Transposed Poisson structures on quasi-filiform Lie algebras of maximum length

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Abstract: *In this work, we explore the* $\frac{1}{2}$*-derivations of quasi-filiform Lie algebras of maximum length, focusing on dimensions 7 and higher. Using these derivations, we then construct non-trivial transposed Poisson algebras. Additionally, we develop a commutative associative multiplication method to build a transposed Poisson algebra associated with each Lie algebra under consideration.*

Keywords: *Lie algebra, transposed Poisson algebra,* $\frac{1}{2}$*-derivation.*

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INTRODUCTION

Bai et.al. [3] introduced a dual notion of the Poisson algebra, which is called a *transposed Poisson algebra*, by changing the roles of the two multiplications in the Leibniz rule. Poisson algebra is introduced to commutative associative algebras by their derivation. Similarly, on a Lie algebra, the concept of a transposed Poisson algebra is defined by the $\frac{1}{2}$-derivation. This way of defining not only shares some properties of a Poisson algebra, such as the closedness under tensor products and the Koszul self-duality as an operation, but also admits a rich class of identities [3, 5, 8, 10, 25–27].

The description of all Poisson algebras with a fixed Lie or associative part [11, 15, 30] is one of the natural tasks in the theory. In this paper, we present transposed Poisson algebras that demonstrate quasi-filiform Lie algebras of maximum length. Abdurasulov et.al. [1], obtained the description of all transposed Poisson algebra structures on solvable Lie algebras with filiform nilradical.

It is worth noting that any unital transposed Poisson algebra is a particular case of a "contact bracket" algebra and a quasi-Poisson algebra [5]. Every transposed Poisson algebra is a commutative Gelfand-Dorfman algebra [27], and is also an algebra of Jordan brackets [8]. Ferreira et. al. [7] established a relation between transposed Poisson algebras and $\frac{1}{2}$-derivations of Lie algebras. These ideas played an active role in describing all transposed Poisson structures on Witt and Virasoro algebras in [7]. Twisted Heisenberg-Virasoro, Schrödinger-Virasoro and extended Schrödinger-Virasoro algebras were studied in [32]. The fact that the original deformative Schrödinger-Virasoro algebras have nontrivial $\frac{1}{2}$-derivations, indicating that they possess nontrivial transposed Poisson structures is shown in [28]. In [23] Transposed Poisson structures on deformative Schrödinger-Witt algebras, non-finitely graded Witt algebras, and non-finitely graded Heisenberg-Witt algebras are classified.

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Moreover, in [24] authors computed $\frac{1}{2}$-derivations on the deformed generalized Heisenberg-Virasoro algebras and on not-finitely graded Heisenberg-Virasoro algebras, and classified all transposed Poisson structures on such algebras. δ-derivations of simple Jordan algebras with values in irreducible bimodules studied in [33] Meanwhile, Schrödinger algebra in $(n+1)$-dimensional space-time was discussed in [29]. Then, in [17] Witt type Lie algebras were discussed and [18] studied generalized Witt algebras. The papers [16, 18] studied Block Lie algebras, while [19] examined Lie algebras of upper triangular matrices and a transposed Poisson structures on Lie algebra of upper triangular matrices. Additionally, [20] was devoted to the study of Lie incidence algebras. In [6] considered a new family of modified double Poisson brackets and mixed double Poisson algebras. It was proved in [22] that any complex finite-dimensional solvable Lie algebra admits a non-trivial transposed Poisson structure. In [4] authors gave the algebraic and geometric classification of three-dimensional transposed Poisson algebras. The list of actual open questions on transposed Poisson algebras can be seen in [5]. Recently, all transposed Poisson algebra structures on oscillator Lie algebras, i.e., on one-dimensional solvable extensions of $(2n+1)$-dimensional Heisenberg algebra; on solvable Lie algebras with naturally graded filiform nilpotent radical; on $(n+1)$-dimensional solvable extensions of the $(2n+1)$-dimensional Heisenberg algebra; and on n-dimensional solvable extensions of n-dimensional algebra with the trivial multiplication were described in [21]. This paper also gave an example of a finite-dimensional Lie algebra with non-trivial $\frac{1}{2}$-derivations without non-trivial transposed Poisson algebra structures. In [4], the algebraic and geometric classification of all complex 3-dimensional transposed Poisson algebras was obtained, whereas [31] focused on the algebraic classification of all complex 3-dimensional transposed Poisson 3-Lie algebras.

The purpose of this article is to find all transposed Poisson algebras that demonstrate quasi-filiform Lie algebras of maximum length. Building on our previous work [2], where we addressed the case of quasi-filiform Lie algebras in dimensions 5 and 6, this paper extends the results to dimensions 7 and higher, thereby providing a general solution to the problem. In order to achieve our goal, we have organized this paper as follows: in Section 2, we describe $\frac{1}{2}$-derivations of quasi-filiform Lie algebras of maximum length and Section 3, describes all non-trivial transposed Poisson algebras with quasi-filiform Lie algebras of maximum length. Next, by using descriptions of $\frac{1}{2}$-derivations of Lie algebras, we establish commutative associative multiplication to construct a transposed Poisson algebra associated with given Lie algebra.

1. PRELIMINARIES

This section discusses the concepts and known results on algebras over the field $\mathbb{C}$ unless the otherwise is stated.

A Poisson algebra consists of a vector space L equipped with two bilinear operations, denoted $\cdot$ and $[\cdot,\cdot]$, which map $L\times L$ to L. In this structure, $(L,\cdot)$ is a commutative associative algebra, while $(L,[\cdot,\cdot])$ forms a Lie algebra. The operations must satisfy a compatibility condition known as the Leibniz rule, expressed as

$$[x, y\cdot z] = [x, y]\cdot z + y\cdot [x, z],$$

where the adjoint operators of the Lie algebra act as derivations on the commutative associative algebra.

Poisson algebras arose from the study of Poisson geometry in the 1970s and have appeared in an extremely wide range of areas in mathematics and physics. Below we give the definition of transposed Poisson algebra, which was given in the paper by Bai, Bai, Guo, and Wu [3].

⟨TPA⟩ **Definition 1.** *Let $\mathfrak{L}$ be a vector space equipped with two bilinear operations*

$$\cdot,\ [-,-] : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}.$$

The triple $(\mathfrak{L}, \cdot, [-,-])$ is called a ***transposed Poisson algebra*** *if $(\mathfrak{L}, \cdot)$ is a commutative associative algebra and $(\mathfrak{L}, [-,-])$ is a Lie algebra which satisfies the following compatibility condition*

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]. \tag{1}$$ eq:dualp

Eq. (1) is called **transposed Leibniz rule** because the roles of two binary operations in the Leibniz rule in a Poisson algebra are switched. Further, the resulting operation is rescaled by introducing a factor 2 on the left-hand side.

⟨halfderiv⟩ **Definition 2.** *Let $(\mathfrak{L}, [-,-])$ be an algebra with a multiplication $[-,-]$, and φ be a bilinear map. Then φ is a $\frac{1}{2}$-**derivation** if it satisfies:*

$$\varphi([x, y]) = \frac{1}{2}([\varphi(x), y] + [x, \varphi(y)]). \tag{2}$$ halfderiv

Note that $\frac{1}{2}$-derivations are a particular case of δ-derivations introduced by Filippov in [9] (see, [13,14] and references therein). It is easy to see from Definition 2 that $[\mathfrak{L}, \mathfrak{L}]$ and $Ann(\mathfrak{L})$ are invariant under any $\frac{1}{2}$-derivation of $\mathfrak{L}$. Definitions 1 and 2 immediately imply the following lemma.

?⟨lemma1⟩? **Lemma 3.** *Let $(\mathfrak{L}, [-,-])$ be a Lie algebra and $\cdot$ a new binary (bilinear) operation on $\mathfrak{L}$. Then $(\mathfrak{L}, \cdot, [-,-])$ is a transposed Poisson algebra if and only if $\cdot$ is commutative and associative, and for every $z \in \mathfrak{L}$ the multiplication by z in $(\mathfrak{L}, \cdot)$ is a $\frac{1}{2}$-derivation of $(\mathfrak{L}, [-,-])$.*

Lower central series for a given Lie algebra $\mathfrak{L}$ is defined as follows:

$$\mathfrak{L}^1 = \mathfrak{L},\ \mathfrak{L}^{k+1} = [\mathfrak{L}^k, \mathfrak{L}],\ k \geq 1.$$

Definition 4. *An n-dimensional Lie algebra is called* ***quasi-filiform*** *if $\mathfrak{L}^{n-2} \neq \{0\}$ and $\mathfrak{L}^{n-1} = \{0\}$.*

A Lie algebra $\mathfrak{L}$ is $\mathbb{Z}$-graded, if $\mathfrak{L} = \bigoplus_{i \in \mathbb{Z}} \mathbf{V}_i$, where $[V_i, V_j] \subseteq V_{i+j}$ for any $i, j \in \mathbb{Z}$ with a finite number of non-null spaces V_i. We say that a nilpotent Lie algebra $\mathfrak{L}$ admits **the connected gradation** $\mathfrak{L} = V_{k_1} \oplus \cdots \oplus V_{k_t}$, if $V_{k_i} \neq \{0\}$ for any i $(1 \leq i \leq t)$.

Definition 5. *The number $l(\oplus\mathfrak{L}) = l(V_{k_1} \oplus \cdots \oplus V_{k_t}) = k_t - k_1 + 1$ is called* ***the length of gradation****. A gradation is said to be* ***of maximum length****, if $l(\oplus\mathfrak{L}) = \dim(\mathfrak{L})$.*

We denote $l(\mathfrak{L}) = \max\{l(\oplus L)$ such that $L = V_{k_1} \oplus \cdots \oplus V_{k_t}$ is a connected gradation$\}$ of **the length of an algebra** $\mathfrak{L}$.

Definition 6. *A Lie algebra $\mathfrak{L}$ is called of* ***maximum length*** *if $l(\mathfrak{L}) = \dim(\mathfrak{L})$.*

Since we covered the dimensions 5 and 6 in our previous work [2], we will consider $n \geq 7$ in this work. In the following theorem we present the classification of quasi-filiform Lie algebras of maximum length given in [12].

?⟨thm26⟩? **Theorem 7.** *Let L be an n-dimensional ($n \geq 7$) quasi-filiform Lie algebra of maximum length. Then the algebra L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$g^1_{(n,1)} : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_{n-i}] = (-1)^i e_n, & 2 \ \leq i \leq \frac{n-1}{2}, \ n \ is \ odd; \end{cases}$$

$$g^2_{(n,1)} : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3; \end{cases} \qquad g^3_{(n,1)} : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-2, \\ [e_i, e_n] = e_{i+2}, & 2 \leq i \leq n-3 \\ [e_2, e_i] = e_{i+3}, & 3 \leq i \leq n-4; \end{cases}$$

$$g^1_7 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 5, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_i, e_{7-i}] = (-1)^i e_7, & 2 \leq i \leq 3; \end{cases} \qquad g^2_9 : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 7, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_5] = 3e_7, & \\ [e_2, e_6] = 5e_8, & \\ [e_3, e_i] = -2e_{i+3}, & 4 \leq i \leq 5, \\ [e_i, e_{9-i}] = (-1)^i e_9, & 2 \leq i \leq 4; \end{cases}$$

$$g^3_{11} : \begin{cases} [e_1, e_i] = e_{i+1}, & 2 \leq i \leq 9, \\ [e_2, e_i] = e_{i+2}, & 3 \leq i \leq 4, \\ [e_2, e_i] = -e_{i+2}, & 6 \leq i \leq 7 \\ [e_3, e_7] = -e_{10}, & \\ [e_3, e_i] = e_{i+3}, & 4 \leq i \leq 5, \\ [e_4, e_i] = e_{i+4}, & 5 \leq i \leq 6, \\ [e_i, e_{11-i}] = (-1)^i e_{11}, & 2 \leq i \leq 5, \end{cases}$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra.

2. $\frac{1}{2}$-DERIVATION OF QUASI-FILIFORM LIE ALGEBRAS OF MAXIMUM LENGTH

In this section, we will derive and calculate $\frac{1}{2}$-derivation of quasi-filiform Lie algebras of maximum length with dimension $n \geq 7$. We start with the following theorem.

⟨halfderiv1⟩ **Theorem 8.** *Any $\frac{1}{2}$-derivations of the algebra $g^1_{(n,1)}$ has the form*

$$\begin{cases} \varphi(e_1) = \sum\limits_{i=1}^{n} \alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2} e_{n-2} + \beta_{n-1} e_{n-1} + \beta_n e_n, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2} e_{n-1} - \frac{1}{2}\alpha_{n-2} e_n, \\ \varphi(e_i) = \alpha_1 e_i + \frac{(-1)^i}{2}\alpha_{n-i+1} e_n, \ 4 \leq i \leq n-1, \\ \varphi(e_n) = \alpha_1 e_n. \end{cases}$$

Proof. It is easy to see that $g^1_{(n,1)}$ has 2 generators. We use these generators to calculate $\frac{1}{2}$-derivation.

$$\varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad \varphi(e_2) = \sum_{i=1}^{n} \beta_i e_i.$$

Now consider the condition of $\frac{1}{2}$-derivation for the elements e_1 and e_2 :

$$\varphi(e_3) = \varphi([e_1, e_2]) = \tfrac{1}{2}([\varphi(e_1), e_2] + [e_1, \varphi(e_2)])$$

$$= \tfrac{1}{2}([\sum_{i=1}^{n} \alpha_i e_i, e_2] + [e_1, \sum_{i=1}^{n} \beta_i e_i]) = \tfrac{1}{2}((\alpha_1 + \beta_2)e_3 + \sum_{i=4}^{n-1} \beta_{i-1} e_i - \alpha_{n-2} e_n).$$

We prove the following equality for $3 \leq i \leq n-1$ by induction:

$$\varphi(e_i) = \tfrac{1}{2^{i-2}}\Big((2^{i-2} - 1)\alpha_1 + \beta_2\Big)e_i + \tfrac{1}{2^{i-2}} \sum_{t=i+1}^{n-1} \beta_{t-i+2} e_t + \tfrac{(-1)^i}{2}\alpha_{n-i+1} e_n.$$

If $i = 3$, the relationship holds according to the above equality. Now, we prove that it is true for i and $i+1$. By considering the condition of $\frac{1}{2}$-derivation for the elements e_1, e_i we have

$$\varphi(e_{i+1}) = \varphi([e_1, e_i]) = \tfrac{1}{2}([\varphi(e_1), e_i] + [e_1, \varphi(e_i)])$$

$$= \tfrac{1}{2}\Big([\sum_{k=1}^{n} \alpha_k e_k, e_i] + [e_1, \tfrac{1}{2^{i-2}}\Big((2^{i-2} - 1)\alpha_1 + \beta_2\Big)e_i + \tfrac{1}{2^{i-2}} \sum_{t=i+1}^{n-1} \beta_{t-i+2} e_t + \tfrac{(-1)^i}{2}\alpha_{n-i+1} e_n]\Big)$$

$$= \tfrac{1}{2}\Big(\alpha_1 e_{i+1} + (-1)^{i+1}\alpha_{n-i} e_n + \tfrac{1}{2^{i-2}}\Big((2^{i-2} - 1)\alpha_1 + \beta_2\Big)e_{i+1} + \tfrac{1}{2^{i-2}} \sum_{t=i+1}^{n-2} \beta_{t-i+2} e_{t+1}\Big)$$

$$= \tfrac{1}{2^{i-1}}\Big((2^{i-1} - 1)\alpha_1 + \beta_2\Big)e_{i+1} + \tfrac{1}{2^{i-1}} \sum_{t=i+2}^{n-1} \beta_{t-i+1} e_t + \tfrac{(-1)^{i+1}}{2}\alpha_{n-i} e_n.$$

Now, consider the condition of $\frac{1}{2}$-derivation for the elements e_2, e_{n-2} :

$$\varphi(e_n) = \varphi([e_2, e_{n-2}]) = \tfrac{1}{2}([\varphi(e_2), e_{n-2}] + [e_2, \varphi(e_{n-2})])$$

$$= \tfrac{1}{2}\Big([\sum_{i=1}^{n} \beta_i e_i, e_{n-2}] + [e_2, \tfrac{1}{2^{n-4}}\Big((2^{n-4} - 1)\alpha_1 + \beta_2\Big)e_{n-2} + \tfrac{1}{2^{n-4}}\beta_3 e_{n-1} + \tfrac{(-1)^{n-2}}{2}\alpha_3 e_n]\Big)$$

$$= \tfrac{1}{2}\Big(\beta_1 e_{n-1}+\beta_2 e_n+\tfrac{1}{2^{n-4}}\Big((2^{n-4}-1)\alpha_1+\beta_2\Big)e_n\Big) = \tfrac{1}{2}\beta_1 e_{n-1}+\tfrac{1}{2^{n-3}}\Big((2^{n-4}-1)\alpha_1+(2^{n-4}+1)\beta_2\Big)e_n.$$

Thus, we obtain that

$$\varphi(e_n) = \frac{1}{2}\beta_1 e_{n-1} + \frac{1}{2^{n-3}}\Big((2^{n-4}-1)\alpha_1 + (2^{n-4}+1)\beta_2\Big)e_n. \qquad (3)\ \text{fien1}$$

Now, we consider the condition of $\frac{1}{2}$-derivation for the elements e_2 and e_i, $3 \leq i \leq n-3$:

$$0 = \varphi([e_2, e_i]) = \tfrac{1}{2}([\varphi(e_2), e_i] + [e_2, \varphi(e_i)])$$

$$= \tfrac{1}{2}\Big([\sum_{i=1}^{n}\beta_i e_i, e_i] + [e_2, \tfrac{1}{2^{i-2}}\Big((2^{i-2}-1)\alpha_1+\beta_2\Big)e_i + \tfrac{1}{2^{i-2}}\sum_{t=i+1}^{n-1}\beta_{t-i+2}e_t + \tfrac{(-1)^i}{2}\alpha_{n-i+1}e_n]\Big)$$

$$= \tfrac{1}{2}\Big([\beta_1 e_{i+1} - (-1)^i\beta_{n-i}e_n + \tfrac{1}{2^{i-2}}\beta_{n-i}e_n\Big).$$

By the comparison of coefficients at the basis elements we obtain that

$$\beta_1 = \beta_i = 0,\ 3 \leq i \leq n-3.$$

By considering the condition of $\frac{1}{2}$-derivation for the elements e_3 and e_{n-3} we have

$$\varphi([e_3, e_{n-3}]) = \tfrac{1}{2}([\varphi(e_3), e_{n-3}] + [e_3, \varphi(e_{n-3})])$$

$$= \tfrac{1}{2}\Big([\tfrac{1}{2}((\alpha_1+\beta_2)e_3 + \beta_{n-2}e_{n-1} - \alpha_{n-2}e_n), e_{n-3}] + [e_3, \tfrac{1}{2^{n-5}}\Big((2^{n-5}-1)\alpha_1+\beta_2\Big)e_{n-3} + \tfrac{1}{2}\alpha_4 e_n]\Big)$$

$$= \tfrac{1}{2}\Big(-\tfrac{1}{2}((\alpha_1+\beta_2)e_n - \tfrac{1}{2^{n-5}}\Big((2^{n-5}-1)\alpha_1+\beta_2\Big)e_n\Big).$$

On the other hand,

$$\varphi([e_3, e_{n-3}]) = -\varphi(e_n) = -\Big(\frac{1}{2^2}((\alpha_1+\beta_2) + \frac{1}{2^{n-4}}\Big((2^{n-5}-1)\alpha_1+\beta_2\Big)\Big)e_n. \qquad (4)\ \text{fien2}$$

By comparing the coefficients at the basis elements in expressions (3) and (4) we obtain that $\alpha_1 = \beta_2$.

This completes the proof of the theorem. $\square$

Now we study the $\frac{1}{2}$-derivation of the algebra $g^2_{(n,1)}$.

⟨halfderiv2⟩? **Theorem 9.** *Any $\frac{1}{2}$-derivations of the algebra $g^2_{(n,1)}$ has the form*

$$\begin{cases} \varphi(e_1) = \sum\limits_{i=1}^{n-1}\alpha_i e_i, \\ \varphi(e_2) = \alpha_1 e_2 + \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_{n-2}e_{n-1}, \\ \varphi(e_i) = \alpha_1 e_i,\ 4 \leq i \leq n-1, \\ \varphi(e_n) = -\sum\limits_{i=3}^{n-2}\alpha_{i-1}e_i + \gamma_{n-1}e_{n-1} + \alpha_1 e_n. \end{cases}$$

Proof. From the multiplication table of the algebra $g^2_{(n,1)}$ we conclude that e_1, e_2 and e_n are the generator basis elements of the algebra. We use these generators to calculate $\frac{1}{2}$-derivation:

$$\varphi(e_1)=\sum_{i=1}^{n}\alpha_i e_i,\ \varphi(e_2)=\sum_{i=1}^{n}\beta_i e_i,\ \varphi(x)=\sum_{i=1}^{n}\gamma_i e_i.$$

Now consider the condition of $\frac{1}{2}$-derivation for the elements e_1 and e_2 :

$$\varphi(e_3)=\varphi([e_1,e_2])=\tfrac{1}{2}([\varphi(e_1),e_2]+[e_1,\varphi(e_2)])$$

$$=\tfrac{1}{2}([\sum_{i=1}^{n}\alpha_i e_i,e_2]+[e_1,\sum_{i=1}^{n}\beta_i e_i])=\tfrac{1}{2}\Big((\alpha_1+\beta_2)e_3+(\beta_3-\alpha_n)e_4+\sum_{i=5}^{n-1}\beta_{i-1}e_i\Big).$$

We prove the following equality for $3\le i\le n-2$ by induction:

$$\varphi(e_i)=\tfrac{1}{2^{i-2}}\Big((2^{i-2}-1)\alpha_1+\beta_2\Big)e_i+\tfrac{1}{2^{i-2}}\Big(\beta_3-(2^{i-2}-1)\alpha_n\Big)e_{i+1}+\tfrac{1}{2^{i-2}}\sum_{t=i+2}^{n-1}\beta_{t-i+2}e_t$$

and

$$\varphi(e_{n-1})=\tfrac{1}{2^{n-3}}\Big((2^{n-3}-1)\alpha_1+\beta_2\Big)e_{n-1}.$$

If $i=3$, the relationship holds according to the above equality. Now, we prove that it is true for i and $i+1$. By considering the condition of $\frac{1}{2}$-derivation for the elements e_1, e_i , we have the following:

$$\varphi(e_{i+1})=\varphi([e_1,e_i])=\tfrac{1}{2}([\varphi(e_1),e_i]+[e_1,\varphi(e_i)])$$

$$=\tfrac{1}{2}\Big([\sum_{k=1}^{n}\alpha_k e_k,e_i]+[e_1,\tfrac{1}{2^{i-2}}\Big((2^{i-2}-1)\alpha_1+\beta_2\Big)e_i+\tfrac{1}{2^{i-2}}\Big(\beta_3-(2^{i-2}-1)\alpha_n\Big)e_{i+1}+\tfrac{1}{2^{i-2}}\sum_{t=i+2}^{n-1}\beta_{t-i+2}e_t\Big)$$

$$=\tfrac{1}{2}\Big(\alpha_1 e_{i+1}-\alpha_n e_{i+2}+\tfrac{1}{2^{i-2}}\Big((2^{i-2}-1)\alpha_1+\beta_2\Big)e_{i+1}+\tfrac{1}{2^{i-2}}\Big(\beta_3-(2^{i-2}-1)\alpha_n\Big)e_{i+2}$$
$$+\tfrac{1}{2^{i-2}}\sum_{t=i+2}^{n-2}\beta_{t-i+2}e_{t+1}\Big)$$

$$=\tfrac{1}{2^{i-1}}\Big((2^{i-1}-1)\alpha_1+\beta_2\Big)e_{i+1}+\tfrac{1}{2^{i-1}}\Big(\beta_3-(2^{i-1}-1)\alpha_n\Big)e_{i+2}+\tfrac{1}{2^{i-1}}\sum_{t=i+3}^{n-1}\beta_{t-i+1}e_t.$$

From

$$0=\varphi([e_1,e_n])=\frac{1}{2}([\varphi(e_1),e_n]+[e_1,\varphi(e_n)])=\frac{1}{2}\Big([\sum_{k=1}^{n}\alpha_k e_k,e_n]+[e_1,\sum_{k=1}^{n}\gamma_k e_k]\Big)$$

$$=\frac{1}{2}\Big(\sum_{k=2}^{n-3}\alpha_k e_{k+2}+\sum_{k=2}^{n-2}\gamma_k e_{k+1}\Big)=\frac{1}{2}\Big(\sum_{k=4}^{n-1}\alpha_{k-2}e_k+\sum_{k=3}^{n-1}\gamma_{k-1}e_k\Big),$$

we obtain,

$$\gamma_2=0,\ \gamma_i=-\alpha_{i-1},\ 3\le i\le n-2.$$

Now consider the condition of $\frac{1}{2}$-derivation for the elements e_2, e_n :

$$\varphi([e_2,e_n]) = \tfrac{1}{2}([\varphi(e_2),e_n]+[e_2,\varphi(e_n)]) = \tfrac{1}{2}\Big([\sum_{i=1}^{n}\beta_i e_i, e_n]+[e_2,\sum_{i=1}^{n}\gamma_i e_i]\Big)$$

$$= \tfrac{1}{2}\Big(\sum_{i=2}^{n-3}\beta_i e_{i+2} - \gamma_1 e_3 + \gamma_n e_4\Big) = \tfrac{1}{2}\Big(-\gamma_1 e_3 + (\beta_2+\gamma_n)e_4 + \sum_{i=5}^{n-1}\beta_{i-2}e_i\Big).$$

Thus,

$$\varphi([e_2,e_n]) = \varphi(e_4).$$

By Comparing the coefficients at the basis elements, we obtain that

$$\gamma_1 = 0,\ 2\gamma_n = 3\alpha_1 - \beta_2;\ \beta_3 = -3\alpha_n,\ \beta_i = 0,\ 4 \le i \le n-3.$$

By using the property of $\frac{1}{2}$-derivation for the products $[e_2,e_3]=0$ and $[e_3,e_n]=e_5$, we have

$$\beta_1 = 0;\ \beta_n = 0,\ \beta_2 = \alpha_1;\ \beta_3 = \alpha_n = 0.$$

This completes the proof of the theorem. □

Now, we will study the $\frac{1}{2}$-derivation of the algebra $g^3_{(n,1)}$.

⟨halfderiv3⟩ **Theorem 10.** *Any $\frac{1}{2}$-derivations of the algebra $g^3_{(n,1)}$ has the form*

- *for $n=7$:*

$$\begin{cases}\varphi(e_1) = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6,\\ \varphi(e_2) = \alpha_1 e_2 + 4\alpha_2 e_4 + \beta_5 e_5 + \beta_6 e_6,\\ \varphi(e_3) = \alpha_1 e_3 + 2\alpha_2 e_5 + \frac{1}{2}(\beta_5-\alpha_3)e_6,\\ \varphi(e_4) = \alpha_1 e_4 + \frac{3}{2}\alpha_2 e_6,\\ \varphi(e_5) = \alpha_1 e_5,\ \varphi(e_6) = \alpha_1 e_6,\\ \varphi(e_7) = -\alpha_2 e_3 - \alpha_3 e_4 - \alpha_4 e_5 + \gamma_6 e_6 + \alpha_1 e_7,\end{cases}$$

- *for $n \ge 8$:*

$$\begin{cases}\varphi(e_1) = \alpha_1 e_1 + \sum\limits_{i=3}^{n-1}\alpha_i e_i,\\ \varphi(e_2) = \alpha_1 e_2 + \sum\limits_{i=5}^{n-3}\alpha_{i-2}e_i + \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1},\\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}(\beta_{n-2}-\alpha_{n-4})e_{n-1},\\ \varphi(e_i) = \alpha_1 e_i,\ 4 \le i \le n-1,\\ \varphi(e_n) = -\sum\limits_{i=4}^{n-2}\alpha_{i-1}e_i + \gamma_{n-1}e_{n-1} + \alpha_1 e_n.\end{cases}$$

Proof. The algebra $g^3_{(n,1)}$ has generators e_1, e_2 and e_n. Then, we have the following

$$\varphi(e_1)=\sum_{i=1}^{n}\alpha_ie_i,\ \varphi(e_2)=\sum_{i=1}^{n}\beta_ie_i,\ \varphi(e_n)=\sum_{i=1}^{n}\gamma_ie_i.$$

Now, consider the condition of $\frac{1}{2}$-derivation for the elements e_2 and e_n we obtain

$$\varphi(e_3)=\varphi([e_1,e_2])=\tfrac{1}{2}([\varphi(e_1),e_2]+[e_1,\varphi(e_2)])=\tfrac{1}{2}([\sum_{i=1}^{n}\alpha_ie_i,e_2]+[e_1,\sum_{i=1}^{n}\beta_ie_i])$$

$$=\tfrac{1}{2}\Big(\alpha_1e_3-\sum_{i=3}^{n-4}\alpha_ie_{i+3}-\alpha_ne_4+\sum_{i=2}^{n-2}\beta_ie_{i+1}\Big)$$

$$=\tfrac{1}{2}\Big((\alpha_1+\beta_2)e_3+(\beta_3-\alpha_n)e_4+\beta_4e_5+\sum_{i=6}^{n-1}(\beta_{i-1}-\alpha_{i-3})e_i\Big).$$

We can prove the following equalities by induction:

$$\varphi(e_i)=\tfrac{1}{2^{i-2}}\Big((2^{i-2}-1)\alpha_1+\beta_2\Big)e_i+\tfrac{1}{2^{i-2}}\Big(\beta_3-(2^{i-2}-1)\alpha_n\Big)e_{i+1}$$

$$+\tfrac{1}{2^{i-2}}\Big((2^{i-2}-2)\alpha_2+\beta_4\Big)e_{i+2}+\tfrac{1}{2^{i-2}}\sum_{t=i+3}^{n-1}(\beta_{t-i+2}-\alpha_{t-i})e_t$$

for $3\le i\le n-4$ and

$$\varphi(e_{n-3})=\tfrac{1}{2^{n-5}}\Big((2^{n-5}-1)\alpha_1+\beta_2\Big)e_{n-3}+\tfrac{1}{2^{n-5}}\Big(\beta_3-(2^{n-5}-1)\alpha_n\Big)e_{n-2}$$

$$+\tfrac{1}{2^{n-5}}\Big((2^{n-5}-2)\alpha_2+\beta_4\Big)e_{n-1},$$

$$\varphi(e_{n-2})=\tfrac{1}{2^{n-4}}\Big((2^{n-4}-1)\alpha_1+\beta_2\Big)e_{n-2}+\tfrac{1}{2^{n-4}}\Big(\beta_3-(2^{n-4}-1)\alpha_n\Big)e_{n-1},$$

$$\varphi(e_{n-1})=\tfrac{1}{2^{n-3}}\Big((2^{n-3}-1)\alpha_1+\beta_2\Big)e_{n-1}.$$

From

$$0=\varphi([e_1,e_n])=\frac{1}{2}([\varphi(e_1),e_n]+[e_1,\varphi(e_n)])=\frac{1}{2}\Big([\sum_{k=1}^{n}\alpha_ke_k,e_n]+[e_1,\sum_{k=1}^{n}\gamma_ke_k]\Big)$$

$$=\frac{1}{2}\Big(\sum_{k=2}^{n-3}\alpha_ke_{k+2}+\sum_{k=2}^{n-2}\gamma_ke_{k+1}\Big)=\frac{1}{2}\Big(\sum_{k=4}^{n-1}\alpha_{k-2}e_k+\sum_{k=3}^{n-1}\gamma_{k-1}e_k\Big),$$

we obtain that

$$\gamma_2=0,\ \gamma_i=-\alpha_{i-1},\ 3\le i\le n-2.$$

Now, we consider the condition of $\frac{1}{2}$-derivation for the elements e_2,e_n :

$$\varphi([e_2,e_n])=\tfrac{1}{2}([\varphi(e_2),e_n]+[e_2,\varphi(e_n)])=\tfrac{1}{2}\Big([\sum_{i=1}^{n}\beta_ie_i,e_n]+[e_2,\sum_{i=1}^{n}\gamma_ie_i]\Big)$$

$$= \frac{1}{2}\Big(\sum_{i=2}^{n-3}\beta_i e_{i+2} - \gamma_1 e_3 + \sum_{i=3}^{n-4}\gamma_i e_{i+3} + \gamma_n e_4\Big)$$

$$= \frac{1}{2}\Big(- \gamma_1 e_3 + (\beta_2 + \gamma_n)e_4 + \beta_3 e_5 + \sum_{i=6}^{n-1}(\beta_{i-2} - \gamma_{i-3})e_i\Big).$$

Thus, we have

$$\varphi([e_2, e_n]) = \varphi(e_4) = \tfrac{1}{4}(3\alpha_1 + \beta_2)e_4 + \tfrac{1}{4}(\beta_3 - 3\alpha_n)e_5 + \tfrac{1}{4}(2\alpha_2 + \beta_4)e_6 + \tfrac{1}{4}\sum_{t=7}^{n-1}(\beta_{t-2} - \alpha_{t-4})e_t.$$

Comparing the coefficients at the basis elements, we obtain

$$\gamma_1 = 0,\ 2\gamma_n = 3\alpha_1 - \beta_2,\ \beta_3 = -3\alpha_n,\ \beta_4 = 4\alpha_2,\ \beta_i = \alpha_{i-2},\ 5 \le i \le n-3.$$

By using the property of $\frac{1}{2}$-derivation for the products $[e_2, e_3] = e_6$ and $[e_3, e_n] = e_5$, we have

$$\beta_1 = 0,\ \beta_n = 0,\ \beta_2 = \alpha_1;$$

$$\beta_3 = \alpha_n = 0, \text{ and for } n \ge 8 \text{ we have } \beta_4 = \alpha_2 = 0.$$

This completes the proof of the theorem. □

The following theorems, we present an analogous descriptions of $\frac{1}{2}$-derivation for quasi-filiform Lie algebras of maximum length g_7^1, g_9^2 and g_{11}^3.

⟨halfderiv4⟩ **Theorem 11.** *Any $\frac{1}{2}$-derivations of the algebra g_7^1 has the form*

$$\begin{cases}
\varphi(e_1) = \alpha_1 e_1 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7,\\
\varphi(e_2) = \alpha_1 e_2 - \frac{1}{3}\alpha_3 e_4 + \beta_5 e_5 + \beta_6 e_6 + \beta_7 e_7,\\
\varphi(e_3) = \alpha_1 e_3 - \frac{2}{3}\alpha_3 e_5 + \frac{1}{2}(\beta_5 - \alpha_4)e_6 - \frac{1}{2}\alpha_5 e_7,\\
\varphi(e_4) = \alpha_1 e_4 - \frac{1}{3}\alpha_3 e_6 + \frac{1}{2}\alpha_4 e_7,\\
\varphi(e_5) = \alpha_1 e_5 - \frac{1}{2}\alpha_3 e_7,\\
\varphi(e_6) = \alpha_1 e_6,\\
\varphi(e_7) = \alpha_1 e_7.
\end{cases}$$

alfderiv12⟩? **Theorem 12.** *Any $\frac{1}{2}$-derivations of the algebra g_9^2 has the form*

$$\begin{cases}
\varphi(e_1) = \alpha_1 e_1 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9,\\
\varphi(e_2) = \alpha_1 e_2 + \frac{1}{3}\alpha_5 e_6 + \beta_7 e_7 + \beta_8 e_8 + \beta_9 e_9,\\
\varphi(e_3) = \alpha_1 e_3 - \frac{4}{3}\alpha_5 e_7 + \frac{1}{2}(\beta_7 - 5\alpha_6)e_8 - \frac{1}{2}\alpha_7 e_9,\\
\varphi(e_4) = \alpha_1 e_4 + \frac{1}{3}\alpha_5 e_8 + \frac{1}{2}\alpha_6 e_9,\\
\varphi(e_5) = \alpha_1 e_5 - \frac{1}{2}\alpha_5 e_9,\\
\varphi(e_i) = \alpha_1 e_i,\ \ 6 \le i \le 9.
\end{cases}$$

2halfderiv3⟩ **Theorem 13.** *Any $\frac{1}{2}$-derivations of the algebra g_{11}^3 has the form*

$$\begin{cases} \varphi(e_1) = \alpha_1 e_1 + \alpha_6 e_6 + \alpha_7 e_7 + \alpha_8 e_8 + \alpha_9 e_9 + \alpha_{10} e_{10} + \alpha_{11} e_{11}, \\ \varphi(e_2) = \alpha_1 e_2 - \alpha_6 e_7 - \alpha_7 e_8 + \beta_9 e_9 + \beta_{10} e_{10} + \beta_{11} e_{11}, \\ \varphi(e_3) = \alpha_1 e_3 + \frac{1}{2}\beta_9 e_{10} - \frac{1}{2}\alpha_9 e_{11}, \\ \varphi(e_4) = \alpha_1 e_4 + \frac{1}{2}\alpha_7 e_{10} + \frac{1}{2}\alpha_8 e_{11}, \\ \varphi(e_5) = \alpha_1 e_5 - \frac{1}{2}\alpha_6 e_{10} - \frac{1}{2}\alpha_7 e_{11}, \\ \varphi(e_6) = \alpha_1 e_6 + \frac{1}{2}\alpha_6 e_{11}, \\ \varphi(e_i) = \alpha_1 e_i, \ \ 7 \le i \le 11. \end{cases}$$

3. Transposed Poisson structure for quasi-filiform Lie algebras of maximum length

In this section, we describe all transposed Poisson algebra structures on quasi-filiform Lie algebras of maximum length.

Theorem 14. *Let $(g^1_{(n,1)}, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on the Lie algebra $g^1_{(n,1)}$. Then, the multiplication of $(g^1_{(n,1)}, \cdot)$ has the following form:*

$$\mathbf{TP}_1(g^1_{(n,1)}): \begin{cases} e_1 \cdot e_1 = \sum\limits_{j=4}^{n} \alpha_j e_j, \ e_1 \cdot e_2 = \beta_1 e_{n-2} + \beta_2 e_{n-1} + \beta_3 e_n, \\ e_1 \cdot e_3 = \frac{1}{2}\beta_1 e_{n-1} - \frac{1}{2}\alpha_{n-2} e_n, \ e_1 \cdot e_j = \frac{(-1)^j}{2}\alpha_{n-j+1} e_n, \ 4 \le j \le n-3, \\ e_2 \cdot e_2 = \beta_4 e_{n-2} + \beta_5 e_{n-1} + \beta_6 e_n, \ e_2 \cdot e_3 = \frac{1}{2}\beta_4 e_{n-1} - \frac{1}{2}\beta_1 e_n; \end{cases}$$

$$\mathbf{TP}_2(g^1_{(n,1)}): \begin{cases} e_1 \cdot e_1 = e_3 + \sum\limits_{j=4}^{n} \alpha_j e_j, \ e_1 \cdot e_2 = \beta_1 e_{n-2} + \beta_2 e_{n-1} + \beta_3 e_n, \\ e_1 \cdot e_3 = \frac{1}{2}\beta_1 e_{n-1} - \frac{1}{2}\alpha_{n-2} e_n, \ e_1 \cdot e_j = \frac{(-1)^j}{2}\alpha_{n-j+1} e_n, \ 4 \le j \le n-3, \\ e_1 \cdot e_{n-2} = -\frac{1}{2} e_n, \ e_2 \cdot e_2 = \beta_4 e_{n-1} + \beta_5 e_n, \ e_2 \cdot e_3 = -\frac{1}{2}\beta_1 e_n; \end{cases}$$

$$\mathbf{TP}_3(g^1_{(n,1)}): \begin{cases} e_1 \cdot e_1 = e_2 + \sum\limits_{j=3}^{n} \alpha_j e_j, \ e_1 \cdot e_2 = 2\beta_1 e_{n-1} + \beta_2 e_n, \\ e_1 \cdot e_3 = -\frac{1}{2}\alpha_{n-2} e_n, \ e_1 \cdot e_j = \frac{(-1)^j}{2}\alpha_{n-j+1} e_n, \ 4 \le j \le n-2, \\ e_1 \cdot e_{n-1} = \frac{1}{2} e_n, \ e_2 \cdot e_2 = \beta_1 e_n; \end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and the remaining products are equal to zero.

Proof. Let $(g^1_{(n,1)}, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on the Lie algebra $g^1_{(n,1)}$. Then, for any element $x \in g^1_{(n,1)}$, the operator of multiplication $\varphi_x(y) = x \cdot y$ is a $\frac{1}{2}$-derivation. Hence, by Theorem 8, we derive the following relations for $1 \le i \le n$:

$$\begin{cases}\varphi_{e_i}(e_1)=\sum\limits_{j=1}^{n}\alpha_{i,j}e_j,\\ \varphi_{e_i}(e_2)=\alpha_{i,1}e_2+\beta_{i,n-2}e_{n-2}+\beta_{i,n-1}e_{n-1}+\beta_{i,n}e_n,\\ \varphi_{e_i}(e_3)=\alpha_{i,1}e_3+\frac{1}{2}\beta_{i,n-2}e_{n-1}-\frac{1}{2}\alpha_{i,n-2}e_n,\\ \varphi_{e_i}(e_j)=\alpha_{i,1}e_j+\frac{(-1)^j}{2}\alpha_{i,n-j+1}e_n,\ 4\leq j\leq n-1,\\ \varphi_{e_i}(e_n)=\alpha_{i,1}e_n.\end{cases}$$

By considering equality $\varphi_{e_i}(e_j)=e_i\cdot e_j=e_j\cdot e_i=\varphi_{e_j}(e_i)$, we have following restrictions:

$$\begin{array}{lll}
\{e_1,e_2\}, & \Rightarrow & \alpha_{2,1}=0,\ \alpha_{2,2}=\alpha_{1,1},\ \alpha_{2,t}=0,\ 3\leq t\leq n-3,\\
 & & \alpha_{2,n-2}=\beta_{1,n-2},\ \alpha_{2,n-1}=\beta_{1,n-1},\ \alpha_{2,n}=\beta_{1,n},\\
\{e_1,e_3\}, & \Rightarrow & \alpha_{3,1}=\alpha_{3,2}=0,\ \alpha_{3,3}=\alpha_{1,1},\ \alpha_{3,t}=0,\ 4\leq t\leq n-2,\\
 & & \alpha_{3,n-1}=\frac{1}{2}\beta_{1,n-2},\ \alpha_{3,n}=-\frac{1}{2}\beta_{1,n},\\
\{e_1,e_j\}, & \Rightarrow & \alpha_{j,t}=0,\ 1\leq t\leq j-1,\ \alpha_{j,j}=\alpha_{1,1},\\
4\leq j\leq n-1, & & \alpha_{j,n}=\frac{(-1)^j}{2}\alpha_{1,n-j+1},\ \alpha_{j,t}=0,\ j+1\leq t\leq n-1,\\
\{e_1,e_n\}, & \Rightarrow & \alpha_{n,t}=0,\ 1\leq t\leq n-1,\ \alpha_{n,n}=\alpha_{1,1},\\
\{e_2,e_3\}, & \Rightarrow & \beta_{3,n-2}=0,\ \beta_{3,n-1}=\frac{1}{2}\beta_{2,n-2},\ \beta_{3,n}=-\frac{1}{2}\beta_{1,n-2},\\
\{e_2,e_j\}, & \Rightarrow & \beta_{j,n-2}=\beta_{j,n-1}=\beta_{j,n}=0,\\
4\leq j\leq n-1, & & \\
\{e_2,e_n\}, & \Rightarrow & \beta_{n,n-2}=\beta_{n,n-1}=\beta_{n,n}=0,\\
\{e_3,e_j\}, & \Rightarrow & \alpha_{3,t}=0,\ 2\leq t\leq n-3,\\
4\leq j\leq n-1. & &
\end{array}$$

Thus, we have the following table of commutative multiplications of the transposed Poisson algebra structure $(g^1_{(n,1)},,\cdot,[-,-])$:

$$\begin{cases}e_1\cdot e_1=\sum\limits_{j=2}^{n}\alpha_je_j,\\ e_1\cdot e_2=\beta_1e_{n-2}+\beta_2e_{n-1}+\beta_3e_n,\\ e_1\cdot e_3=\frac{1}{2}\beta_1e_{n-1}-\frac{1}{2}\alpha_{n-2}e_n,\\ e_1\cdot e_j=\frac{(-1)^j}{2}\alpha_{n-j+1}e_n,\ 4\leq j\leq n-1,\\ e_2\cdot e_2=\beta_4e_{n-2}+\beta_5e_{n-1}+\beta_6e_n,\\ e_2\cdot e_3=\frac{1}{2}\beta_4e_{n-1}-\frac{1}{2}\beta_1e_n.\end{cases} \tag{5}$$

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Considering the identity $x\cdot(y\cdot z)=(x\cdot y)\cdot z$ in (5), we obtain the following restrictions on structure constants:

$\{e_1,e_2,e_3\}, \quad \Rightarrow \quad \alpha_2\beta_4=0,$
$\{e_1,e_2,e_2\}, \quad \Rightarrow \quad \alpha_3\beta_4-\alpha_2\beta_5=0,$
$\{e_1,e_1,e_3\}, \quad \Rightarrow \quad \alpha_2\beta_1=0,$
$\{e_1,e_1,e_2\}, \quad \Rightarrow \quad 2\alpha_2\beta_5+\alpha_3\beta_4=0,\ \alpha_2(2\beta_6-\beta_2)=0,$
$\{e_1,e_1,e_i\}, \quad \Rightarrow \quad \alpha_2\beta_4=0,\ \alpha_3\beta_4=0,\ \alpha_2\beta_5=0,\ \alpha_2\beta_1=0,\ \alpha_2(2\beta_6-\beta_2)=0,$
$4\leq i\leq n-1.$

Now we consider the general change of basis:

$$e_1'=\sum_{j=1}^{n}A_je_j,\ e_2'=\sum_{j=1}^{n}B_je_j,\ e_{i+1}'=[e_1',e_i'],\ 2\leq i\leq n-2,\ e_n'=[e_2',e_{n-2}'],$$

where $B_1=0$ and $A_1B_2\neq 0$. Then, from the multiplication $e_1\cdot e_1=\sum\limits_{j=2}^{n}\alpha_je_j$, we discover that the structure constant α_2 changes as follows:

$$\alpha_2'=\frac{A_1^2}{B_2}\alpha_2.$$

Thus, we have the following cases.

(1) If $\alpha_2=0$. Again, by using a change of basis and from the product $e_1\cdot e_1=\sum\limits_{j=3}^{n}\alpha_je_j$, we obtain the following relation:

$$\alpha_3'=\frac{A_1}{B_2}\alpha_3.$$

Now, we consider the following subcases.

(a) If $\alpha_3=0$. Then, we have the algebra $\mathbf{TP}_1(g^1_{(n,1)})$.

(b) If $\alpha_3\neq 0$, then, we have $\beta_4=0$, and via transformation

$$\phi(e_1)=e_1,\ \phi(e_i)=\alpha_3e_i,\ 2\leq i\leq n-1,\ \phi(e_n)=\alpha_3^2e_n,$$

we get the algebra $\mathbf{TP}_2(g^1_{(n,1)})$.

(2) If $\alpha_2\neq 0$, then we derive $\beta_4=\beta_5=\beta_1=0,\ \beta_2=2\beta_6$, and via the transformation

$$\phi(e_1)=e_1,\ \phi(e_i)=\alpha_2e_i,\ 2\leq i\leq n-1,\ \phi(e_n)=\alpha_2^2e_n,$$

we obtain the algebra $\mathbf{TP}_3(g^1_{(n,1)})$.

□

Theorem 15. *Let $(g^2_{(n,1)},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^2_{(n,1)}$. Then, the multiplication of $(g^2_{(n,1)},\cdot)$ has the following form:*

$$\mathbf{TP}_1(g^2_{(n,1)}):\ \begin{cases} e_1\cdot e_1=e_2+\sum\limits_{i=3}^{n-1}\alpha_ie_i,\ e_1\cdot e_2=2\alpha_1e_{n-2}+\alpha_ne_{n-1},\ e_1\cdot e_3=\alpha_1e_{n-1},\\ e_1\cdot e_n=-e_3-\sum\limits_{i=4}^{n-2}\alpha_{i-1}e_i+\alpha_{n+1}e_{n-1},\ e_2\cdot e_n=-\alpha_1\alpha_3e_{n-1},\\ e_n\cdot e_n=\sum\limits_{i=4}^{n-2}\alpha_{i-1}e_i+\alpha_{n+2}e_{n-1};\end{cases}$$

$$\mathbf{TP}_2(g^2_{(n,1)}):\ \begin{cases} e_1\cdot e_1=e_3+\sum\limits_{i=4}^{n-1}\alpha_ie_i,\ e_1\cdot e_2=\alpha_ne_{n-1},\ e_1\cdot e_n=-e_4-\sum\limits_{i=5}^{n-2}\alpha_{i-1}e_i+\alpha_{n+1}e_{n-1},\\ e_2\cdot e_2=\alpha_{n+3}e_{n-1},\ e_2\cdot e_n=\alpha_{n+4}e_{n-1},\ e_n\cdot e_n=e_4+\sum\limits_{i=5}^{n-2}\alpha_{i-1}e_i+\alpha_{n+5}e_{n-1};\end{cases}$$

$$\mathbf{TP}_3(g^2_{(n,1)}):\ \begin{cases} e_1\cdot e_1=\sum\limits_{i=4}^{n-1}\alpha_ie_i,\ e_1\cdot e_2=2\alpha_1e_{n-2}+\alpha_ne_{n-1},\ e_1\cdot e_3=\alpha_1e_{n-1},\\ e_1\cdot e_n=-\sum\limits_{i=5}^{n-2}\alpha_{i-1}e_i+\alpha_{n+1}e_{n-1},\ e_2\cdot e_2=2\alpha_{n+2}e_{n-2}+\alpha_{n+3}e_{n-1},\\ e_2\cdot e_3=\alpha_{n+2}e_{n-1},\ e_2\cdot e_n=\alpha_{n+4}e_{n-1},\ e_n\cdot e_n=\sum\limits_{i=5}^{n-2}\alpha_{i-1}e_i+\alpha_{n+5}e_{n-1};\end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and the remaining products are equal to zero.

Proof. Let $(g^2_{(n,1)},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^2_{(n,1)}$. Then, for any element $x\in g^2_{(n,1)}$, we have operator of multiplication $\varphi_x(y)=x\cdot y$ is a $\frac{1}{2}$-derivation. Hence, for $1\le i\le n$, we derive

$$\begin{cases}\varphi_{e_i}(e_1)=\sum\limits_{j=1}^{n-1}\alpha_{i,j}e_j,\\ \varphi_{e_i}(e_2)=\alpha_{i,1}e_2+\beta_{i,n-2}e_{n-2}+\beta_{i,n-1}e_{n-1},\\ \varphi_{e_i}(e_3)=\alpha_{i,1}e_3+\frac{1}{2}\beta_{i,n-2}e_{n-1},\\ \varphi_{e_i}(e_j)=\alpha_{i,1}e_j,\ 4\le j\le n-1,\\ \varphi_{e_i}(e_n)=-\sum\limits_{j=3}^{n-2}\alpha_{i,j-1}e_j+\gamma_{i,n-1}e_{n-1}+\alpha_{i,1}e_n.\end{cases}$$

It is known that $\varphi_{e_i}(e_j)=e_i\cdot e_j=e_j\cdot e_i=\varphi_{e_j}(e_i)$. Then for some pair elements, we derive the following restrictions:

$$\begin{array}{lll}\{e_1,e_2\}, & \Rightarrow & \alpha_{2,1}=0,\ \alpha_{2,2}=\alpha_{1,1},\ \alpha_{2,j}=0, 3\le j\le n-3,\\ & & \alpha_{2,n-2}=\beta_{1,n-2},\ \alpha_{2,n-1}=\beta_{1,n-1},\\ \{e_1,e_3\}, & \Rightarrow & \alpha_{3,1}=\alpha_{3,2}=0,\ \alpha_{3,3}=\alpha_{1,1},\ \alpha_{3,j}=0, 4\le j\le n-2,\\ & & \alpha_{3,n-1}=\frac{1}{2}\beta_{1,n-2},\\ \{e_1,e_j\}, & \Rightarrow & \alpha_{j,j}=\alpha_{1,1},\ \alpha_{j,t}=0,\ 1\le t\ne j\le n-1,\\ 4\le j\le n-1, & & \end{array}$$

$$\begin{array}{lll}
\{e_1, e_n\}, & \Rightarrow & \alpha_{n,1} = \alpha_{n,2} = 0,\ \alpha_{n,t} = -\alpha_{1,t-1},\ 3 \leq t \leq n-2, \\
 & & \alpha_{n,n-1} = \gamma_{1,n-1},\ \alpha_{1,1} = 0, \\
\{e_2, e_3\}, & \Rightarrow & \beta_{3,n-2} = 0,\ \beta_{3,n-1} = \frac{1}{2}\beta_{2,n-2}, \\
\{e_2, e_j\}, & \Rightarrow & \beta_{j,n-2} = \beta_{j,n-1} = 0, \\
4 \leq j \leq n-1, & & \\
\{e_2, e_n\}, & \Rightarrow & \beta_{n,n-2} = 0,\ \beta_{n,n-1} = \gamma_{2,n-1}, \\
\{e_i, e_n\}, & \Rightarrow & \gamma_{i,n-1} = 0, \quad 3 \leq i \leq n-1.
\end{array}$$

Thus, we have the following table of commutative multiplications of the transposed Poisson algebra structure defined on $g^2_{(n,1)}$:

$$\begin{cases}
e_1 \cdot e_1 = \sum\limits_{i=2}^{n-1} \alpha_i e_i,\ e_1 \cdot e_2 = 2\alpha_1 e_{n-2} + \alpha_n e_{n-1},\ e_1 \cdot e_3 = \alpha_1 e_{n-1}, \\
e_1 \cdot e_n = -\sum\limits_{i=3}^{n-2} \alpha_{i-1} e_i + \alpha_{n+1} e_{n-1},\ e_2 \cdot e_2 = 2\alpha_{n+2} e_{n-2} + \alpha_{n+3} e_{n-1}, \\
e_2 \cdot e_3 = \alpha_{n+2} e_{n-1},\ e_2 \cdot e_n = \alpha_{n+4} e_{n-1},\ e_n \cdot e_n = \sum\limits_{i=4}^{n-2} \alpha_{i-1} e_i + \alpha_{n+5} e_{n-1}.
\end{cases} \tag{6}$$ 2klas

Considering the associative identity $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ in (6), we obtain the following restrictions on structure constants:

$$\begin{array}{lll}
\{e_1, e_1, e_2\}, & \Rightarrow & \alpha_2 \alpha_{n+2} = 0,\ \alpha_2 \alpha_{n+3} + \alpha_3 \alpha_{n+2} = 0, \\
\{e_1, e_1, e_n\}, & \Rightarrow & \alpha_1 \alpha_3 + \alpha_2 \alpha_{n+4} = 0.
\end{array}$$

Similarly, by using the multiplication of Lie algebra $g^2_{(n,1)}$ we consider the general basis change:

$$e'_1 = \sum_{t=1}^{n} A_t e_t,\ e'_2 = \sum_{t=1}^{n} B_t e_t,\ e'_n = \sum_{t=1}^{n} C_t e_t.$$

Then the product $e'_1 \cdot e'_1 = \sum\limits_{i=2}^{n-1} \alpha'_i e'_i$ gives

$$\alpha'_2 = \frac{A_1^2}{B_2} \alpha_2.$$

We have the following cases.

(1) Let $\alpha_2 \neq 0$. Then by choosing $B_2 = A_1^2 \alpha_2$, we put $\alpha'_2 = 1$. In this case we obtain the algebra $\mathbf{TP}_1(g^2_{(n,1)})$.

(2) Let $\alpha_2 = 0$. Then from above restrictions we have $\alpha_3 \alpha_{n+2} = \alpha_1 \alpha_3 = 0$. If we apply a general change of basis, then we have

$$\alpha'_3 = \frac{A_1}{B_2} \alpha_3.$$

  (a) If $\alpha_3 \neq 0$, then $\alpha'_3 = 1$ and we have the algebra $\mathbf{TP}_2(g^2_{(n,1)})$.

  (b) If $\alpha_3 = 0$, then we have the algebra $\mathbf{TP}_3(g^2_{(n,1)})$.

□

Theorem 16. *Let $(g^3_{(7,1)}, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^3_{(7,1)}$. Then, the multiplication of $(g^3_{(7,1)}, \cdot)$ has the following form:*

$$\mathbf{TP}_1(g^3_{(7,1)}): \begin{cases} e_1 \cdot e_1 = \alpha_4 e_4 + \alpha_5 e_5,\ e_1 \cdot e_2 = 2\alpha_6 e_5 + \alpha_7 e_6,\ e_1 \cdot e_3 = \alpha_6 e_6, \\ e_1 \cdot e_7 = -\alpha_4 e_5 + \alpha_8 e_6,\ e_2 \cdot e_2 = 2\alpha_1 e_5 + \alpha_9 e_6,\ e_2 \cdot e_3 = \alpha_1 e_6, \\ e_2 \cdot e_7 = \alpha_{10} e_6,\ e_7 \cdot e_7 = \alpha_{11} e_6; \end{cases}$$

$$\mathbf{TP}_2(g^3_{(7,1)}): \begin{cases} e_1 \cdot e_1 = 2e_3 + \alpha_4 e_4 + \alpha_5 e_5,\ e_1 \cdot e_2 = 2\alpha_6 e_5 + \alpha_7 e_6,\ e_1 \cdot e_3 = (\alpha_6 - 1)e_6, \\ e_1 \cdot e_7 = -2e_4 - \alpha_4 e_5 + \alpha_8 e_6,\ e_2 \cdot e_2 = \alpha_9 e_6,\ e_2 \cdot e_7 = \alpha_{10} e_6, \\ e_7 \cdot e_7 = 2e_5 + \alpha_{11} e_6; \end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and remaining products are equal to zero.

Proof. Let $(g^3_{(7,1)}, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^3_{(7,1)}$. Then for any element $x \in g^3_{(7,1)}$, the operator of multiplication $\varphi_x(y) = x \cdot y$ is a $\frac{1}{2}$-derivation. Hence, for $1 \le i \le 7$ by Theorem 10 we derive the following:

$$\begin{cases} \varphi_{e_i}(e_1) = \alpha_{i,1}e_1 + \alpha_{i,2}e_2 + \alpha_{i,3}e_3 + \alpha_{i,4}e_4 + \alpha_{i,5}e_5 + \alpha_{i,6}e_6, \\ \varphi_{e_i}(e_2) = \alpha_{i,1}e_2 + 4\alpha_{i,2}e_4 + \beta_{i,5}e_5 + \beta_{i,6}e_6, \\ \varphi_{e_i}(e_3) = \alpha_{i,1}e_3 + 2\alpha_{i,2}e_5 + \frac{1}{2}(\beta_{i,5} - \alpha_{i,3})e_6, \\ \varphi_{e_i}(e_4) = \alpha_{i,1}e_4 + \frac{3}{2}\alpha_{i,2}e_6, \\ \varphi_{e_i}(e_5) = \alpha_{i,1}e_5, \\ \varphi_{e_i}(e_6) = \alpha_{i,1}e_6, \\ \varphi_{e_i}(e_7) = -\alpha_{i,2}e_3 - \alpha_{i,3}e_4 - \alpha_{i,4}e_5 + \gamma_{i,6}e_6 + \alpha_{i,1}e_7, \end{cases}$$

Considering the property $\varphi_{e_i}(e_j) = e_i \cdot e_j = e_j \cdot e_i = \varphi_{e_j}(e_i)$, we obtain the following restrictions:

$$\begin{array}{lll}
\{e_1,e_2\}, & \Rightarrow & \alpha_{2,1}=0,\ \alpha_{2,2}=\alpha_{1,1},\ \alpha_{2,3}=0,\ \alpha_{2,4}=4\alpha_{1,2},\ \alpha_{2,5}=\beta_{1,5},\ \alpha_{2,6}=\beta_{1,6},\\
\{e_1,e_3\}, & \Rightarrow & \alpha_{3,1}=\alpha_{3,2}=0,\ \alpha_{3,3}=\alpha_{1,1},\ \alpha_{3,4}=-2\alpha_{1,6},\ \alpha_{3,5}=2\alpha_{1,2},\ \alpha_{3,6}=\frac{1}{2}(\beta_{1,5}-\alpha_{1,3}),\\
\{e_1,e_4\}, & \Rightarrow & \alpha_{4,1}=\alpha_{4,2}=\alpha_{4,3}=0,\ \alpha_{4,4}=\alpha_{1,1},\ \alpha_{4,5}=0,\ \alpha_{4,6}=\frac{3}{2}\alpha_{1,2},\\
\{e_1,e_5\}, & \Rightarrow & \alpha_{5,1}=\alpha_{5,2}=\alpha_{5,3}=\alpha_{5,4}=0,\ \alpha_{5,5}=\alpha_{1,1},\ \alpha_{5,6}=0,\\
\{e_1,e_6\}, & \Rightarrow & \alpha_{6,1}=\alpha_{6,2}=\alpha_{6,3}=\alpha_{6,4}=\alpha_{6,5}=0,\ \alpha_{6,6}=\alpha_{1,1},\\
\{e_1,e_7\}, & \Rightarrow & \alpha_{1,1}=0,\ \alpha_{7,1}=\alpha_{7,2}=0,\ \alpha_{7,3}=-\alpha_{1,2},\ \alpha_{7,4}=-\alpha_{1,3},\ \alpha_{7,5}=-\alpha_{1,4},\ \alpha_{7,6}=\gamma_{1,6},\\
\{e_2,e_3\}, & \Rightarrow & \beta_{3,5}=0,\ \beta_{3,6}=\frac{1}{2}\beta_{2,5},\\
\{e_2,e_4\}, & \Rightarrow & \beta_{4,5}=\beta_{4,6}=0,\\
\{e_2,e_5\}, & \Rightarrow & \beta_{5,5}=\beta_{5,6}=0,\\
\{e_2,e_6\}, & \Rightarrow & \beta_{6,5}=\beta_{6,6}=0,\\
\{e_2,e_7\}, & \Rightarrow & \beta_{7,5}=-4\alpha_{1,2},\ \beta_{7,6}=\gamma_{2,6},\\
\{e_3,e_7\}, & \Rightarrow & \alpha_{1,6}=0,\ \gamma_{3,6}=-\frac{3}{2}\alpha_{1,2},\\
\{e_4,e_7\}, & \Rightarrow & \gamma_{4,6}=0,\\
\{e_5,e_7\}, & \Rightarrow & \gamma_{5,6}=0,\\
\{e_6,e_7\}, & \Rightarrow & \gamma_{6,6}=0.
\end{array}$$

Thus, we have the following table of commutative multiplications of the transposed Poisson algebra structure $(g^3_{(7,1)},,\cdot,[-,-])$:

$$\begin{cases}
e_1\cdot e_1=2\alpha_2e_2+2\alpha_3e_3+\alpha_4e_4+\alpha_5e_5,\ e_1\cdot e_2=8\alpha_2e_4+2\alpha_6e_5+\alpha_7e_6,\\
e_1\cdot e_3=4\alpha_2e_5+(\alpha_6-\alpha_3)e_6,\ e_1\cdot e_4=3\alpha_2e_6,\ e_1\cdot e_7=-2\alpha_2e_3-2\alpha_3e_4-\alpha_4e_5+\alpha_8e_6,\\
e_2\cdot e_2=2\alpha_1e_5+\alpha_9e_6,\ e_2\cdot e_3=\alpha_1e_6,\ e_2\cdot e_7=-8\alpha_2e_5+\alpha_{10}e_6,\\
e_3\cdot e_7=-3\alpha_2e_6,\ e_7\cdot e_7=2\alpha_2e_4+2\alpha_3e_5+\alpha_{11}e_6.
\end{cases}$$

Considering the associative identity for the triples $\{e_1,e_1,e_7\}$ and $\{e_1,e_1,e_2\}$, we obtain the following restrictions, respectively:

$$\alpha_2=0,\ \alpha_1\alpha_3=0.$$

Using multiplication of Lie algebra $g^3_{(7,1)}$ and multiplication $e_1\cdot e_1=2\alpha_3e_3+\alpha_4e_4+\alpha_5e_5$, and considering a general change of basis, we can show that α_3 is an invariant.

If $\alpha_3=0$. Then, we have the algebra $\mathbf{TP}_1(g^3_{(7,1)})$.

If $\alpha_3\neq 0$. Then, we obtain $\alpha_1=0$. By choosing change of basis, we can assume that $\alpha_3=1$ and obtain the algebra $\mathbf{TP}_2(g^3_{(7,1)})$. □

?⟨thm3klas⟩? **Theorem 17.** *Let $(g^3_{(n,1)},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^3_{(n,1)}$ and $n\geq 8$. Then, the multiplication of $(g^3_{(n,1)},\cdot)$ has the following form:*

$$\mathbf{TP}(g^3_{(8,1)}):\begin{cases} e_1\cdot e_1=\sum\limits_{t=4}^{7}\alpha_te_t,\ e_1\cdot e_2=\beta_1e_6+\beta_2e_7,\ e_1\cdot e_3=\frac{1}{2}(\beta_1-\alpha_4)e_7,\\ e_1\cdot e_8=-\alpha_4e_5-\alpha_5e_6+\gamma_1e_7,\ e_2\cdot e_2=\beta_3e_6+\beta_4e_7,\\ e_2\cdot e_3=\frac{1}{2}\beta_3e_7,\ e_2\cdot e_8=\gamma_2e_7,\ e_8\cdot e_8=\alpha_4e_6+\gamma_3e_7;\end{cases}$$

$$\mathbf{TP}_1(g^3_{(9,1)}):\begin{cases} e_1\cdot e_1=\sum\limits_{t=4}^{8}\alpha_te_t,\ e_1\cdot e_2=\beta_1e_7+\beta_2e_8,\ e_1\cdot e_3=\frac{1}{2}(\beta_1-\alpha_5)e_8,\\ e_1\cdot e_9=-\sum\limits_{t=5}^{7}\alpha_{t-1}e_t+\gamma_1e_8,\ e_2\cdot e_2=\beta_3e_7+\beta_4e_8,\ e_2\cdot e_3=\frac{1}{2}\beta_3e_8,\\ e_2\cdot e_9=-\alpha_4e_7+\gamma_2e_8,\ e_3\cdot e_9=\frac{1}{2}\alpha_4e_8,\ e_9\cdot e_9=\sum\limits_{t=6}^{7}\alpha_{t-2}e_t+\gamma_3e_8;\end{cases}$$

$$\mathbf{TP}_2(g^3_{(9,1)}):\begin{cases} e_1\cdot e_1=e_3+\sum\limits_{t=5}^{8}\alpha_te_t,\ e_1\cdot e_2=e_5+\beta_1e_7+\beta_2e_8,\\ e_1\cdot e_3=\frac{1}{2}(\beta_1-\alpha_5)e_8,\ e_1\cdot e_9=-e_4-\sum\limits_{t=6}^{7}\alpha_{t-1}e_t+\gamma_1e_8,\\ e_2\cdot e_2=e_7+\beta_4e_8,\ e_2\cdot e_9=-e_6+\gamma_2e_8,\ e_9\cdot e_9=e_5+\alpha_5e_7+\gamma_3e_8;\end{cases}$$

$$\mathbf{TP}_1(g^3_{(n,1)}):\begin{cases} e_1\cdot e_1=\sum\limits_{t=4}^{n-1}\alpha_te_t,\ e_1\cdot e_2=\sum\limits_{t=6}^{n-3}\alpha_{t-2}e_t+\beta_1e_{n-2}+\beta_2e_{n-1},\\ e_1\cdot e_3=\frac{1}{2}(\beta_1-\alpha_{n-4})e_{n-1},\ e_1\cdot e_n=-\sum\limits_{t=5}^{n-2}\alpha_{t-1}e_t+\gamma_1e_{n-1},\\ e_2\cdot e_2=\sum\limits_{t=8}^{n-3}\alpha_{t-4}e_t+\beta_3e_{n-2}+\beta_4e_{n-1},\ e_2\cdot e_3=\frac{1}{2}(\beta_3-\alpha_{n-6})e_{n-1},\\ e_2\cdot e_n=-\sum\limits_{t=7}^{n-2}\alpha_{t-3}e_t+\gamma_2e_{n-1},\ e_3\cdot e_n=\frac{1}{2}\alpha_{n-5}e_{n-1},\ e_n\cdot e_n=\sum\limits_{t=6}^{n-2}\alpha_{t-2}e_t+\gamma_3e_{n-1};\end{cases}$$

$$\mathbf{TP}_2(g^3_{(n,1)}):\begin{cases} e_1\cdot e_1=e_3+\sum\limits_{t=4}^{n-6}\alpha_te_t+\sum\limits_{t=n-4}^{n-1}\alpha_te_t,\ e_1\cdot e_2=e_5+\sum\limits_{t=5}^{n-4}\alpha_{t-2}e_t+\beta_1e_{n-2}+\beta_2e_{n-1},\\ e_1\cdot e_3=\frac{1}{2}(\beta_1-\alpha_{n-4})e_{n-1},\ e_1\cdot e_n=-e_4-\sum\limits_{t=5}^{n-5}\alpha_{t-1}e_t-\sum\limits_{t=n-3}^{n-2}\alpha_{t-1}e_t+\gamma_1e_{n-1},\\ e_2\cdot e_2=e_7+\sum\limits_{t=8}^{n-2}\alpha_{t-4}e_t+\beta_4e_{n-1},\ e_2\cdot e_n=-e_6-\sum\limits_{t=7}^{n-3}\alpha_{t-3}e_t+\gamma_2e_{n-1},\\ e_n\cdot e_n=e_5+\sum\limits_{t=6}^{n-4}\alpha_{t-2}e_t+\alpha_{n-4}e_{n-2}+\gamma_3e_{n-1};\end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and remaining products are equal to zero.

Proof. Let $(g^3_{(n,1)},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on Lie algebra $g^3_{(n,1)}$. Then for any element $x\in g^3_{(n,1)}$ the operator of multiplication $\varphi_x(y)=x\cdot y$ is a $\frac{1}{2}$ -derivation. Hence, for $1\leq i\leq n$ we derive

$$\begin{cases} \varphi_{e_i}(e_1) = \alpha_{i,1}e_1 + \sum\limits_{t=3}^{n-1} \alpha_{i,t}e_t, \\ \varphi_{e_i}(e_2) = \alpha_{i,1}e_2 + \sum\limits_{t=5}^{n-3} \alpha_{i,t-2}e_t + \beta_{i,n-2}e_{n-2} + \beta_{i,n-1}e_{n-1}, \\ \varphi_{e_i}(e_3) = \alpha_{i,1}e_3 + \frac{1}{2}(\beta_{i,n-2} - \alpha_{i,n-4})e_{n-1}, \\ \varphi_{e_i}(e_j) = \alpha_{i,1}e_j, \ 4 \leq j \leq n-1, \\ \varphi_{e_i}(e_n) = -\sum\limits_{t=4}^{n-2} \alpha_{i,t-1}e_t + \gamma_{i,n-1}e_{n-1} + \alpha_{i,1}e_n. \end{cases}$$

It is known that $\varphi_{e_i}(e_j) = e_i \cdot e_j = e_j \cdot e_i = \varphi_{e_j}(e_i)$. Then for some pairs of elements, we derive the following restrictions:

$$\begin{array}{lll} \{e_1, e_2\}, & \Rightarrow & \alpha_{1,1} = \alpha_{2,1} = \alpha_{2,3} = \alpha_{2,4} = 0, \ \alpha_{2,t} = \alpha_{1,t-2}, \ 5 \leq t \leq n-3, \\ & & \alpha_{2,n-2} = \beta_{1,n-2}, \ \alpha_{2,n-1} = \beta_{1,n-1}, \\ \{e_1, e_3\}, & \Rightarrow & \alpha_{3,1} = 0, \ \alpha_{3,t} = 0, \ 3 \leq t \leq n-2, \ \alpha_{3,n-1} = \frac{1}{2}(\beta_{1,n-2} - \alpha_{1,n-4}), \\ \{e_1, e_j\}, & \Rightarrow & \alpha_{j,1} = 0, \ \alpha_{j,t} = 0, \ 3 \leq t \leq n-1, \\ 4 \leq j \leq n-1, & & \\ \{e_1, e_n\}, & \Rightarrow & \alpha_{n,1} = \alpha_{n,3} = 0, \ \alpha_{n,t} = -\alpha_{1,t-1}, \ 4 \leq t \leq n-2, \ \alpha_{n,n-1} = \gamma_{1,n-1}, \\ \{e_2, e_3\}, & \Rightarrow & \beta_{3,n-2} = 0, \ \beta_{3,n-1} = \frac{1}{2}(\beta_{2,n-2} - \alpha_{1,n-6}), \\ \{e_2, e_j\}, & \Rightarrow & \beta_{j,n-2} = \beta_{j,n-1} = 0, \\ 4 \leq j \leq n-1, & & \\ \{e_2, e_n\}, & \Rightarrow & \beta_{n,n-2} = -\alpha_{1,n-5}, \ \beta_{n,n-1} = \gamma_{2,n-1}, \\ \{e_3, e_n\}, & \Rightarrow & \gamma_{3,n-1} = \frac{1}{2}\alpha_{1,n-5}, \\ \{e_i, e_n\}, & \Rightarrow & \gamma_{i,n-1} = 0, \\ 4 \leq i \leq n-1. & & \end{array}$$

So, we have the following table of commutative multiplications of the transposed Poisson algebra structure $(g^3_{(n,1)}, , \cdot, [-,-])$

For $n = 8$:

$$\begin{cases} e_1 \cdot e_1 = \sum\limits_{t=3}^{7} \alpha_t e_t, \ e_1 \cdot e_2 = \alpha_3 e_5 + \beta_1 e_6 + \beta_2 e_7, \ e_1 \cdot e_3 = \frac{1}{2}(\beta_1 - \alpha_4)e_7, \\ e_1 \cdot e_8 = -\sum\limits_{t=4}^{6} \alpha_{t-1}e_t + \gamma_1 e_7, \ e_2 \cdot e_2 = \beta_3 e_6 + \beta_4 e_7, \ e_2 \cdot e_3 = \frac{1}{2}\beta_3 e_7, \\ e_2 \cdot e_8 = -\alpha_3 e_6 + \gamma_2 e_7, \ e_3 \cdot e_8 = \frac{1}{2}\alpha_3 e_7, \ e_8 \cdot e_8 = \sum\limits_{t=5}^{6} \alpha_{t-2}e_t + \gamma_3 e_7; \end{cases}$$

For $n \geq 9$:

$$\begin{cases} e_1 \cdot e_1 = \sum\limits_{t=3}^{n-1} \alpha_t e_t, \ e_1 \cdot e_2 = \sum\limits_{t=5}^{n-3} \alpha_{t-2} e_t + \beta_1 e_{n-2} + \beta_2 e_{n-1}, \ e_1 \cdot e_3 = \frac{1}{2}(\beta_1 - \alpha_{n-4}) e_{n-1}, \\ e_1 \cdot e_n = -\sum\limits_{t=4}^{n-2} \alpha_{t-1} e_t + \gamma_1 e_{n-1}, \ e_2 \cdot e_2 = \sum\limits_{t=7}^{n-3} \alpha_{t-4} e_t + \beta_3 e_{n-2} + \beta_4 e_{n-1}, \\ e_2 \cdot e_3 = \frac{1}{2}(\beta_3 - \alpha_{n-6}) e_{n-1}, \ e_2 \cdot e_n = -\sum\limits_{t=6}^{n-2} \alpha_{t-3} e_t + \gamma_2 e_{n-1}, \\ e_3 \cdot e_n = \frac{1}{2}\alpha_{n-5} e_{n-1}, \ e_n \cdot e_n = \sum\limits_{t=5}^{n-2} \alpha_{t-2} e_t + \gamma_3 e_{n-1}. \end{cases}$$

Considering the associative identity for the triples $\{e_1, e_1, e_2\}$ and $\{e_1, e_1, e_n\}$, we obtain the following restrictions, respectively:

$$\alpha_3(\beta_3 - \alpha_{n-6}) = 0, \ \alpha_3 \alpha_{n-5} = 0. \tag{7}$$ gn3

We have the following cases.

(1) Let $n = 8$. Then we get $\alpha_3 = 0$ and we have the algebra $\mathbf{TP}(g^3_{(8,1)})$.

(2) Let $n \geq 9$. Then by using the multiplication of the Lie algebra $g^3_{(n,1)}$ we consider the general basis change:

$$e_1' = \sum_{t=1}^{n} A_t e_t, \ e_2' = \sum_{t=1}^{n} B_t e_t, \ e_n' = \sum_{t=1}^{n} C_t e_t.$$

The product $e_1' \cdot e_1' = \sum\limits_{i=3}^{n-1} \alpha_i' e_i'$ gives

$$\alpha_3' = \frac{A_1}{B_2} \alpha_3.$$

(a) If $\alpha_3 = 0$. Then for $n = 9$ we derive the algebra $\mathbf{TP}_1(g^3_{(9,1)})$ and for the case when $n \geq 10$ we obtain $\mathbf{TP}_1(g^3_{(n,1)})$.

(b) If $\alpha_3 \neq 0$. Then by choosing $B_2 = A_1 \alpha_3$ we can assume $\alpha_3' = 1$ and from the restrictions (7) we get $\alpha_{n-5} = 0, \beta_3' = \alpha_{n-6}$. In this case for the $n = 9$ we obtain $\mathbf{TP}_2(g^3_{(9,1)})$ and for $n \geq 10$ we have the algebra $\mathbf{TP}_2(g^3_{(n,1)})$.

□

?⟨thm4klas⟩? **Theorem 18.** *Let $(g_7^1, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on Lie algebra g_7^1. Then, the multiplication of $(g_7^1, \cdot)$ has the following form:*

$$\mathbf{TP}_1(g_7^1): \begin{cases} e_1 \cdot e_1 = \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7, \ e_1 \cdot e_2 = \alpha_1 e_5 + \alpha_2 e_6 + \alpha_8 e_7, \\ e_1 \cdot e_3 = \frac{1}{2}(\alpha_1 - \alpha_4) e_6 - \frac{1}{2}\alpha_5 e_7, \ e_1 \cdot e_4 = \frac{1}{2}\alpha_4 e_7, \\ e_2 \cdot e_2 = \alpha_9 e_5 + \alpha_{10} e_6 + \alpha_{11} e_7, \ e_2 \cdot e_3 = \frac{1}{2}\alpha_9 e_6 - \frac{1}{2}\alpha_1 e_7; \end{cases}$$

$$\mathbf{TP}_2(g_7^1):\ \begin{cases} e_1\cdot e_1=6e_3+\alpha_4e_4+\alpha_5e_5+\alpha_6e_6+\alpha_7e_7,\ e_1\cdot e_2=-2e_4+\alpha_1e_5+\alpha_2e_6+\alpha_8e_7,\\ e_1\cdot e_3=-4e_5+\frac{1}{2}(\alpha_1-\alpha_4)e_6-\frac{1}{2}\alpha_5e_7,\ e_1\cdot e_4=-2e_6+\frac{1}{2}\alpha_4e_7,\ e_1\cdot e_5=-3e_7,\\ e_2\cdot e_2=-\frac{2}{3}e_5+\alpha_{10}e_6+\alpha_{11}e_7,\ e_2\cdot e_3=\frac{2}{3}e_6-\frac{1}{2}\alpha_1e_7,\ e_2\cdot e_4=-e_7,\ e_3\cdot e_3=2e_7;\end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and remaining products are equal to zero.

Proof. Let $(g_7^1,\cdot,[-,-])$ be a transposed Poisson algebra structure defined on the Lie algebra $g^3_{(n,1)}$. Then for any element of $x\in g_7^1$, the operator of multiplication $\varphi_x(y)=x\cdot y$ is a $\frac{1}{2}$-derivation. Hence, for $1\leq i\leq 7$ by using Theorem 11 we derive

$$\begin{cases}\varphi_{e_i}(e_1)=\alpha_{i,1}e_1+\alpha_{i,3}e_3+\alpha_{i,4}e_4+\alpha_{i,5}e_5+\alpha_{i,6}e_6+\alpha_{i,7}e_7,\\ \varphi_{e_i}(e_2)=\alpha_{i,1}e_2-\frac{1}{3}\alpha_{i,3}e_4+\beta_{i,5}e_5+\beta_{i,6}e_6+\beta_{i,7}e_7,\\ \varphi_{e_i}(e_3)=\alpha_{i,1}e_3-\frac{2}{3}\alpha_{i,3}e_5+\frac{1}{2}(\beta_{i,5}-\alpha_{i,4})e_6-\frac{1}{2}\alpha_{i,5}e_7,\\ \varphi_{e_i}(e_4)=\alpha_{i,1}e_4-\frac{1}{3}\alpha_{i,3}e_6+\frac{1}{2}\alpha_{i,4}e_7,\ \varphi_{e_i}(e_5)=\alpha_{i,1}e_5-\frac{1}{2}\alpha_{i,3}e_7,\\ \varphi_{e_i}(e_6)=\alpha_{i,1}e_6,\ \varphi_{e_i}(e_7)=\alpha_{i,1}e_7.\end{cases}$$

By considering equality $\varphi_{e_i}(e_j)=e_i\cdot e_j=e_j\cdot e_i=\varphi_{e_j}(e_i)$ for the some elements e_i and e_j, we have the following restrictions:

$$\begin{array}{lll}
\{e_1,e_2\}, & \Rightarrow & \alpha_{1,1}=\alpha_{2,1}=\alpha_{2,3}=0,\ \alpha_{2,4}=-\frac{1}{3}\alpha_{1,3},\ \alpha_{2,5}=\beta_{1,5},\ \alpha_{2,6}=\beta_{1,6},\ \alpha_{2,7}=\beta_{1,7},\\
\{e_1,e_3\}, & \Rightarrow & \alpha_{3,1}=\alpha_{3,3}=\alpha_{3,4}=0,\ \alpha_{3,5}=-\frac{2}{3}\alpha_{1,3},\ \alpha_{3,6}=\frac{1}{2}(\beta_{1,5}-\alpha_{1,4}),\ \alpha_{3,7}=-\frac{1}{2}\alpha_{1,5},\\
\{e_1,e_4\}, & \Rightarrow & \alpha_{4,1}=\alpha_{4,3}=\alpha_{4,4}=\alpha_{4,5}=0,\ \alpha_{4,6}=-\frac{1}{3}\alpha_{1,3},\ \alpha_{4,7}=\frac{1}{2}\alpha_{1,4},\\
\{e_1,e_5\}, & \Rightarrow & \alpha_{5,1}=\alpha_{5,3}=\alpha_{5,4}=\alpha_{5,5}=\alpha_{5,6}=0,\ \alpha_{5,7}=-\frac{1}{2}\alpha_{1,3},\\
\{e_1,e_6\}, & \Rightarrow & \alpha_{6,1}=\alpha_{6,3}=\alpha_{6,4}=\alpha_{6,5}=\alpha_{6,6}=\alpha_{6,7}=0,\\
\{e_1,e_7\}, & \Rightarrow & \alpha_{7,1}=\alpha_{7,3}=\alpha_{7,4}=\alpha_{7,5}=\alpha_{7,6}=\alpha_{7,7}=0,\\
\{e_2,e_3\}, & \Rightarrow & \beta_{3,5}=0,\ \beta_{3,6}=\frac{1}{6}(3\beta_{2,5}+\alpha_{1,3}),\ \beta_{3,7}=-\frac{1}{2}\beta_{1,5},\\
\{e_2,e_4\}, & \Rightarrow & \beta_{4,5}=\beta_{4,6}=0,\ \beta_{4,7}=-\frac{1}{6}\alpha_{1,3},\\
\{e_2,e_5\}, & \Rightarrow & \beta_{5,5}=\beta_{5,6}=\beta_{5,7}=0,\\
\{e_2,e_6\}, & \Rightarrow & \beta_{6,5}=\beta_{6,6}=\beta_{6,7}=0,\\
\{e_2,e_7\}, & \Rightarrow & \beta_{7,5}=\beta_{7,6}=\beta_{7,7}=0.
\end{array}$$

Thus, we have the following table of commutative multiplications of the transposed Poisson algebra structure $(g^1_{(7)},,\cdot,[-,-])$

$$\begin{cases} e_1 \cdot e_1 = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6 + \alpha_7 e_7,\ e_1 \cdot e_2 = -\frac{1}{3}\alpha_3 e_4 + \alpha_1 e_5 + \alpha_2 e_6 + \alpha_8 e_7, \\ e_1 \cdot e_3 = -\frac{2}{3}\alpha_3 e_5 + \frac{1}{2}(\alpha_1 - \alpha_4)e_6 - \frac{1}{2}\alpha_5 e_7,\ e_1 \cdot e_4 = -\frac{1}{3}\alpha_3 e_6 + \frac{1}{2}\alpha_4 e_7,\ e_1 \cdot e_5 = -\frac{1}{2}\alpha_3 e_7, \\ e_2 \cdot e_2 = \alpha_9 e_5 + \alpha_{10} e_6 + \alpha_{11} e_7,\ e_2 \cdot e_3 = \frac{1}{6}(3\alpha_9 + \alpha_3)e_6 - \frac{1}{2}\alpha_1 e_7, \\ e_2 \cdot e_4 = -\frac{1}{6}\alpha_3 e_7,\ e_3 \cdot e_3 = \frac{1}{3}\alpha_3 e_7, \end{cases}$$

The equality $e_1 \cdot (e_1 \cdot e_2) = (e_1 \cdot e_1) \cdot e_2$ gives $\alpha_3(\alpha_3 + 9\alpha_9) = 0$. Using a general change of basis in a similar way as above, we can show that α_3 is an invariant.

(1) If $\alpha_3 = 0$, then we have the algebra $\mathbf{TP}_1(g_7^1)$;
(2) If $\alpha_3 \neq 0$, then we get the algebra $\mathbf{TP}_2(g_7^1)$.

□

?⟨thm4klas⟩? **Theorem 19.** *Let $(g_9^2, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on the Lie algebra g_9^2. Then the multiplication of $(g_9^2, \cdot)$ has the following form:*

$$\mathbf{TP}(g_9^2): \begin{cases} e_1 \cdot e_1 = \alpha_1 e_5 + \alpha_2 e_6 + \alpha_3 e_7 + \alpha_4 e_8 + \alpha_5 e_9,\ e_1 \cdot e_2 = \frac{1}{3}\alpha_1 e_6 + \alpha_6 e_7 + \alpha_7 e_8 + \alpha_8 e_9, \\ e_1 \cdot e_3 = -\frac{4}{3}\alpha_1 e_7 + \frac{1}{2}(\alpha_6 - 5\alpha_2)e_8 - \frac{1}{2}\alpha_3 e_9,\ e_1 \cdot e_4 = \frac{1}{3}\alpha_1 e_8 + \frac{1}{2}\alpha_2 e_9, \\ e_1 \cdot e_5 = -\frac{1}{2}\alpha_1 e_9,\ e_2 \cdot e_2 = \alpha_9 e_7 + \alpha_{10} e_8 + \alpha_{11} e_9, \\ e_2 \cdot e_3 = \frac{1}{6}(3\alpha_9 - 5\alpha_1)e_8 - \frac{1}{2}\alpha_6 e_9,\ e_2 \cdot e_4 = \frac{1}{6}\alpha_1 e_9,\ e_3 \cdot e_3 = \frac{2}{3}\alpha_1 e_9, \end{cases}$$

where the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and remaining products are equal to zero.

Proof. Let $(g_9^2, \cdot, [-,-])$ be a transposed Poisson algebra structure defined on Lie algebra g_9^2. Then for any element $x \in g_9^2$, the operator of multiplication $\varphi_x(y) = x \cdot y$ is a $\frac{1}{2}$-derivation. Hence, for $1 \leq i \leq 9$ we derive

$$\begin{cases} \varphi_{e_i}(e_1) = \alpha_{i,1}e_1 + \alpha_{i,5}e_5 + \alpha_{i,6}e_6 + \alpha_{i,7}e_7 + \alpha_{i,8}e_8 + \alpha_{i,9}e_9, \\ \varphi_{e_i}(e_2) = \alpha_{i,1}e_2 + \frac{1}{3}\alpha_{i,5}e_6 + \beta_{i,7}e_7 + \beta_{i,8}e_8 + \beta_{i,9}e_9, \\ \varphi_{e_i}(e_3) = \alpha_{i,1}e_3 - \frac{4}{3}\alpha_{i,5}e_7 + \frac{1}{2}(\beta_{i,7} - 5\alpha_{i,6})e_8 - \frac{1}{2}\alpha_{i,7}e_9, \\ \varphi_{e_i}(e_4) = \alpha_{i,1}e_4 + \frac{1}{3}\alpha_{i,5}e_8 + \frac{1}{2}\alpha_{i,6}e_9,\ \varphi_{e_i}(e_5) = \alpha_{i,1}e_5 - \frac{1}{2}\alpha_{i,5}e_9,\ \varphi_{e_i}(e_j) = \alpha_{i,1}e_j,\ \ 6 \leq j \leq 9. \end{cases}$$

It is known that $\varphi_{e_i}(e_j) = e_i \cdot e_j = e_j \cdot e_i = \varphi_{e_j}(e_i)$. Then for some pairs of elements, we derive the following restrictions:

$$\begin{array}{lll}
\{e_1,e_2\}, & \Rightarrow & \alpha_{1,1}=\alpha_{2,1}=\alpha_{2,5}=0,\ \alpha_{2,6}=\frac{1}{3}\alpha_{1,5},\ \alpha_{2,7}=\beta_{1,7},\ \alpha_{2,8}=\beta_{1,8},\ \alpha_{2,9}=\beta_{1,9},\\
\{e_1,e_3\}, & \Rightarrow & \alpha_{3,1}=\alpha_{3,5}=\alpha_{3,6}=0,\ \alpha_{3,7}=-\frac{4}{3}\alpha_{1,5},\ \alpha_{3,8}=\frac{1}{2}(\beta_{1,7}-5\alpha_{1,6}),\ \alpha_{3,9}=-\frac{1}{2}\alpha_{1,7},\\
\{e_1,e_4\}, & \Rightarrow & \alpha_{4,1}=\alpha_{4,5}=\alpha_{4,6}=\alpha_{4,7}=0,\ \alpha_{4,8}=\frac{1}{3}\alpha_{1,5},\ \alpha_{4,9}=\frac{1}{2}\alpha_{1,6},\\
\{e_1,e_5\}, & \Rightarrow & \alpha_{5,1}=\alpha_{5,5}=\alpha_{5,6}=\alpha_{5,7}=\alpha_{5,8}=0,\ \alpha_{5,9}=-\frac{1}{2}\alpha_{1,5},\\
\{e_1,e_j\}, & \Rightarrow & \alpha_{j,1}=\alpha_{j,5}=\alpha_{j,6}=\alpha_{j,7}=\alpha_{j,8}=\alpha_{j,9},\ 6\leq j\leq 9,\\
\{e_2,e_3\}, & \Rightarrow & \beta_{3,7}=0,\ \beta_{3,8}=\frac{1}{6}(3\beta_{2,7}-5\alpha_{1,5}),\ \beta_{3,9}=-\frac{1}{2}\beta_{1,7},\\
\{e_2,e_4\}, & \Rightarrow & \beta_{4,7}=\beta_{4,8}=0,\ \beta_{4,9}=\frac{1}{6}\alpha_{1,3},\\
\{e_2,e_j\}, & \Rightarrow & \beta_{j,7}=\beta_{j,8}=\beta_{j,9}=0,\ 5\leq j\leq 9.
\end{array}$$

So, we have the following table of multiplications of the transposed Poisson algebra structure $(g^2_{(9)},,\cdot,[-,-])$. Note that the identity of associativity holds for any triple.

$$\begin{cases}
e_1\cdot e_1=\alpha_1e_5+\alpha_2e_6+\alpha_3e_7+\alpha_4e_8+\alpha_5e_9,\ e_1\cdot e_2=\frac{1}{3}\alpha_1e_6+\alpha_6e_7+\alpha_7e_8+\alpha_8e_9,\\
e_1\cdot e_3=-\frac{4}{3}\alpha_1e_7+\frac{1}{2}(\alpha_6-5\alpha_2)e_8-\frac{1}{2}\alpha_3e_9,\ e_1\cdot e_4=\frac{1}{3}\alpha_1e_8+\frac{1}{2}\alpha_2e_9,\\
e_1\cdot e_5=-\frac{1}{2}\alpha_1e_9,\ e_2\cdot e_2=\alpha_9e_7+\alpha_{10}e_8+\alpha_{11}e_9,\\
e_2\cdot e_3=\frac{1}{6}(3\alpha_9-5\alpha_1)e_8-\frac{1}{2}\alpha_6e_9,\ e_2\cdot e_4=\frac{1}{6}\alpha_1e_9,\ e_3\cdot e_3=\frac{2}{3}\alpha_1e_9.
\end{cases}$$

□

?⟨thm5klas⟩? **Theorem 20.** *Let $(g^3_{11},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on the Lie algebra g^3_{11}. Then the multiplication of $(g^3_{11},\cdot)$ has the following form:*

$$\mathbf{TP}(g^3_{11}):\ \begin{cases}
e_1\cdot e_1=\alpha_6e_6+\alpha_7e_7+\alpha_8e_8+\alpha_9e_9+\alpha_{10}e_{10}+\alpha_{11}e_{11},\\
e_1\cdot e_2=-\alpha_6e_7-\alpha_7e_8+\alpha_1e_9+\alpha_2e_{10}+\alpha_3e_{11},\ e_1\cdot e_3=\frac{1}{2}\alpha_1e_{10}-\frac{1}{2}\alpha_9e_{11},\\
e_1\cdot e_4=\frac{1}{2}\alpha_7e_{10}+\frac{1}{2}\alpha_8e_{11},\ e_1\cdot e_5=-\frac{1}{2}\alpha_6e_{10}-\frac{1}{2}\alpha_7e_{11},\ e_1\cdot e_6=\frac{1}{2}\alpha_6e_{11},\\
e_2\cdot e_2=\alpha_6e_8+\alpha_4e_9+\alpha_5e_{10}+\alpha_{12}e_{11},\ e_2\cdot e_3=\frac{1}{2}\alpha_4e_{10}-\frac{1}{2}\alpha_1e_{11},\\
e_2\cdot e_4=-\frac{1}{2}\alpha_6e_{10}-\frac{1}{2}\alpha_7e_{11},\ e_2\cdot e_5=\frac{1}{2}\alpha_6e_{11},
\end{cases}$$

where it is taken into account that the transposed Poisson algebra has its products with respect to the bracket $[-,-]$, and the remaining products are equal to zero.

Proof. Let $(g^3_{11},\cdot,[-,-])$ be a transposed Poisson algebra structure defined on Lie algebra g^3_{11}. Then for any element of $x\in g^3_{11}$ the operator of multiplication $\varphi_x(y)=x\cdot y$ is a $\frac{1}{2}$-derivation. Further, using Theorem 13 for $1 \leq i \leq 11$ we derive

$$\begin{cases} \varphi_{e_i}(e_1) = \alpha_{i,1}e_1 + \alpha_{i,6}e_6 + \alpha_{i,7}e_7 + \alpha_{i,8}e_8 + \alpha_{i,9}e_9 + \alpha_{i,10}e_{10} + \alpha_{i,11}e_{11}, \\ \varphi_{e_i}(e_2) = \alpha_{i,1}e_2 - \alpha_{i,6}e_7 - \alpha_{i,7}e_8 + \beta_{i,9}e_9 + \beta_{i,10}e_{10} + \beta_{i,11}e_{11}, \\ \varphi_{e_i}(e_3) = \alpha_{i,1}e_3 + \frac{1}{2}\beta_{i,9}e_{10} - \frac{1}{2}\alpha_{i,9}e_{11}, \\ \varphi_{e_i}(e_4) = \alpha_{i,1}e_4 + \frac{1}{2}\alpha_{i,7}e_{10} + \frac{1}{2}\alpha_{i,8}e_{11}, \\ \varphi_{e_i}(e_5) = \alpha_{i,1}e_5 - \frac{1}{2}\alpha_{i,6}e_{10} - \frac{1}{2}\alpha_{i,7}e_{11}, \\ \varphi_{e_i}(e_6) = \alpha_{i,1}e_6 + \frac{1}{2}\alpha_{i,6}e_{11}, \\ \varphi_{e_i}(e_j) = \alpha_{i,1}e_j, \ \ 7 \leq j \leq 11. \end{cases}$$

Considering the property $\varphi_{e_i}(e_j) = e_i \cdot e_j = e_j \cdot e_i = \varphi_{e_j}(e_i)$, we obtain the following restrictions:

$$\begin{array}{lll} \{e_1, e_2\}, & \Rightarrow & \alpha_{1,1} = \alpha_{2,1} = \alpha_{2,6} = 0, \\ & & \alpha_{2,7} = -\alpha_{1,6}, \ \alpha_{2,8} = -\alpha_{1,7}, \ \alpha_{2,9} = \beta_{1,9}, \ \alpha_{2,10} = \beta_{1,10}, \ \alpha_{2,11} = \beta_{1,11}, \\ \{e_1, e_3\}, & \Rightarrow & \alpha_{3,1} = \alpha_{3,6} = \alpha_{3,7} = \alpha_{3,8} = \alpha_{3,9} = 0, \ \alpha_{3,10} = \frac{1}{2}\beta_{1,9}, \ \alpha_{3,11} = -\frac{1}{2}\alpha_{1,9}, \\ \{e_1, e_4\}, & \Rightarrow & \alpha_{4,1} = \alpha_{4,6} = \alpha_{4,7} = \alpha_{4,8} = \alpha_{4,9} = 0, \ \alpha_{4,10} = \frac{1}{2}\alpha_{1,7}, \ \alpha_{4,11} = \frac{1}{2}\alpha_{1,8}, \\ \{e_1, e_5\}, & \Rightarrow & \alpha_{5,1} = \alpha_{5,6} = \alpha_{5,7} = \alpha_{5,8} = \alpha_{5,9} = 0, \ \alpha_{5,10} = -\frac{1}{2}\alpha_{1,6}, \ \alpha_{5,11} = -\frac{1}{2}\alpha_{1,7}, \\ \{e_1, e_6\}, & \Rightarrow & \alpha_{6,1} = \alpha_{6,6} = \alpha_{6,7} = \alpha_{6,8} = \alpha_{6,9} = \alpha_{6,10} = 0, \ \alpha_{6,11} = \frac{1}{2}\alpha_{1,6}, \\ \{e_1, e_j\}, & \Rightarrow & \alpha_{j,1} = \alpha_{j,6} = \alpha_{j,7} = \alpha_{j,8} = \alpha_{j,9} = \alpha_{j,10} = \alpha_{j,11} = 0, \ 7 \leq j \leq 9, \\ \{e_2, e_3\}, & \Rightarrow & \beta_{3,9} = 0, \ \beta_{3,10} = \frac{1}{2}\beta_{2,9}, \ \beta_{3,11} = -\frac{1}{2}\beta_{1,9}, \\ \{e_2, e_4\}, & \Rightarrow & \beta_{4,9} = 0, \ \beta_{4,10} = -\frac{1}{2}\alpha_{1,6}, \ \beta_{4,11} = -\frac{1}{2}\alpha_{1,7}, \\ \{e_2, e_5\}, & \Rightarrow & \beta_{5,9} = \beta_{5,10} = 0, \ \beta_{5,11} = \frac{1}{2}\alpha_{1,6}, \\ \{e_2, e_j\}, & \Rightarrow & \beta_{j,9} = \beta_{j,10} = \beta_{j,11} = 0, \ 6 \leq j \leq 11. \end{array}$$

Thus, from the above restrictions we obtain the transposed Poisson algebra structure $\mathbf{TP}(g_{11}^3)$ defined on Lie algebra g_{11}^3. □

Declarations

Conflict of interest There is no conflict of interest.

REFERENCES

Kobiljon [1] Abdurasulov K., Adashev J., Eshmeteva S. *Transposed Poisson structures on solvable Lie algebras with filiform nilradical,* Communications in Mathematics, 32(3), (2024), 441-483.

Kobiljon1 [2] Abdurasulov K., Deraman F., Saydaliyev A., Sapar.S.H. *Transposed Poisson Structures on Low Dimensional Quasi-Filiform Lie Algebras of Maximum Length,* Lobachevskii Journal of Mathematics, 45(11), (2024), 5735-5749.

Bai [3] Bai C., Bai R., Guo L., Wu Y., *Transposed Poisson algebras, Novikov-Poisson algebras, and 3-Lie algebras,* Journal of Algebra, 632 (2023), 535-566.

bfk23 [4] Beites P., Fernández Ouaridi A., Kaygorodov I., *The algebraic and geometric classification of transposed Poisson algebras,* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 117 (2023), 2, 55.

bfk22 [5] Beites P., Ferreira B. L. M., Kaygorodov I., *Transposed Poisson structures,* Results in Mathematics, 79 (2024), 93, 29.

MF [6] Fairon M., Modified double brackets and a conjecture of S. Arthamonov, Communications in Mathematics, 33 (2025), 3, 5.

FKL [7] Ferreira B. L. M., Kaygorodov I., Lopatkin V., $\frac{1}{2}$*-derivations of Lie algebras and transposed Poisson algebras,* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 115 (2021), 3, 142.

fer23 [8] Fernández Ouaridi A., *On the simple transposed Poisson algebras and Jordan superalgebras,* Journal of Algebra, 641 (2024), 173-198.

fil1 [9] Filippov V., δ*-Derivations of Lie algebras,* Siberian Mathematical Journal, 39 (1998), 6, 1218-1230.

conj [10] Huang J., Chen X., Chen Z., Ding M., *On a conjecture of transposed Poisson* n*-Lie algebras*, AIMS Mathematics, 9 (2024), 3, 6709-6733.

jawo [11] Jaworska-Pastuszak A., Pogorzały Z., *Poisson structures for canonical algebras*, Journal of Geometry and Physics, 148 (2020), 103564.

Gomez [12] Gómez J.R. , Jiménez-Merchán A. , Reyes J., *Quasi-filiform Lie algebras of maximum length.* Linear Algebra and its Applications, 335, (2001), 119-135.

k07 [13] Kaygorodov I., δ*-derivations of simple finite-dimensional Jordan superalgebras*, Algebra and Logic, 46 (2007), 5, 318-329.

k10 [14] Kaygorodov I., δ*-superderivations of simple finite-dimensional Jordan and Lie superalgebra*, Algebra and Logic, 49 (2010), 2, 130-144.

kk21 [15] Kaygorodov I., Khrypchenko M., *Poisson structures on finitary incidence algebras,* Journal of Algebra, 578 (2021), 402-420.

kk22 [16] Kaygorodov I., Khrypchenko M., *Transposed Poisson structures on Block Lie algebras and superalgebras,* Linear Algebra and Its Applications, 656 (2023), 167-197.

kk23 [17] Kaygorodov I., Khrypchenko M., *Transposed Poisson structures on Witt type algebras,* Linear Algebra and its Applications, 665 (2023), 196-210.

kkg23 [18] Kaygorodov I., Khrypchenko M., *Transposed Poisson structures on generalized Witt algebras and Block Lie algebras,* Results in Mathematics, 78 (2023), 5, 186.

KK7 [19] Kaygorodov I., Khrypchenko M., *Transposed Poisson structures on the Lie algebra of upper triangular matrices,* Portugaliae Mathematica, 81 (2024), 1-2, 135-149.

kkinc [20] Kaygorodov I., Khrypchenko M., *Transposed Poisson structures on Lie incidence algebras*, Journal of Algebra, 647 (2024), 458-491.

KKh [21] Kaygorodov I., Khudoyberdiyev A., *Transposed Poisson structures on solvable and perfect Lie algebras,* Journal of Physics A: Mathematical and Theoretical, 57 (2024), 035205.

klv22 [22] Kaygorodov I., Lopatkin V., Zhang Z., *Transposed Poisson structures on Galilean and solvable Lie algebras,* Journal of Geometry and Physics, 187 (2023), 104781.

klv221 [23] Kaygorodov I., Khudoyberdiyev A., Shermatova Z., *Transposed Poisson structures on not-finitely graded Witt-type algebras,* Boletin de la Sociedad Matematica Mexicana, 31(1), 22 (2025).

klv222 [24] Kaygorodov I., Khudoyberdiyev A., Shermatova Z., *Transposed Poisson structures on Virasoro-type algebras,* Journal of Geometry and Physics, 207, 105356 (2025).

lb23 [25] Liu G., Bai C., *A bialgebra theory for transposed Poisson algebras via anti-pre-Lie bialgebras and anti-pre-Lie-Poisson bialgebras*, Communications in Contemporary Mathematics, (2023), 2350050.

bl23 [26] Liu G., Bai C., *New splittings of operations of Poisson algebras and transposed Poisson algebras and related algebraic structures*, In: Hounkonnou, M.N., Mitrović, M., Abbas, M., Khan, M. (eds) Algebra without Borders

– Classical and Constructive Nonassociative Algebraic Structures. STEAM-H: Science, Technology, Engineering, Agriculture, Mathematics & Health. Springer, Cham., 2023, 49-96.

kms [27] Sartayev B., *Some generalizations of the variety of transposed Poisson algebras,* Communications in Mathematics, 32 (2024), 2, 55-62.

SZ [28] Shermatova Z., Transposed Poisson structures on the extended Schrödinger-Virasoro and the original deformative Schrödinger-Virasoro algebras, Siberian Electronic Mathematical Reports, 21 (2024), 2, 1385-1399.

ytk [29] Yang Ya., Tang X., Khudoyberdiyev A., *Transposed Poisson structures on Schrodinger algebra in* $(n+1)$*-dimensional space-time,* arXiv:2303.08180.

YYZ07 [30] Yao Y., Ye Y., Zhang P., *Quiver Poisson algebras*, Journal of Algebra, 312 (2007), 2, 570-589.

ch24 [31] Yaxi J., Chuangchuang K., Jiafeng L., *Classification of Transposed Poisson* 3*-Lie algebras of dimension* 3, arXiv:2401.02593.

yh21 [32] Yuan L., Hua Q., $\frac{1}{2}$*-(bi)derivations and transposed Poisson algebra structures on Lie algebras,* Linear and Multilinear Algebra, 70 (2022), 22, 7672-7701.

zz [33] Zohrabi A., Zusmanovich P., A δ-first Whitehead lemma for Jordan algebras, Communications in Mathematics, 33 (2025), 1, 2.