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THE SOBOLEV–POINCARÉ INEQUALITY AND THE $L_{q,p}$-COHOMOLOGY OF TWISTED CYLINDERS

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Abstract. We establish a vanishing result for the $L_{q,p}$-cohomology ($q \geq p$) of a twisted cylinder, which is a generalization of a warped cylinder. The result is new even for warped cylinders. We base on the methods for proving the (p, q)-Sobolev–Poincaré inequality developed by L. Shartser.

Keywords: differential form, Sobolev–Poincaré inequality, $L_{q,p}$-cohomology, twisted cylinder, homotopy operator

1. Introduction

The $L_{q,p}$*-cohomology* $H^k_{q,p}(M)$ of a Riemannian manifold (M, g) is, by definition, the quotient of the space of closed p-integrable differential k-forms by the exterior differentials of q-integrable k-forms. If $p = q$ then $L_{q,p}$-cohomology is usually referred to simply as L_p-cohomology and the index p is used instead of p, p in all the notations.

A *twisted product* $X \times_h Y$ of two Riemannian manifolds (X, g_X) and (Y, g_Y) is the direct product manifold $X \times_g Y$ endowed with a Riemannian metric of the form

$$g := g_X + h^2(x, y)g_Y, \tag{1.1}$$

where $h : X \times Y \to \mathbb{R}$ is a smooth positive function (see [5]). If X is a half-interval $[a, b)$ then the twisted product $X \times_h Y$ is called a *twisted cylinder*.

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We refer to an m-dimensional Riemannian manifold (M, g_M) as an *asymptotic twisted product* (respectively, as an *asymptotic twisted cylinder*) if, outside an m-dimensional compact submanifold, it is bi-Lipschitz equivalent to a twisted product (respectively, to a twisted cylinder).

In this paper, we prove some vanishing results for the $L_{q,p}$-cohomology of twisted cylinders $[a, b) \times_h N$ for a positive smooth function $h : [a, b) \times N \to \mathbb{R}$ in the case where the base N is a closed manifold and $p \geq q > 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N+1)}$.

If in (1.1) the function h depends only on x then we obtain the familiar notion of a *warped product* (see [1]). Twisted products were the object of recent investigations [4, 6, 8, 9, 10, 16, 21]. The $L_{q,p}$-cohomology of warped cylinders $[a, b) \times_h N$, i.e., of product manifolds $[a, b) \times N$ endowed with a warped product metric

$$g = dt^2 + h^2(t)g_N,$$

where g_N is the Riemannian metric of N and $h : [a, b) \to \mathbb{R}$ is a positive smooth function, was studied by Gol′dshtein, Kuz′minov, and Shvedov [11], Kuz′minov and Shvedov [19, 20] (for $p = q$), and Kopylov [17, 18] for $p, q \in [1, \infty)$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{\dim N+1}$. In particular, in [18], the second author found a sufficient condition for the L_{qp}-cohomology of a warped cylinder to be nontrivial (and even infinite-dimensional) in terms of Hardy's inequality.

The main result of the paper (Theorem 7.1) states that the $L_{q,p}$-cohomology $H^k_{q,p}(C^h_{a,b}N)$ of the twisted cylinder $C^h_{a,b}N$ with $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(\dim N+1)}$ is zero provided that the de Rham cohomology $H^k_{\mathrm{DR}}(N)$ of the base N is trivial and some integral conditions on the twisting function involving p, q and an auxiliary parameter $\overline{p}$ are fulfilled.

The paper is organized as follows: In Section 2, we recall some basic definitions concerning the $L_{q,p}$-cohomology of Riemannian manifolds. Section 3 describes the representations of differential forms on a twisted cylinder obtained in [10] and analogous to the representations of forms on a warped product proposed by Gol′dshtein, Kuz′minov, and Shvedov in [12]. In Section 4, we develop a version of the weighted Sobolev–Poncaré inequality for convex sets in $\mathbb{R}^n$ by introducing a homotopy operator and consider some of its consequences; the exposition is based on the ideas of Shartser suggested in [22] and [23]. In Section 5, we consider a new homotopy operator A_α on differential forms defined on a convex domain in $\mathbb{R}^n$ and show that it guarantees the fulfillment of an inequality of Sobolev–Poincaré-type for $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. In Section 6, using the ideas of Shartser's article [23], we "glue" local homotopy operators on a twisted cylinder to obtain a global homotopy operator. In Section 7, we use this global homotopy operator for proving our above-metioned main result on the triviality of the $L_{q,p}$-cohomology of a twisted cylinder (Theorem 7.1), and in Section 8, we extend this theorem to asymptotic twisted cylinders (Theorem 8.2). Section 9 contains some examples.

2. Basic Definitions

We recall the main definitions and notations.

Below we tacitly assume all manifolds to be oriented.

Let M be a smooth oriented Riemannian manifold. Denote by $\mathcal{D}^k(M) := C_0^\infty(M, \Lambda^k)$ the space of all smooth differential k-forms with compact support contained in $M \backslash \partial M$ denote by $L^1_{loc}(M, \Lambda^k)$ the space of locally integrable differential forms.

Denote by $L^p(M,\Lambda^k)$ the Banach space of locally integrable differential k-forms endowed with the norm $\|\theta\|_{L^p(M,\Lambda^k)} := \left(\int_M |\theta|^p dx\right)^{\frac{1}{p}} < \infty$ (as usual, we identify forms coinciding outside a set of measure zero). Of course, we can add a positive (smooth) weight $\sigma : M \to \mathbb{R}$ and thus integrate $|\theta|^p\sigma^p$ to obtain the weighted L^p-space $L^p(M,\Lambda^k,\sigma)$.

Definition 2.1. We call a differential $(k+1)$-form $\theta \in L^1_{loc}(M,\Lambda^{k+1})$ *the weak exterior derivative* (or *differential*) of a differential k-form $\phi \in L^1_{\mathrm{loc}}(M,\Lambda^k)$ and write $d\phi = \theta$ if

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M)$.

Remark 2.2. Note that the orientability of M is not substantial in this definition since one can take integrals over orientable domains on M instead of integrals over M.

We then introduce an analog of Sobolev spaces for differential k-forms, i.e., the space of q-integrable forms with p-integrable weak exterior derivative:

$$\Omega^k_{q,p}(M) = \left\{ \omega \in L^q(M,\Lambda^k) \mid d\omega \in L^p(M,\Lambda^{k+1}) \right\}.$$

This is a Banach space for the graph norm

$$\|\omega\|_{q,p} = \left(\|\omega\|^2_{L^q(M,\Lambda^k)} + \|d\omega\|^2_{L^p(M,\Lambda^{k+1})} \right)^{1/2}.$$

The space $\Omega^k_{q,p}(M)$ is a reflexive Banach space for any $1 < q,p < \infty$. This can be proved using standard arguments of functional analysis.

We now define our basic ingredients (for three parameters r,q,p).

Definition 2.3. Put
(a) $Z^k_{p,r}(M) = \mathrm{Ker}[d : \Omega^k_{p,r}(M) \to L^r(M,\Lambda^{k+1})]$.
(b) $B^k_{q,p}(M) = \mathrm{Im}[d : \Omega^{k-1}_{q,p}(M) \to L^p(M,\Lambda^k)]$.

The subspace $Z^k_{p,r}(M)$ does not depend on r and is a closed subspace in $L^p(M,\Lambda^k)$ (see Lemma [14, Lemma 2.4(i)]). This allows us to use the notation $Z^k_p(M)$ for all $Z^k_{p,r}(M)$. Note that $Z^k_p(M) \subset L^p(M,\Lambda^k)$ is always a closed subspace but that is in general not true for $B^k_{q,p}(M)$. Denote by $\overline{B}^k_{q,p}(M)$ its closure in the L^p-topology. Observe also that since $d \circ d = 0$, one has $\overline{B}^k_{q,p}(M) \subset Z^k_p(M)$. Thus,

$$B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M,\Lambda^k).$$

Definition 2.4. Suppose that $1 \le q,p \le \infty$. The $L_{q,p}$*-cohomology* of (M,g) is defined as the quotient

$$H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M),$$

and the *reduced* $L_{q,p}$*-cohomology* of (M,g) is, by definition, the space

$$\overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M).$$

Since $B^k_{p,q}$ is not always closed, the L_p-cohomology is in general a (non-Hausdorff) semi-normed space, while the reduced L_p-cohomology is a Banach space.

Below $|X|$ stands for the volume of a Riemannian manifold (X, g).

It follows from the results of [13] that, under suitable assumptions on p, q, the $L_{q,p}$-cohomology of a Riemannian manifold M can be expressed in terms of smooth forms.

Let $C^\infty(M, \Lambda^k)$ be the space of smooth k-forms on M.

Introduce the notations:

$$\begin{aligned}
C^\infty L^p(M, \Lambda^k) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k);\\
C^\infty L^p(M, \Lambda^k, \sigma) &:= C^\infty(M, \Lambda^k) \cap L^p(M, \Lambda^k, \sigma);\\
C^\infty \Omega^k_{q,p}(M) &:= C^\infty(M, \Lambda^k) \cap \Omega^k_{q,p}(M);\\
C^\infty H^k_{q,p}(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z^k_p(M)}{C^\infty(M, \Lambda^k) \cap B^k_{q,p}(M)};\\
C^\infty \overline{H}^k_{q,p}(M) &:= \frac{C^\infty(M, \Lambda^k) \cap Z^k_p(M)}{C^\infty(M, \Lambda^k) \cap \overline{B}^k_{q,p}(M)}.
\end{aligned}$$

Theorem 2.5. [13, Theorem 12.5 and 12.8, Corollary 12.9]. *Let (M, g) be a n-dimensional Riemannian manifold and suppose the fulfillment of one of the following conditions:*

- *$p, q \in (1, \infty)$ and $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n}$;*
- *$p, q \in [1, \infty)$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.*

Then the cohomology $H^*_{q,p}(M)$ can be represented by smooth forms, and thus $H^*_{q,p}(M) = C^\infty H^*_{q,p}(M)$.

More exactly, any closed form in $Z^k_p(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M, \Lambda^k) \cap Z^k_p(M)$ are cohomologous modulo $d\Omega^{k-1}_{q,p}(M)$ then they are cohomologous modulo $dC^\infty\Omega^{k-1}_{q,p}(M)$.

Similarly, any reduced cohomology class can be represented by a smooth form.

3. Differential Forms on a Twisted Cylinder

From now on, $C^h_{a,b}N$ is the twisted cylinder $[a, b) \times_h N$, that is, the product of a half-interval $[a, b)$ and a closed smooth n-dimensional Riemannian manifold (N, g_N) equipped with the Riemannian metric $dt^2 + h^2(t, x)g_N$, where $h : [a, b) \times N \to \mathbb{R}$ is a smooth positive function.

Every differential form on $[a, b) \times N$ admits a unique representation of the form $\omega = \omega_A + dt \wedge \omega_B$, where the forms ω_0 and ω_1 do not contain dt (cf. [12]). It means that ω_0 and ω_1 can be viewed as one-parameter families $\omega_A(t)$ and $\omega_B(t)$, $t \in I$, of differential forms on N.

The modulus of a form ω of degree k on $C^h_{a,b}N$ is expressed via the moduli of $\omega_A(t)$ and $\omega_B(t)$ on N as follows:

$$|\omega(t,x)|_{C^h_{a,b}N} = \left[h^{-2k}(t,x)|\omega_A(t,x)|^2_N + h^{-2(k+1)}(t,x)|\omega_B(t,x)|^2_N\right]^{1/2} \tag{3.1}$$

Consequently,

(3.2)

$$\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}=\left[\int_a^b\int_N\left(h^{2(\frac{n}{p}-k)}(t,x)|\omega_A(t,x)|^2_N+h^{2(\frac{n}{p}-k+1)}(t,x)|\omega_B(t,x)|^2_N\right)^{\frac{p}{2}}dxdt\right]^{\frac{1}{p}}.$$

Put

$$f_{k,p}(t)=\min_{x\in N}\left\{h^{\frac{n}{p}-k}(t,x)\right\}$$

and

$$F_{k,p}(t)=\max_{x\in N}\left\{h^{\frac{n}{p}-k}(t,x)\right\}.$$

4. The Weighted Sobolev–Poincare Inequality for Convex Sets in $\mathbb{R}^n$

Denote by $\Omega^*_{loc}(M)$ the space all locally integrable differential forms with locally integrable weak differential.

Suppose that $D\subset R^n$ is a convex set and $\psi_y : D\times[0,1]\to D$, $\psi_y(x,t):=tx+(1-t)y$, is the homotopy induced by the convex structure. For a k-form $\omega\in\Omega^k_{loc}(D)$ the pullback $\psi^*_y\omega$ can be written as

$$\psi^*_y\omega(x,t)=\left(\psi^*_y\omega\right)_0(x,t)+dt\wedge\left(\psi^*_y\omega\right)_1(x,t),$$

where $\left(\psi^*_y\omega\right)_0$ and $\left(\psi^*_y\omega\right)_1$ do not contain dt.

For each $y\in D$ define a homotopy operator

$$K_y:\Omega^k_{loc}(D)\to\Omega^{k-1}_{loc}(D)$$

as follows:

$$K_y\omega(x):=\int_0^1\left(\psi^*_y\omega\right)_1(t)\,dt$$

It is easy to see that K_y takes smooth forms to smooth forms. It is proved in [15] that $K_yd\omega+dK_y\omega=\omega$ The following proposition is a generalization of results from [2] and Shartser's thesis [22] (see also [23]) to the weighted case and to unbounded convex domains.

Proposition 4.1. *Suppose that D is a convex set in $\mathbb{R}^n$, $q\geq p\geq 1$, and $\beta : D\to\mathbb{R}$ is a positive smooth function.*

If the inequality

$$C(k,p,q,n,\beta):=\int_0^1\sup_{z\in D}\|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)}t^k(1-t)^{-n/p}dt<\infty$$

holds then the inequality

$$\left\|\beta(x)\left\|\frac{K_yd\omega(x)}{|x-y|}\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}\leq C(k,p,q,n,\beta)\,\|d\omega\|_{L^p(D,\Lambda^{k+1})}\,.$$

is valid for every $\omega\in\Omega^k_{loc}(D)$ such that $d\omega\in L^p(D,\Lambda^{k+1})$. Here $\mathbf{1}_{xt+(1-t)D}$ is the characteristic function of the set $xt+(1-t)D$.

Proof. By the definition of K_y, we have

$$\left\|\beta(x)\left\|\frac{K_yd\omega(x)}{|x-y|}\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}=\left\|\beta(x)\left\|\int_0^1\frac{\left(\psi^*_yd\omega\right)_1(x,t)}{|x-y|}dt\right\|_{L^p(D,dy)}\right\|_{L^q(D,dx)}$$

$$\le \int_0^1 \left\| \beta(x) \left\| \frac{\left(\psi_y^* d\omega\right)_1 (x,t)}{|x-y|} \right\|_{L^p(D,dy)} \right\|_{L^q(D,dx)} dt$$

$$\le \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt.$$

As usual, we identify the tangent space to $\mathbb{R}^n$ at any of its points with $\mathbb{R}^n$. By easy calculations,

$$\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right| \le |x-y|\, t^k \left| d\omega\left(\psi_y(x,t)\right)\right|.$$

Therefore,

$$\int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D \frac{\left|\left(\psi_y^* d\omega\right)_1 (x,t)\right|^p}{|x-y|^p} dy \right]^{q/p} dx \right\}^{1/q} dt$$

$$\le \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} \left| d\omega\left(\psi_y(x,t)\right)\right|^p dy \right]^{q/p} dx \right\}^{1/q} dt$$

$$= \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_D t^{kp} \left| d\omega\left(tx+(1-t)y\right)\right|^p dy \right]^{q/p} dx \right\}^{1/q} dt := I.$$

The change of variables $z = tx + (1-t)y$ in the inner integral yields

$$I = \int_0^1 \left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p \, dz \right]^{q/p} dx \right\}^{1/q} t^k (1-t)^{-n/p} dt$$

Since D is convex, the set $tx + (1-t)D$ is contained in D for all $x \in D$ and $t \in [0,1]$. Using Minkowski's integral inequality, we infer

$$\left\{ \int_D \beta^q(x) \left[\int_{tx+(1-t)D} |d\omega(z)|^p \, dz \right]^{q/p} dx \right\}^{1/q}$$

$$= \left\{ \int_D \beta^q(x) \left[\int_D \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^p \, dz \right]^{q/p} dx \right\}^{1/q}$$

$$= \left\{ \left(\int_D \left[\int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^p \, dz \right]^{q/p} dx \right)^{p/q} \right\}^{1/p}$$

$$= \left\{ \left\| \int_D \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^p \, dz \right\|_{L^{q/p}(D,dx)} \right\}^{1/p}$$

$$\le \left\{ \int_D \left\| \beta^p(x) \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^p \right\|_{L^{q/p}(D,dx)} dz \right\}^{1/p}$$

$$= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) \, |d\omega(z)|^q \, dx \right)^{p/q} dz \right\}^{1/p}$$
$$= \left\{ \int_D \left(\int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{p/q} |d\omega(z)|^p \, dz \right\}^{1/p}$$
$$\leq \left(\sup_{z \in D} \int_D \beta^q(x) \mathbf{1}_{tx+(1-t)D}(z) dx \right)^{1/q} \left(\int_D |d\omega(z)|^p \, dz \right)^{1/p}$$
$$= \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} \, \|d\omega\|_{L^p(D,\Lambda^{k+1})} \, .$$

The proposition follows. □

Estimate
$$C(k,p,q,n,\beta) = \int_0^1 \sup_{z \in D} \|\beta(x) \mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt$$
in particular cases.

Corollary 4.2. *Suppose that D is a convex set of finite measure in $\mathbb{R}^n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, and the weight $\beta(x) \equiv 1$. Then*
$$C(k,p,q,n,1) \leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt.$$

Remark 4.3. It is easy to see that the integral of the corollary exists because of the conditions imposed on p and q.

Proof. Using the change of variables $u = tx$, we obtain
$$\int_0^1 \sup_{z \in D} \left\| \mathbf{1}_{tx+(1-t)D}(z) \right\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt$$
$$= \int_0^1 \sup_{z \in D} \left\| \mathbf{1}_{u+(1-t)D}(z) \right\|_{L^q(tD,du)} t^{k-n/q} (1-t)^{-n/p} dt.$$

Note that $|tD \cap \{u + (1-t)D\}| \leq |D| \min(t^n, (1-t)^n)$. It follows that
$$\left\| \mathbf{1}_{u+(1-t)D}(z) \right\|_{L^q(tD,du)} \leq |D|^{1/q} \min(t^{n/q}, (1-t)^{n/q});$$
$$C(k,p,q,n,1) \leq |D|^{1/q} \int_0^1 t^{k-n/q} (1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt$$
□

Corollary 4.4. *Suppose that U is a convex set of finite measure $|U|$ in $\mathbb{R}^n$, $D = [a,b) \times U$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, and $\beta : [a,b) \to \mathbb{R}$ is an integrable positive function. If $\|\beta\|_{L^q([a,b))} < \infty$ then*
$$C(k,p,q,n,\beta) \leq |U|^{1/q} \, \|\beta\|_{L^q([a,b))} \, .$$

Proof. If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a,b)$ and $w \in U$. Using the special type of the weight $\beta(x) := \beta(\tau)$ and representing $z \in D$ as $z = (\eta, \zeta)$ with $\eta \in [a,b)$ and $\zeta \in U$, we obtain
$$\int_0^1 \sup_{z \in D} \left\| \beta(x) \mathbf{1}_{tx+(1-t)D}(z) \right\|_{L^q(D,dx)} t^k (1-t)^{-\frac{n+1}{p}} dt$$

$$\leq \int_0^1 \sup_{a\leq\eta<b} \left(\int_a^b \beta^q(\tau) \mathbf{1}_{t\tau+(1-t)[a,b)}(\eta) d\tau \right)^{\frac{1}{q}} \sup_{\zeta\in U} \left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k (1-t)^{-\frac{n+1}{p}} dt,$$

where $x = (\tau, w)$.

Using the change of variables $u = tw$ and the estimate

$$|tU \cap \{u + (1-t)U\}| \leq |U| \min(t^n, (1-t)^n),$$

we finally get

$$\int_0^1 \sup_{a\leq\eta<b} \left(\int_a^b \beta^q(\tau) \mathbf{1}_{t\tau+(1-t)[a,b)}(\eta) d\tau \right)^{\frac{1}{q}} \sup_{\zeta\in U} \left(\int_U \mathbf{1}_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k (1-t)^{-\frac{n+1}{p}} dt$$
$$\leq |U|^{1/q} \|\beta\|_{L^q([a,b))} \int_0^1 t^{k-n/q} (1-t)^{-(n+1)/p} \min(t^{n/q}, (1-t)^{n/q}) dt$$

The conditions on p and q imply the finiteness of the last integral. □

Corollary 4.4 is a key ingredient in the proof of out main result, Theorem 7.1. Unfortunately, for being able to "separate" the variable t, we have to impose the stronger constraint $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1} - \frac{1}{q(n+1)}$ than the condition $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1}$ given by Proposition 4.1.

5. A New Homotopy Operator for $q \geq p$. The Case of a Convex Domain in $\mathbb{R}^n$

In the previous section, we considered the homotopy operator on Ω^*_{loc} of the form

$$A_\alpha = \int_D \alpha(y) K_y \omega(x) dy$$

for a convex set D in $\mathbb{R}^n$. We will need to modify A for obtaining some estimates.

Consider the same operator K_y as in the previous section:

$$\psi_y(x,t) = tx + (1-t)y, \quad K_y\omega(x) = \int_0^1 (\psi_y)_1^* \omega dt.$$

Recall that $dK_y\omega + K_y d\omega = \omega$. Choose a smooth positive function $\alpha : D \to \mathbb{R}$ such that $\int_D \alpha(x)dx = 1$ and put

$$A_\alpha\omega(x) := \int_D \alpha(y) K_y \omega(x) dy, \quad \omega \in \Omega^*_{\text{loc}}.$$

By a straightforward calculation,

$$dA_\alpha\omega = d\left(\int_D \alpha(y) K_y \omega(x) dy \right) = \int_D \alpha(y) d_x K_y \omega(x) dy;$$

$$A_\alpha d\omega = \int_D \alpha(y) K_y d\omega(x) dy;$$

$$dA_\alpha\omega + A_\alpha d\omega = \int_D \alpha(y) \left[d_x K_y \omega(x) + K_y d\omega(x) \right] dy = \int_D \alpha(y) \omega(x) dy = \omega.$$

In particular, if $d\omega = 0$ then

$$dA_\alpha\omega = \omega.$$

The definition of A_α easily implies the following

Proposition 5.1. *The homotopy operator A_α takes smooth forms to smooth forms.*

Definition 5.2. Call a smooth positive function $\alpha : D \to \mathbb{R}$ an *admissible weight* for a convex domain $D \subset \mathbb{R}^n$ and $p \geq 1$ if

$$\int_D \alpha(x)dx = 1; \quad \|\alpha\|_{L^{p'}(D)} < \infty; \quad \|\alpha(y)|y|\,\|_{L^{p'}(D)} < \infty.$$

For $p \geq 1$, we as usual put

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

Theorem 5.3. *Suppose that $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ is a convex set, $\beta : D \to \mathbb{R}$ is a positive smooth function, and $\alpha : D \to \mathbb{R}$ is an admissible weight. If*

$$C_1(k,p,q,n,\beta) := \int_0^1 \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt < \infty;$$

$$C_2(k,p,q,n,\beta) := \int_0^1 \sup_{z\in D} \||x|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k (1-t)^{-n/p} dt < \infty$$

then for any $\omega \in C^\infty L^p(D,\Lambda^k)$ we have

$$\|A_\alpha \omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq C(k,p,q,\alpha,\beta,n)\|\omega\|_{L^p(D,\Lambda^k)}$$

where

$$C(k,p,q,\alpha,\beta,n) = \|\alpha(y)|y|\|_{L^{p'}(D)} C_1(k,p,q,n,\beta) + \|\alpha\|_{L^{p'}(D)} C_2(k,p,q,n,\beta).$$

Proof. Put $\xi := A_\alpha \omega$. If $p > 1$ then, by Hölder's inequality, we infer

$$\|A_\alpha\omega\|_{L^q(D,\Lambda^{k-1},\beta)} = \left\|\beta(x)\int_D \alpha(y)K_y\omega(x)dy\right\|_{L^q(D,\Lambda^{k-1},dx)}$$
$$\leq \left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)} \|\alpha(y)|x-y|\|_{L^{p'}(D,dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}.$$

The above estimate also obviously holds for $p = 1$.

By the triangle inequality,

$$\|\alpha(y)|x-y|\|_{L^{p'}(D,dy)} \leq |x|\|\alpha(y)\|_{L^{p'}(D,dy)} + \|\alpha(y)|y|\,\|_{L^{p'}(D,dy)}.$$

Therefore,

$$\|A_\alpha\omega\|_{L^q(\beta,D,\Lambda^{k-1})}$$
$$\leq \|\alpha(y)|y|\,\|_{L^{p'}(D,dy)} \left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}$$
$$+ \|\alpha(y)\|_{L^{p'}(D,dy)} \left\|\beta(x)|x|\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)}.$$

By Proposition 4.1,

$$\left\|\beta(x)\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)} \leq C_1(k,p,q,n,\beta)\,\|\omega\|_{L^p(D,\Lambda^k)};$$

$$\left\|\beta(x)|x|\left\|\frac{K_y\omega(x)}{|x-y|}\right\|_{L^p(D,\Lambda^{k-1},dy)}\right\|_{L^q(D,\Lambda^{k-1},dx)} \leq C_2(k,p,q,n,\beta)\,\|\omega\|_{L^p(D,\Lambda^k)}.$$

The theorem is proved. □

Corollary 5.4. *Suppose that* $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ *is a convex set,* $\alpha : [a,b) \to \mathbb{R}$ *is an admissible weight,* $\beta, \gamma : D \to \mathbb{R}$ *are positive smooth functions. If the conditions*

$$C_1(k,\overline{p},q,n,\beta) := \int_0^1 \sup_{z\in D} \|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k(1-t)^{-n/\overline{p}}dt < \infty;$$

$$C_2(k,\overline{p},q,n,\beta) := \int_0^1 \sup_{z\in D} \||x|\beta(x)\mathbf{1}_{tx+(1-t)D}(z)\|_{L^q(D,dx)} t^k(1-t)^{-n/\overline{p}}dt < \infty;$$

$$Q(k,\overline{p},p,\gamma) := \left\|\gamma^{-1}\right\|_{L^{p\overline{p}/(p-\overline{p})}(D)} < \infty$$

are fulfilled for some $\overline{p}$, $1 \leq \overline{p} \leq p$ *(for* $\overline{p} = p$, *we put* $\frac{p\overline{p}}{p-\overline{p}} = \infty$*), then the inequality*

$$\|A_\alpha \omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq C(k,p,q,\alpha,\beta,\gamma,n)\|\omega\|_{L^p(D,\Lambda^k,\gamma)},$$

where

$$C(k,p,q,\alpha,\beta,\gamma,n) = Q(k,\overline{p},p,\gamma)\, C(k,\overline{p},q,n,\alpha,\beta),$$

holds for any $\omega \in C^\infty L^p(D,\Lambda^k)$.

Proof. By Theorem 5.3,

$$\|A_\alpha \omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq C(k,\overline{p},q,n,\alpha,\beta)\, \|\omega\|_{L^{\overline{p}}(D,\Lambda^k)}\,.$$

If $\overline{p} < p$ then, using Hölder's inequality, we have

$$\|\omega\|_{L^{\overline{p}}(D,\Lambda^k)} \leq \|\gamma\omega\|_{L^p(D,\Lambda^k)} \left\|\gamma^{-1}\right\|_{L^{p\overline{p}/(p-\overline{p})}(D)}\,. \tag{5.1}$$

Inequality (5.1) also holds for $\overline{p} = p$.

The corollary follows. □

Corollary 5.5. *Suppose that* $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, U *is a bounded convex set in* $\mathbb{R}^n$, $D = [a,b) \times U$, $\alpha : [a,b) \to \mathbb{R}$ *is an admissible weight, and* $\beta, \gamma : [a,b) \to \mathbb{R}$ *are positive smooth functions. If the conditions* $\|\beta\|_{L^q([a,b))}<\infty$, $\|\tau\beta(\tau)\|_{L^q([a,b))}<\infty$, *and* $\|\gamma^{-1}\|_{L^{p\overline{p}/(p-\overline{p})}([a,b))} < \infty$ *are fulfilled for some* $\overline{p}$, $1 \leq \overline{p} \leq p$ *(for* $\overline{p} = p$, *we put* $\frac{p\overline{p}}{p-\overline{p}} = \infty$*), then the inequality*

$$\|A_\alpha \omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq \text{const}\, \|\omega\|_{L^p(D,\Lambda^k,\gamma)}$$

with some constant depending k,p,q,n,α,β, and γ holds for any $\omega \in C^\infty L^p(D,\Lambda^k,\gamma)$.

Proof. Suppose that a number $\overline{p} \leq p$ satisfies the conditions of the corollary.

If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a,b)$ and $w \in U$. By Corollary 4.4, since $\frac{1}{\overline{p}} - \frac{1}{q} < \frac{q-1}{q(n+1)}$ and $\|\beta\|_{L^q([a,b))} < \infty$, we have

$$\int_0^1 \sup_{z\in D} \left\|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$
$$\leq |U|^{1/q}\|\beta\|_{L^q([a,b))} \int_0^1 t^{k-n/q}(1-t)^{-(n+1)/p} \min(t^{n/q},(1-t)^{n/q})dt.$$

On the other hand, since $\|\tau\beta(\tau)\|_{L^q([a,b))} < \infty$, we have by Corollary 4.4:

$$\int_0^1 \sup_{z\in D} \left\|\, |x|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$
$$= \int_0^1 \sup_{z\in D} \left\| \sqrt{\tau^2+w^2}\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\right\|_{L^q(D,dx)} t^k(1-t)^{-(n+1)/\overline{p}}dt$$

$$\leq \sqrt{2}\int_0^1 \sup_{z\in D}\big\| (|\tau|+|w|)\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\big\|_{L^q(D,dx)}\, t^k(1-t)^{-(n+1)/\overline{p}}dt$$

$$\leq \sqrt{2}\int_0^1 \sup_{z\in D}\big\| |\tau|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\big\|_{L^q(D,dx)}\, t^k(1-t)^{-(n+1)/\overline{p}}dt$$

$$+\sqrt{2}\int_0^1 \sup_{z\in D}\big\| |w|\beta(\tau)\mathbf{1}_{tx+(1-t)D}(z)\big\|_{L^q(D,dx)}\, t^k(1-t)^{-(n+1)/\overline{p}}dt$$

$$\leq \sqrt{2}|U|^{1/q}\|\tau\beta(\tau)\|_{L^q([a,b))}\int_0^1 t^{k-n/q}(1-t)^{-(n+1)/\overline{p}}\min(t^{n/q},(1-t)^{n/q})dt$$

$$+\sqrt{2}\sup_{w\in U}|w|\,\|\beta\|_{L^q([a,b))}\int_0^1 t^{k-n/q}(1-t)^{-(n+1)/\overline{p}}\min(t^{n/q},(1-t)^{n/q})dt<\infty.$$

The relations $\|\tau\beta(\tau)\|_{L^q([a,b))}<\infty$ and $\||x|\beta(\tau)\|_{L^q(D)}<\infty$ enable us to apply Corollary 5.4 and obtain the desired assertion. □

6. Globalization: the Sobolev–Poincare Inequality on a Cylinder

Here we globalize the Sobolev–Poincaré inequality to cylinders. The main assertion of the section is

Theorem 6.1. *Suppose that M is the cylinder $[a,b)\times N$, where N is a closed manifold of dimension n, $q\geq p\geq 1$, $\frac{1}{p}-\frac{1}{q}<\frac{q-1}{q(n+1)}$, and $\beta,\gamma:[a,b)\to\mathbb{R}$ be positive smooth functions. Let ω be an exact k-form in $C^\infty L^p(M,\Lambda^k,\gamma)$. If the conditions $\|\beta\|_{L^q([a,b))}<\infty$, $\|t\beta(t)\|_{L^q([a,b))}<\infty$, and $\|\gamma^{-1}\|_{L^{p\overline{p}/(p-\overline{p})}([a,b))}<\infty$ are fulfilled for some $\overline{p}$, $1\leq\overline{p}\leq p$ (for $\overline{p}=p$, we put $\frac{p\overline{p}}{p-\overline{p}}=\infty$), then there exists a $(k-1)$-form $\xi\in C^\infty L^q(M,\Lambda^{k-1},\beta)$ such that*

$$(6.1)\qquad d\xi=\omega \quad\textit{and}\quad \|\xi\|_{L^q(M,\Lambda^{k-1},\beta)}\leq \mathrm{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.$$

Let $\tilde{\mathcal{U}}=\{\tilde{U}_x\}$, $x\in N$, be a coordinate open cover of the base N. At each point $x\in N$, consider a geodesic ball U_x that is geodesically convex (small balls are geodesically convex, see [7, Proposition 4.2]) and such that its closure (a compact set) is contained in $\tilde{U}_x$. Then $\mathcal{U}'=\{\tilde{U}_x\}$ is an open cover of N. Extract a finite subcover $\mathcal{U}=\{U_i\}$, $i=1,\dots,l$, from U^0. Since $\mathcal{U}$ consists of geodesic balls, it is a *good cover*, i.e., all finite intersections $U_I=U_{i_0}\cap\dots\cap U_{i_{s-1}}$, $I=(i_0,\dots,i_{s-1})$, are bi-Lipschitz diffeomorphic to convex open sets with compact closure in $\mathbb{R}^n$. With such a cover $\mathcal{U}$, we associate the corresponding cover $\mathcal{V}=\{V_i=[a,b)\times U_i\}$, $i=1,\dots,l$, of M and put $V_I=V_{i_0}\cap\dots\cap V_{i_{s-1}}$ for $I=(i_0,\dots,i_{s-1})$. Then each intersection V_I is bi-Lipschitz diffeomorphic to a cylinder of the form $[a,b)\times U_{\mathbb{R}^n}$, where $U_{\mathbb{R}^n}$ is a convex set with compact closure in $\mathbb{R}^n$. By analogy with [23], we put

$$K^{k,0}:=C^\infty(M,\Lambda^k);\quad K^{k,s}:=\bigoplus_{i_0<\dots<i_{s-1}}C^\infty(V_I,\Lambda^k).$$

Given $\varkappa\in K^{r,s}$, denote by $\varkappa_I$, $I=(i_0,\dots,i_{s-1})$, $i_0<\dots<i_s$, the components of $\varkappa$. Define a coboundary operator $\delta:K^{k,s}\to K^{k,s+1}$ as follows:

$$(\delta\varkappa)_J=\left(\sum_{r=0}^{s}(-1)^r\varkappa_{j_0\dots\hat{j}_r\dots j_s}\right)\Bigg|_{V_J},\qquad J=(j_0,\dots,j_s).$$

Let $L^q(K^{k,s})$ be the space of elements $\varkappa \in K^{k,s}$ with the finite norm

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0<\dots i_{s-1}} \|\varkappa_I\|_{L^q(V_I,\Lambda^k,\beta)}.$$

As usual, if $\varkappa \in K^{k,s}$ has components $\varkappa_I$, $I=(i_0,\dots,i_{s-1})$, $i_0<\dots<i_s$, and ν is a permutation of the set $\{0,\dots,s-1\}$ then $\alpha_{\nu(I)}=\alpha_I \operatorname{sign}\nu$.

The following proposition is a modification for our case of [23, Proposition 3.6], which is in turn an adaptation of [3, Propositions 8.3 and 8.5].

Proposition 6.2. *$(K^{k,\bullet},\delta)$ is an exact complex. Moreover, if $\lambda \in L^q(K^{k,s+1},\beta)$ satisfies $\delta\lambda = 0$ then there exists $\varkappa \in L^q(K^{k,s},\beta)$ such that $\lambda=\delta\varkappa$ and*

- $\|\varkappa\|_{L^q(K^{k,s},\beta)} \le \operatorname{const} \|\lambda\|_{L^q(K^{k,s+1},\beta)}$
- $\|d\varkappa\|_{L^q(K^{k+1,s},\beta)} \le \operatorname{const} \big(\|\lambda\|_{L^q(K^{k,s+1},\beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1},\beta)}\big)$.

Proof. The fact that $(K^{k,\bullet},\delta)$ is an exact complex was established in [3, Propositions 8.3 and 8.5] but we will give the standard argument for completeness. If $\varkappa \in L^q(K^{k,s},\beta)$ then

$$\begin{aligned}(\delta(\delta\varkappa))_{i_0\cdots_{s+1}} &= \sum_r (-1)^i (\delta\varkappa)_{i_0\dots\hat{i}_r\dots i_{s+1}} \\ &= \sum_{l<r} (-1)^r(-1)^l \varkappa_{i_0\dots\hat{i}_l\dots\hat{i}_r\dots i_{s+1}} + \sum_{l<r} (-1)^r(-1)^{l-1} \varkappa_{i_0\dots\hat{i}_l\dots\hat{i}_r\dots i_{s+1}} = 0.\end{aligned}$$

Suppose that $\lambda \in L^q(K^{k,s+1},\beta)$ is such that $\delta\lambda = 0$. Let $\tilde\rho_j$ be a partition of unity subordinate to the cover $\{U_i\}$ of N. Then the functions $\rho_j : M \to \mathbb{R}$, $\rho_j(t,x) = \tilde\rho_j(x)$ for all $(t,x) \in M = [a,b)\times N$, constitute a partition of unity subordinate to the cover $\{V_i\}$ of M. Put

$$\varkappa_{i_0\dots i_{s-1}} := \sum_j \rho_j \lambda_{ji_0\dots i_{s-1}}. \tag{6.2}$$

Show that $\delta\varkappa = \lambda$.

We have

$$(\delta\varkappa)_{i_0\dots i_s} = \sum_r (-1)^r \varkappa_{i_0\dots\hat{i}_r\dots i_s} = \sum_{r,j} (-1)^r \rho_j \lambda_{ji_0\dots\hat{i}_r\dots i_s}.$$

Since λ is a cocycle,

$$(\delta\lambda)_{ji_0\dots i_s} = \lambda_{i_0\dots i_s} + \sum_r (-1)^{r+1} \lambda_{ji_0\dots\hat{i}_r\dots i_s} = 0$$

Hence,

$$(\delta\varkappa)_{i_0\dots i_s} = \sum_j \rho_j \sum_r (-1)^r \lambda_{ji_0\dots\hat{i}_r\dots i_s} = \sum_j \rho_j \lambda_{i_0\dots i_s} = \lambda_{i_0\dots i_s}.$$

Thus, $(K^{k,\bullet},\delta)$ is indeed an exact complex.

The element $\varkappa$ defined by (6.2) admits the estimates of the norms mentioned in the proposition.

Indeed, we infer

$$\|\varkappa\|_{L^q(K^{k,s},\beta)} = \sum_{i_0<\dots<i_{s-1}} \Big\| \sum_j \rho_j \lambda_{ji_0\dots i_{s-1}} \Big\|_{L^q(U_I,\Lambda^k,\beta)}$$

$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \|\rho_j \lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^k,\beta)}$$

$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \|\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_{j,I},\Lambda^k,\beta)} \leq \|\lambda\|_{L^q(K^{k,s+1},\beta)},$$

which gives the first estimate of the proposition.

Let us prove the second estimate. We have

$$d\varkappa_{i_0\dots i_{s-1}} = \sum_j (d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}} + \rho_j d\lambda_{ji_0\dots i_{s-1}}).$$

Therefore,

$$\|\varkappa\|_{L^q(K^{k+1,s},\beta)} = \sum_{i_0<\dots<i_{s-1}} \left\| \sum_j d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}} + \rho_j d\lambda_{ji_0\dots i_{s-1}} \right\|_{L^q(U_I,\Lambda^{k+1},\beta)}$$

$$\leq \sum_{i_0<\dots<i_{s-1}} \sum_j \left(\|d\rho_j \wedge \lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^{k+1},\beta)} + \|\rho_j d\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda_{k+1}),\beta} \right)$$

$$\leq \text{const} \sum_{i_0<\dots<i_{s-1}} \sum_j \left(\|\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^k,\beta)} + \|d\lambda_{ji_0\dots i_{s-1}}\|_{L^q(U_I,\Lambda^{k+1},\beta)} \right)$$

$$= \text{const} \left(\|\lambda\|_{L^q(K^{k,s+1},\beta)} + \|d\lambda\|_{L^q(K^{k+1,s+1},\beta)} \right).$$

□

Now, applying the general scheme of [23], we first construct some elements $\xi^s \in L^q(K^{k-s-1,s+1},\beta)$ and then elements $x^s \in L^q(K^{k-s-1,s},\beta)$ such that $\xi = x^0 \in C^\infty L^q(M,\Lambda^{k-1},\beta)$ is an element satisfying the claim of Theorem 6.1.

Construction of the elements $\xi^s \in L^q(K^{k-s-1,s+1},\beta)$.

Put $\xi^{-1} = \omega$ and define (by induction) ξ^s by setting its component $(\xi^s)_I$ to be a solution to the equation

$$d\xi^s_I = (\delta\xi^{s-1})_I \tag{6.3}$$

in V_I, $I = (i_0,\dots,i_s)$ such that

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const} \|(\delta\xi^{s-1})_I\|_{L^q(V_I,\Lambda^{k-s},\beta)} \tag{6.4}$$

for $0 \leq s \leq k-1$.

Note that such a solution always exists due to the local Sobolev–Poincaré inequality (Corollary 5.5) since V_I is bi-Lipschitz diffeomorphic to a cylinder over a convex subset in $\mathbb{R}^n$ with compact closure.

We have the following estimate of the weighted q-norm of ξ^s:

Proposition 6.3. *If $I = (i_0,\dots,i_s)$ then*

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const} \|\omega\|_{L^q(M,\Lambda^k,\gamma)}.$$

Proof. Use induction on s. For $s = 0$, the assertion follows from the local Sobolev–Poincaré inequality. Let now $s > 0$. We infer

$$\|\xi^s_I\|_{L^q(V_I,\Lambda^{k-s-1},\beta)} \leq \text{const} \|(\delta\xi^{s-1})_I\|_{L^q(V_I,\Lambda^{k-s},\beta)}$$

$$\leq \text{const} \sum_{r=0}^{s} \|\xi^{s-1}_{i_0\dots\hat{i}_r\dots i_{s-1}}\|_{L^q(V_I,\Lambda^{k-s},\beta)}$$

$$\leq \text{const} \sum_{r=0}^{s} \|\xi^{s-1}_{i_0\dots\hat{i}_r\dots i_{s-1}}\|_{L^q(V_{i_0\dots\hat{i}_r\dots i_{s-1}},\Lambda^{k-s},\beta)}$$
$$\leq \text{const} \sum_{r=0}^{s} \|\omega\|_{L^p(M,\Lambda^k,\gamma)} \leq \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}$$

□

Note that ξ^{k-1} is a collection of 0-forms satisfying the condition $d\delta\xi^{k-1} = 0$. Thus, the functions $(\delta\xi^{k-1})_I$ are constants on each set V_I, $I = (i_0, \dots, i_k)$. The global constant functions $(\delta\xi^{k-1})_I$ on M belong to $L^q(M, \beta)$ due to the hypotheses on β.

The following assertion is Theorem 3.10 in [23]:

Lemma 6.4. *There exists $c \in K^{0,k}$ with constant components c_I, $I = (i_0, \dots, i_{k-1})$, such that*

$$(\delta c)_I = \sum_{r=0}^{k} (-1)^r c_{i_0\dots\hat{i}_r\dots i_k} (\delta\xi^{k-1})_I, \quad I = (i_0, \dots, i_k).$$

In addition, there exist numbers $b_{I,L} \in \mathbb{R}$, $I = (i_0, \dots, i_{k-1})$, $L = (i_0, \dots, i_k)$, such that

$$c_I = \sum_L b_{I,L} (\delta\xi^{k-1})_L,$$

where $b_{i,L}$ depend on the chosen cover $\mathcal{U}$ of N.

We have

Proposition 6.5. *The constants c_I of Lemma 6.4 satisfy the estimate*

$$\|c_I\|_{L^q(V_I,\beta)} \leq \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}$$

Proof. By Lemma 6.4, each c_I is representable as $c_I = \sum_L b_{I,L}(\delta\xi^{k-1})_L$. Hence,

$$\|c_I\|_{L^q(V_I,\beta)} \leq \sum_L |b_{I,L}| \|(\delta\xi^{k-1})_L\|_{L^q(V_I,\beta)}.$$

Since $(\delta\xi^{k-1})_L$ is a globally defined constant function on M as in the proof of Proposition 6.3, we have

$$\|(\delta\xi^{k-1})_L\|_{L^q(V_I,\beta)} = \frac{\|\beta\|_{L^q([a,b))}(\text{vol}(U_I))^{1/q}}{\|\beta\|_{L^q([a,b))}(\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L,\beta)}$$
$$= \frac{(\text{vol}(U_I))^{1/q}}{(\text{vol}(U_L))^{1/q}} \|(\delta\xi^{k-1})_L\|_{L^q(V_L,\beta)} \leq \text{const}\, \|\omega\|_{L^p(M,\Lambda^k,\gamma)}.$$

This gives the estimate of the proposition. □

Construction of the elements $x^s \in L^q(K^{k-s-1,s}, \beta)$.

Let us now glue all the forms ξ^s, $s{-}0, \dots, k{-}1$, into a global form ξ satisfying (6.1). Construct by induction elements $x^s \in L^q(K^{k-s-1,s}, \beta)$, $s = k-1, \dots, 1, 0$, such that $\xi = x^0$ is a desired form on M.

Put $\tilde{\xi}^{k-1}_I = \xi^{k-1}_I - c_I$, where c_I is as in Lemma 6.4, $I = (i_0, \dots, i_{k-1})$. We have $d\tilde{\xi}^{k-1}_I = d\xi^{k-1}_I$ and $\delta\tilde{\xi}^{k-1}_I = 0$. By Proposition 6.2, there exists $x^{k-1} \in L^q(K^{0,k-1}, \beta)$ such that $\delta x^{k-1} = \tilde{\xi}^{k-1}$ and

$$\|dx^{k-1}\|_{L^q(K^{1,k-1},\beta)} \leq \text{const}\, \|\tilde{\xi}^{k-1}\|_{L^q(K^{0,k},\beta)},$$

$$\|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \le \text{const}\left(\|\tilde{\xi}^{k-1}\|_{L^q(K^{0,k},\beta)} + \|d\tilde{\xi}^{k-1}\|_{L^q(K^{1,k},\beta)}\right).$$

Propositions 6.3 and 6.5 yield

$$\|x^{k-1}\|_{L^q(K^{0,k-1},\beta)} \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)} \tag{6.5}$$

and

$$\begin{aligned}\|dx^{k-1}\|_{L^q(K^{1,k-1},\beta)} &\le \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\delta\xi^{k-2}\|_{L^q(K^{1,k},\beta)}\right)\\ &\le \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|\xi^{k-2}\|_{L^q(K^{1,k-1},\beta)}\right) \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.\end{aligned} \tag{6.6}$$

Suppose that $x^{k-(r-1)}$ is already constructed. By Proposition 6.2, there exists x^{k-r} such that

$$\delta x^{k-r} = \xi^{k-r} - dx^{k-r+1},$$

where

$$\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \le \text{const}\,\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} \tag{6.7}$$

and

$$\begin{aligned}&\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)}\\ &\le \text{const}\left(\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|d\xi^{k-r}\|_{L^q(K^{r,k-r+1},\beta)}\right)\\ &\le \text{const}\big(\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\\ &\qquad + \|\delta\xi^{k-r-1}\|_{L^q(K^{r,k-r+1},\beta)}\big)\\ &\le \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right).\end{aligned} \tag{6.8}$$

Here the last inequality stems from the fact that

$$\delta(\xi^{k-r} - dx^{k-r+1}) = \delta\delta x^{k-r} = 0.$$

The above considerations imply the following

Proposition 6.6. *The forms x^s admit the estimates:*
(1) $\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}$;
(2) $\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}$.

Proof. Use induction on r. For $r = 1$, (1) and (2) are just estimates (6.5) and (6.6). Assume that $r > 1$. For proving estimate (2), observe that, by the induction hypothesis and (6.8),

$$\begin{aligned}\|dx^{k-r}\|_{L^q(K^{r,k-r},\beta)} &\le \text{const}\left(\|\omega\|_{L^p(M,\Lambda^k,\gamma)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right)\\ &\le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.\end{aligned}$$

Now, Proposition 6.3 and estimates (6.7) and (2) yield

$$\begin{aligned}&\|x^{k-r}\|_{L^q(K^{r-1,k-r},\beta)} \le \text{const}\,\|\xi^{k-r} - dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\\ &\le \text{const}\left(\|\xi^{k-r}\|_{L^q(K^{r-1,k-r+1},\beta)} + \|dx^{k-r+1}\|_{L^q(K^{r-1,k-r+1},\beta)}\right) \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.\end{aligned}$$

□

Finally, put $\xi = x^0$. Then $d\xi = \omega$. Indeed, we have

$$\delta(\omega - dx^0) = \delta\omega - d\delta x^0 = \delta\omega - d(\xi^0 - dx^1) = \delta\omega - d\xi^0 = 0.$$

Since $\delta(\omega - dx^0)_i = (\omega - dx^0)|_{V_i}$, we infer that $\omega = dx_0$ on M. By Proposition 6.6,

$$\|\xi\|_{L^q(M,\Lambda^{k-1},\beta)} = \|x^0\|_{L^q(K^{k-1,0},\beta)} \le \text{const}\,\|\omega\|_{L^p(M,\Lambda^k,\gamma)}.$$

Theorem 6.1 is completely proved.

7. $L_{q,p}$-Cohomology of a Twisted Cylinder

Theorem 7.1. *Suppose that N is a closed manifold of dimension n, $H^k_{\mathrm{DR}}(N)=0$, $q\geq p\geq 1$, and $\frac{1}{p}-\frac{1}{q}<\frac{q-1}{q(n+1)}$. If*

$$\|\max(F_{k-2,q},F_{k-1,q})\|_{L^q([a,b))}<\infty,\quad \|t\max(F_{k-2,q},F_{k-1,q})(t))\|_{L^q([a,b))}<\infty,$$

and

$$\|\{\min(f_{k-1,p},f_{k,p})\}^{-1}\|_{L^{\frac{p\overline{p}}{p-\overline{p}}}([a,b))}<\infty$$

for some $\overline{p}$, $1\leq\overline{p}\leq p$ (for $\overline{p}=p$, we put $\frac{p\overline{p}}{p-\overline{p}}=\infty$), then $H^k_{q,p}(C^h_{a,b}N)=0$.

Proof. Let $\overline{M}$ be the cylinder $[a,b)\times N$ with the usual product metric. By the Künneth formula for the de Rham cohomology, we have

$$H^k_{\mathrm{DR}}(\overline{M})=H^k_{\mathrm{DR}}(N)=0.$$

Let $\omega\in C^\infty L^p(C^h_{a,b}N,\Lambda^k)$. Using expression (3.2) for the norm and the definition of $f_{l,p}$, we infer

$$\begin{aligned}(7.1)\quad &\|\omega\|_{L^p(\overline{M},\Lambda^k,\min(f_{k-1,p},f_{k,p}))}\\ &=\left[\int_a^b\{\min(f_{k-1,p}(t),f_{k,p}(t))\}^p\int_N(|\omega_A(t,x)|^2_N+|\omega_B(t,x)|^2_N\big)^{\frac{p}{2}}dxdt\right]^{\frac{1}{p}}\\ &\leq\left[\int_a^b\int_N\big(h^{2(\frac{n}{p}-k)}(t,x)|\omega_A(t,x)|^2_N+h^{2(\frac{n}{p}-k+1)}(t,x)|\omega_B(t,x)|^2_N\big)^{\frac{p}{2}}dxdt\right]^{\frac{1}{p}}\\ &=\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}.\end{aligned}$$

Thus, $\omega\in C^\infty L^p(\overline{M},\Lambda^k,\min(f_{k-1,p},f_{k,p}))$. Since the de Rham cohomology $H^k_{\mathrm{DR}}(\overline{M})$ is trivial, ω is exact, and we can apply Theorem 6.1, by which there exists $\xi\in C^\infty L^q(\overline{M},\Lambda^k,\max(F_{k-2,q}(t),F_{k-1,q}(t))$ with

$$(7.2)\qquad \|\xi\|_{L^q(\overline{M},\Lambda^{k-1},\max(F_{k-2,q},F_{k-1,q}))}\leq\mathrm{const}\|\omega\|_{L^p(\overline{M},\Lambda^k,\min(f_{k-1,p},f_{k,p}))}.$$

For this form ξ, we have

$$\begin{aligned}(7.3)\quad &\|\xi\|_{L^q(C^h_{a,b}N,\Lambda^{k-1})}\\ &=\left[\int_a^b\int_N\big(h^{2(\frac{n}{q}-k+1)}(t,x)|\xi_A(t,x)|^2_N+h^{2(\frac{n}{q}-k+2)}(t,x)|\xi_B(t,x)|^2_N\big)^{\frac{q}{2}}dxdt\right]^{\frac{1}{q}}\\ &\leq\left[\int_a^b\{\max(F_{k-2,q}(t),F_{k-1,q}(t))\}^q\int_N(|\xi_A(t,x)|^2_N+|\xi_B(t,x)|^2_N\big)^{\frac{q}{2}}dxdt\right]^{\frac{1}{q}}\\ &=\|\xi\|_{L^q(\overline{M},\Lambda^{k-1},\max(F_{k-2,q},F_{k-1,q}))}.\end{aligned}$$

Combining (7.1),(7.2), and (7.3), we obtain

$$\|\xi\|_{L^q(C^h_{a,b}N,\Lambda^{k-1})}\leq\mathrm{const}\,\|\omega\|_{L^p(C^h_{a,b}N,\Lambda^k)}.$$

Thus, $C^\infty H^k_{q,p}(C^h_{a,b}N)=0$, and hence, by Theorem 2.5, also $H^k_{q,p}(C^h_{a,b}N)=0$. $\square$

Remark 7.2. As was observed in the introduction, in [18], the second author established sufficient conditions for the L_{qp}-cohomology of a warped cylinder to be infinite-dimensional in terms of "almost duality" and Hardy's inequality. In [18], the base manifold must have nontrivial L_p- and L_q-cohomology.

8. $L_{q,p}$-Cohomology of an Asymptotic Twisted Cylinder

Recall the following definition, given in [10]:

Definition 8.1. We refer to a pair (M, X) consisting of an m-dimensional manifold M and an m-dimensional compact submanifold X with boundary as an *asymptotic twisted cylinder* $AC^h_{a,b}\partial X$ if $M\setminus(X\setminus\partial X)$ is bi-Lipschitz diffeomorphically equivalent to the twisted cylinder $C^h_{a,b}\partial X$.

For asymptotic twisted cylinders, Theorem 7.1 gives:

Theorem 8.2. *Let $(M, X) = AC^h_{a,b}\partial X$ be an asymptotic twisted cylinder with $\dim M = \dim X = m = n + 1$. Assume that $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{qm}$, and $H^k_{\mathrm{DR}}(\partial X) = 0$. If*

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b))} < \infty, \quad \|t\max(F_{k-2,q}, F_{k-1,q})(t))\|_{L^q([a,b))} < \infty,$$

and

$$\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^{\frac{p\overline{p}}{p-\overline{p}}}([a,b))} < \infty$$

for some $\overline{p}$, $1 \leq \overline{p} \leq p$ (for $\overline{p} = p$, we put $\frac{p\overline{p}}{p-\overline{p}} = \infty$), then $H^k_{q,p}(M) = 0$.

Proof. Since bi-Lipschitz diffeomorphisms preserve L_{p_1} and L_{p_2} and extension by zero gives a topological isomorphism between the spaces $W^*_{p_1,p_2}(C^h_{a,b}\partial X)$ and $W^*_{p_1,p_2}(M)$ for all p_1, p_2, we have a topological isomorphism

$$H^*_{p_1,p_2}(M) \cong H^*_{p_1,p_2}(C^h_{a,b}\partial X)$$

for all p_1, p_2. The theorem now follows from Theorem 7.1. □

9. Examples

Let us analyze the conditions of the last theorems for comparatively simple cases. Suppose that N is the n-dimensional sphere S^n. Then $H^k_{\mathrm{DR}}(N) = 0$ for any $k \neq 0, n$. By the hypothesess of the theorems, $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$. Put

$$s(t) := \max_{x\in S^n} h(t, x) \quad \text{and} \quad g(t) := \min_{x\in S^n} h(t, x).$$

Then, by definition,

$$I_{1,q,k} := \max(F_{k-2,q}, F_{k-1,q}) = \max(s^{\frac{n}{q}-k+2}, s^{\frac{n}{q}-k+1}),$$

$$I_{2,q,k}(t) := t\max(F_{k-2,q}, F_{k-1,q})(t)) = t\max(s^{\frac{n}{q}-k+2}(t), s^{\frac{n}{q}-k+1}(t))$$

and

$$I_{3,p,k} := \{\min(f_{k-1,p}, f_{k,p})\}^{-1} = \{\min(g^{\frac{n}{p}-k+1}, g^{\frac{n}{p}-k})\}^{-1}.$$

By the hypotheses of the theorems, we must check the integrability of these three functions in the corresponding degrees under the above-mentioned restrictions on p and q.

Suppose for simplicity that $s(t)$ and $g(t)$ are smooth increasing functions tending to ∞ as $t \to b - 0$. Denote the maximal integrability intervals for s^u and g^v by $(-\infty, \alpha)$ and $(-\infty, \beta)$, i.e s^u is integrable on $[a, b)$ for every $u < \alpha$ and is not integrable for every $u > \alpha$ and similarly for g^v. Let also α_1 be the supremum of μ such that $ts^{\mu}(t)$ is integrable on $[a, b)$.

For this case $I_{1,q,k} = s^{\frac{n}{q}-k+2}$, $I_{2,q,k}(t) = ts^{\frac{n}{q}-k+2}(t)$, and $I_{3,p,k} = g^{k-\frac{n}{p}}$.

The conditions of the theorems are fulfilled if

$$\frac{n}{q} - k + 2 < \min(\alpha, \alpha_1), \quad \frac{n}{p} - k > -\beta.$$

Note that these inequalities cannot hold simultaneously if $b = \infty$. In this case, α, α_1, and β are all negative, whence $\frac{n}{p} - k > \frac{n}{q} - k + 2$. We thus have $\frac{1}{p} - \frac{1}{q} > \frac{2}{n}$, which contradicts the hypotheses.

Examine more closely the case of $0 \le a < b < \infty$. The function t is bounded, and hence $\alpha = \alpha_1$. Therefore, the inequalities for $I_{1,q,k}$ and $I_{3,p,k}$ can be combined into one inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \le \frac{1}{p} < \frac{k-\beta}{n}.$$

It means that the additional condition $k - 2 + \alpha \le k - \beta$, i.e., $\alpha + \beta \le 2$, must be fulfilled.

The last condition is $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, i.e., $p \le q < \frac{np}{n+1-p}$.

Summarizing, we conclude that for known integrability limits α and β, we need to check two simple conditions for p and q:

$$\alpha + \beta \le 2, \ p < q < \frac{np}{n+1-p}$$

and the inequality

$$\frac{k-2+\alpha}{n} < \frac{1}{q} \le \frac{1}{p} < \frac{k-\beta}{n}.$$

for the degree k.

Under these conditions, the cohomology of the twisted cylinder $C^h_{[a,b)}S^n$ vanishes.

For example, if $s(t) = g(t) = (b-t)^{-2}$ then $\alpha = \beta = 1/2$. For $p = 2$ we have $2 \le q < 2\frac{n}{n-1}$.

The last inequality yields

$$\frac{k-3/2}{n} < \frac{1}{q} \le \frac{1}{2} < \frac{k-1/2}{n}. \tag{9.1}$$

Let q be an arbitrary number in $(2, \frac{8}{3})$. Then the second inequality $q < \frac{2n}{n-1}$ gives us the constraint $n < q/(q-2)$. Since $q/(q-2) < 4$, we can take $n = 4$. We have $1/2 < (2k-1)/8$, i.e., $k > 2$. For $k = 3$, the leftmost inequality gives us the valid condition $3/8 < 1/q$. Note that if $q = 2$ then always $q < \frac{2n}{n-1}$. If n is even and $k = \frac{n}{2} + 1$ then all inequalities in (9.1) are fulfilled. Thus, we have

$$H^3_{q,2}\big(C^h_{[a,b)}S^4\big) = 0 \quad \text{if } q \in \left[2, \frac{8}{3}\right)$$

and

$$H^{l+1}_{2,2}\big(C^h_{[a,b)}S^{2l}\big) = 0, \quad l \ge 2.$$

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