{cases} \min & G(x) = \dfrac{2x_1 + 5x_2 + \frac{7}{2}}{5x_1 + 2x_2 + 1} + \dfrac{10x_1 + 6x_2 + 2}{\frac{1}{2}x_1 + \frac{3}{4}x_2 + \frac{1}{2}} \\ \text{s.t} & \\ & x_1 + x_2 \leq 10,452, \\ & x_2 \leq 7,3805, \\ & x_1 \leq 5, \\ & x_1, x_2 \geq 0, integer \end{cases} \tag{12}$$

Step 0 ***Initialization.***

We take $G_{opt} = +\infty$, $H^1_{cs} = D_{cs}$ and $l = 1$. Alter solving $\max\{\eta_{c^k} x | x \in D_{cs}\}$, $(k = 1, 2)$, we set $(M^1, M^2) = (7, 5)$.

First iteration

Step 1: An optimal solution of (R^1_{cs2}) is $G(x) = 7.5$ for $x^1 = (0, 0)$a. Let $z(x^1) = (\eta_{c_1} x^1, \eta_{c_2} x^1) = (0, 0)$.

We test the efficiency of x^1 by solving the problem $(P(x^1))$.

$$(Px^1)\begin{cases} \min & \Theta = -\Psi_1 - \Psi_2 \\ s.t. & x \in D_{cs} \\ & -2x_1 + x_2 + \Psi_1 = 0 \\ & x_1 - 2x_2 + \Psi_2 = 0 \\ & \Psi^k \geq 0, \ \ integer \\ & x_1, x_2 \geq 0, integer \end{cases}$$

The optimal value of $(Px^1))$ is -10 , which is achieved at the point $\hat{x}^1 = (4,6)$ and $z(\hat{x}^1) = (-2,-8)$. Thus, $\hat{x}^1 \in \mathbf{E}_{cs}$ and $x^1 \in \mathbf{E}_{cs}$.

Step 2: We solve the problem T^1

$$(T^1)\begin{cases} \min & G(x) = \dfrac{2x_1+5x_2+\frac{7}{2}}{5x_1+2x_2+1} + \dfrac{10x_1+6x_2+2}{\frac{1}{2}x_1+\frac{3}{4}x_2+\frac{1}{2}} \\ \text{s.t} & x \in D_{cs} \\ & -2x_1+x_2 = -2 \\ & x_1 - 2x_2 = -8 \\ & F_L = \dfrac{2x_1+5x_2+\frac{7}{2}}{5x_1+2x_2+1} \\ & F_R = \dfrac{10x_1+6x_2+2}{\frac{1}{2}x_1+\frac{3}{4}x_2+\frac{1}{2}} \\ & x_1, x_2 \geq 0, integer \end{cases}$$

An optimal solution of (T^1) is $\ddot{x}^1 = (4,6)$ and $G(\ddot{x}^1) = 12,5 < G_{opt}$ we initialize $x_{opt} = (4,6)$ and $G_{opt} = 12.5$.

Step 3: Let $l := l+1 = 2$ and solve the problem (R^2_{cs2})

$$(R^2_{cs2})\begin{cases} \min & G(x) = \dfrac{2x_1+5x_2+\frac{7}{2}}{5x_1+2x_2+1} + \dfrac{10x_1+6x_2+2}{\frac{1}{2}x_1+\frac{3}{4}x_2+\frac{1}{2}} \\ \text{s.t} & G(x) < G_{opt} = 12,5 \\ & H^2_{cs} = \begin{cases} X \in H^1_{cs} \\ -2x_1+x_2+10y^1_2 \leq 7 \\ x_1-2x_2+14y^2_2 \leq 5 \\ y^1_2+y^2_2 \geq 1, y^1_2, y^2_2 \in \{0,1\}^2 \end{cases} \end{cases}$$

An optimal solution is $x^2 = (0,5)$, $Z(x^2) = (5,-10)$ and $G(x^2) = 10.12.32$.

We test the efficiency of x^2 by solving the problem $(P(x^2))$.

The remaining iterations (for $l > 1$) are summarized in Tabel-2. $(x_{opt}, G_{opt}) = ((0,7), 10.21884)$ is the optimal solution that is obtained in the fourth iteration. $F(x_{opt}) = [2.566667, 7.652174]$ in the non-dominated solution of $P(E)$.

The table shows the iterative progression of the algorithm, with a gradual improvement in solutions at each step. It demonstrates how the algorithm reaches the optimal solution $(x_{opt} = (0,7), G_{opt} = 10.21884)$, by eliminating dominated solutions while considering stochastic uncertainties.

6 Conclusion

This paper presented an approach for optimizing a linear fractional function with interval coefficients within a multi-objective stochastic programming framework with chance constraints. By transforming probabilistic constraints

Таблица 2. The obtained results at each iteration.

Iteration l	$\underline{(R_{cs2})}$ x^l Z^l $G^l(x)$	$\underline{(P(x^l))}$ Θ, $\hat{x}^l$ $Z(\hat{x}^l)$	$\underline{(T_l)}$ $\ddot{x}^l$ $Z(\ddot{x}^l)$ $G(\ddot{x}^l)$	 $G(\ddot{x}^l)$ F_L F_R	R_{cs2} is unfeasible	$\underline{\text{OPT. SOL.}}$ x_{opt} G_{opt} $F(x_{opt})$
1	$x^l = (0,0)$ $Z^l = (0,0)$ $G^l(x) = 7.5$	$\Theta = -10$, $\hat{x}^l = (4,6)$ $Z(\hat{x}^l) = (-2,-8)$	$\ddot{x}^l = (4,6)$ $Z(\ddot{x}^l) = (-2,-8)$	$G(\ddot{x}^l) = 12.40043$ $F_L = 1.257576$ $F_R = 11.14286$	False	$x_{opt} = (4,6)$ $G_{opt} = 12.40043$ $F(x_{opt}) = [1.257576, 11.14286]$
2	$x^l = (0,5)$ $Z^l = (5,-10)$ $G^l(x) = 10.12032$	$\Theta = -5$, $\hat{x}^l = (3,7)$ $Z(\hat{x}^l) = (1,-11)$	$\ddot{x}^l = (3,7)$ $Z(\ddot{x}^l) = (1,-11)$	$G(\ddot{x}^l) = 11.69023$ $F_L = 1.483333$ $F_R = 10.20690$	False	$x_{opt} = (3,7)$ $G_{opt} = 11.69023$ $F(x_{opt}) = [1.483333, 10.20690]$
3	$x^l = (0,6)$ $Z^l = (6,-12)$ $G^l(x) = 10.17692$	$\Theta = -3$, $\hat{x}^l = (2,7)$ $Z(\hat{x}^l) = (3,-12)$	$\ddot{x}^l = (2,7)$ $Z(\ddot{x}^l) = (3,-12)$	$G(\ddot{x}^l) = 11.18148$ $F_L = 1.700000$ $F_R = 9.481481$	False	$x_{opt} = (2,7)$ $G_{opt} = 11.18148$ $F(x_{opt}) = [1.700000, 9.481481]$
4	$x^l = (0,7)$ $Z^l = (7,-14)$ $G^l = 10.21884$	$\Theta = -3$, $\hat{x}^l = (0,7)$ $Z(\hat{x}^l) = (7,-14)$	$\ddot{x}^l = (0,7)$ $Z(\ddot{x}^l) = (7,-14)$	$G(\ddot{x}^l) = 10.21884$ $F_L = 2.566667$ $F_R = 7.652174$	True	$x_{opt} = (0,7)$ $G_{opt} = 10.21884$ $F(x_{opt}) = [2.566667, 7.652174]$

and stochastic objective functions into deterministic models, the problem was simplified while maintaining result accuracy. The proposed algorithm, avoiding the exhaustive search for all efficient solutions, demonstrated its effectiveness through a detailed numerical example. This example was solved using the LINGO software, confirming the robustness and speed of the proposed method. The approach thus allows better management of uncertainties while providing decision-makers with optimal solutions without compromising result quality.

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