# The algebraic and geometric classification of Jordan superalgebras[*]

### Kobiljon Abdurasulov[†] Roman Lubkov[‡] & Azamat Saydaliyev [§]

**Abstract.** We give the algebraic and geometric classification of complex four-dimensional Jordan superalgebras. In particular, we describe all irreducible components in the corresponding varieties.

## INTRODUCTION

The algebraic classification (up to isomorphism) of algebras of small dimensions from a certain variety defined by a family of polynomial identities is a classic problem in the theory of non-associative algebras. Another interesting approach to studying algebras of a fixed dimension is to study them from a geometric point of view (that is, to study the degenerations and deformations of these algebras). The results in which the complete information about degenerations of a certain variety is obtained are generally referred to as the geometric classification of the algebras of these varieties. There are many results related to the algebraic and geometric classification of Jordan, Lie, Leibniz, Zinbiel, and other algebras (see, [1–7, 9–11, 13–17, 19–26, 35]). Degenerations are related to deformations and cohomology [8, 12, 34].

Superalgebras emerged in physics to provide a unified framework for the study of super symmetry of elementary particles. Jordan algebras, originating from quantum mechanics, gained importance due to their close connection to Lie theory. Finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero were classified by Kac [27] in 1977. One case remained unresolved, which was considered by Kantor [28] in 1990. Recently Racine and Zelmanov [33] presented a classification of finite-dimensional simple Jordan superalgebras over arbitrary fields of characteristic distinct from 2, focusing on cases where the even part is semisimple. For the opposite case, where the even part is not semisimple, a classification was obtained by Martínez and Zelmanov [32] in 2002, completing the entire project.

In [31], the authors focus on the classification of Jordan superalgebras of dimension up to three over an algebraically closed field of characteristic distinct from 2. The mail goal of this paper is

to determine the minimal dimension of exceptional Jordan superalgebras, raised in [29]. In [18], authors obtained all four dimensional indecomposable Jordan superalgebras.

In the present paper, we obtain an algebraic and geometric classification of four-dimensional Jordan superalgebras and find all irreducible components within that variety. Section 1 outlines the foundational concepts and establishes several key preliminary results. In Section 2, we proceed to classify all four-dimensional Jordan superalgebras. Based on this classification, we determine the irreducible components within the variety in Section 3.

Our main results are summarized below.

**Theorem A.** *The variety of complex* 4*-dimensional Jordan superalgebras of type* $(1,3)$ *has dimension* 12. *It is defined by* 11 *rigid superalgebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the superalgebras given in Theorem 13.*

**Theorem B.** *The variety of complex* 4*-dimensional Jordan superalgebras of type* $(2,2)$ *has dimension* 13. *It is defined by* 24 *rigid superalgebras and one one-parametric families of superalgebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the superalgebras given in Theorem 14.*

**Theorem C.** *The variety of complex* 4*-dimensional Jordan superalgebras of type* $(3,1)$ *has dimension* 15. *It is defined by* 21 *rigid superalgebras and can be described as the closure of the union of* $\mathrm{GL}_4(\mathbb{C})$*-orbits of the superalgebras given in Theorem 15.*

## 1. Preliminaries

### 1.1. **Jordan superalgebras.**

**Definition 1.** *A commutative algebra is called a Jordan algebra if it satisfies the identity*

$$(x^2 y)x = x^2(yx).$$

**Definition 2.** *A superalgebra* $\mathscr{A}$ *is an algebra with a* $\mathbb{Z}_2$*-grading. So* $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ *is a direct sum of two vector spaces and*

$$\mathscr{A}_i \mathscr{A}_j \subseteq \mathscr{A}_{i+j}, \;\; where \;\; i,j \in \mathbb{Z}_2.$$

Let $G$ be the Grassmann algebra over $\mathbb{F}$ given by the generators $1, \xi_1, \ldots, \xi_n, \ldots$ and the defining relations $\xi_i^2 = 0$ and $\xi_i \xi_j = -\xi_j \xi_i$. The elements $1, \xi_{i_1} \xi_{i_2} \ldots \xi_{i_k}$, $i_1 < i_2 < \ldots < i_k$, form a basis of the algebra $G$ over $\mathbb{F}$. Denote by $G_0$ and $G_1$ the subspaces spanned by the products of even and odd lengths, respectively; then $G$ can be represented as the direct sum of these subspaces, $G = G_0 \oplus G_1$. Here the relations $G_i G_j \subseteq G_{i+j(mod\ 2)}$, $i,j = 0,1$, hold. In other words, $G$ is a $\mathbb{Z}_2$-graded algebra (or a superalgebra) over $\mathbb{F}$. Suppose now that $A = A_0 \oplus A_1$ is an arbitrary superalgebra over $\mathbb{F}$. Consider the tensor product $G \otimes A$ of $\mathbb{F}$-algebras. The subalgebra

$$G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$$

of $G \otimes A$ is referred to as the Grassmann envelope of the superalgebra $A$. Let $\Omega$ be a variety of algebras over $\mathbb{F}$. A superalgebra $A = A_0 \oplus A_1$ is referred to as an $\Omega$-superalgebra if its Grassmann envelope $G(A)$ is an algebra in $\Omega$. In particular, $A = A_0 \oplus A_1$ is referred to as a Jordan superalgebra if its Grassmann envelope $G(A)$ is a Jordan algebra.

**Definition 3.** *A Jordan superalgebra is a superalgebra* $\mathfrak{J} = \mathfrak{J}_0 + \mathfrak{J}_1$ *satisfying the graded identities:*

$$xy = (-1)^{|x||y|} yx,$$

$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x =$$
$$(xy)(zt) + (-1)^{|t|(|y|+|z|)}(xt)(yz) + (-1)^{|y||z|}(xz)(yt),$$

*where $|x| = i$ for $x \in \mathfrak{J}_i$.*

For convenience, we use the following notation in the next sections.

$$J(x,y,z,t) = ((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x - (xy)(zt) - (-1)^{|t|(|y|+|z|)}(xt)(yz) - (-1)^{|y||z|}(xz)(yt).$$

**Definition 4.** *For arbitrary elements $x, y \in \mathfrak{J}_0 \cup \mathfrak{J}_1$ of a Jordan superalgebra, the operator $D: \mathfrak{J} \to \mathfrak{J}$ satisfying*

$$D(xy) = D(x)y + (-1)^{|D||x|}xD(y)$$

*is a derivation of $\mathfrak{J}$, where $|D| = 0$ if $D$ preserves the gradation and $|D| = 1$ otherwise.*

In 2013, M.E.Martin [30] described all Jordan algebras up to dimension four over an algebraically closed field. Based on that paper, we present the list of indecomposable Jordan algebras and superalgebras of dimension less than or equal to 3 in the following tables, as they will be used in the sequel.

Table 1: Indecomposable Jordan algebras

| $\mathfrak{J}$ | Multiplication table | | | | | | dim |
|---|---|---|---|---|---|---|---|
| $\mathscr{U}_1$ : | $e^2 = e$ | | | | | | 1 |
| $\mathscr{U}_2$ : | $e^2 = 0$ | | | | | | 1 |
| $\mathscr{B}_1$ : | $e_1^2 = e_1$ | $e_1e_2 = e_2$ | $e_2^2 = 0$ | | | | 2 |
| $\mathscr{B}_2$ : | $e_1^2 = e_1$ | $e_1e_2 = \frac{1}{2}e_2$ | $e_2^2 = 0$ | | | | 2 |
| $\mathscr{B}_3$ : | $e_1^2 = e_2$ | $e_1e_2 = 0$ | $e_2^2 = 0$ | | | | 2 |
| $\mathscr{T}_1$ : | $e_1^2 = e_1$ | $e_2^2 = e_3$ | $e_3^2 = 0$ | $e_1e_2 = e_2$ | $e_1e_3 = e_3$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_2$ : | $e_1^2 = e_1$ | $e_2^2 = 0$ | $e_3^2 = 0$ | $e_1e_2 = e_2$ | $e_1e_3 = e_3$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_3$ : | $e_1^2 = e_2$ | $e_2^2 = 0$ | $e_3^2 = 0$ | $e_1e_2 = e_3$ | $e_1e_3 = 0$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_4$ : | $e_1^2 = e_2$ | $e_2^2 = 0$ | $e_3^2 = 0$ | $e_1e_2 = 0$ | $e_1e_3 = e_2$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_5$ : | $e_1^2 = e_1$ | $e_2^2 = e_2$ | $e_3^2 = e_1 + e_2$ | $e_1e_2 = 0$ | $e_1e_3 = \frac{1}{2}e_3$ | $e_2e_3 = \frac{1}{2}e_3$ | 3 |
| $\mathscr{T}_6$ : | $e_1^2 = e_1$ | $e_2^2 = 0$ | $e_3^2 = 0$ | $e_1e_2 = \frac{1}{2}e_2$ | $e_1e_3 = e_3$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_7$ : | $e_1^2 = e_1$ | $e_2^2 = 0$ | $e_3^2 = 0$ | $e_1e_2 = \frac{1}{2}e_2$ | $e_1e_3 = \frac{1}{2}e_3$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_8$ : | $e_1^2 = e_1$ | $e_2^2 = e_3$ | $e_3^2 = 0$ | $e_1e_2 = \frac{1}{2}e_2$ | $e_1e_3 = 0$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_9$ : | $e_1^2 = e_1$ | $e_2^2 = e_3$ | $e_3^2 = 0$ | $e_1e_2 = \frac{1}{2}e_2$ | $e_1e_3 = e_3$ | $e_2e_3 = 0$ | 3 |
| $\mathscr{T}_{10}$ : | $e_1^2 = e_1$ | $e_2^2 = e_2$ | $e_3^2 = 0$ | $e_1e_2 = 0$ | $e_1e_3 = \frac{1}{2}e_3$ | $e_2e_3 = \frac{1}{2}e_3$ | 3 |

Below we list indecomposable superalgebras denoted by $\mathscr{S}_j^i$ where the exponent $i$ represents its dimension.

Table 2: Indecomposable superalgebras

| $\mathfrak{J}$ | Multiplication table | $\mathfrak{J}$ | Multiplication table |
|---|---|---|---|

| | | | | |
|---|---|---|---|---|
| $\mathscr{S}_1^1$ : $f^2 = 0$ | | | | |

$$
\begin{array}{llll}
\mathscr{S}_1^1 & : & f^2 = 0 \\
\mathscr{S}_1^2 & : & e^2 = e & ef = \tfrac{1}{2}f \\
\mathscr{S}_2^2 & : & e^2 = e & ef = f \\
\mathscr{S}_1^3 & : & ef_1 = f_2 & f_1 f_2 = e \\
\mathscr{S}_2^3 & : & f_1 f_2 = e \\
\mathscr{S}_3^3 & : & ef_1 = f_2 \\
\mathscr{S}_4^3 & : & e^2 = e & ef_1 = f_1 & ef_2 = \tfrac{1}{2}f_2 \\
\mathscr{S}_5^3 & : & e^2 = e & ef_1 = \tfrac{1}{2}f_1 & ef_2 = \tfrac{1}{2}f_2
\end{array}
\qquad
\begin{array}{llll}
\mathscr{S}_6^3 & : & e^2 = e & ef_1 = f_1 & ef_2 = f_2 \\
\mathscr{S}_7^3 & : & e^2 = e & ef_1 = \tfrac{1}{2}f_1 & ef_2 = \tfrac{1}{2}f_2 & f_1 f_2 = e \\
\mathscr{S}_8^3 & : & e^2 = e & ef_1 = f_1 & ef_2 = f_2 & f_1 f_2 = e \\
\mathscr{S}_9^3 & : & e_1^2 = e_1 & e_1 e_2 = e_2 & e_1 f = \tfrac{1}{2}f \\
\mathscr{S}_{10}^3 & : & e_1^2 = e_1 & e_1 e_2 = e_2 & e_1 f = \tfrac{1}{2}f \\
\mathscr{S}_{11}^3 & : & e_1^2 = e_1 & e_1 e_2 = \tfrac{1}{2}e_2 & e_1 f = \tfrac{1}{2}f \\
\mathscr{S}_{12}^3 & : & e_1^2 = e_1 & e_1 e_2 = \tfrac{1}{2}e_2 & e_1 f = f \\
\mathscr{S}_{13}^3 & : & e_1^2 = e_1 & e_2^2 = e_2 & e_1 f = \tfrac{1}{2}f & e_2 f = \tfrac{1}{2}f
\end{array}
$$

### 1.2. Degenerations.

Given an $(m,n)$-dimensional vector superspace $V = V_0 \oplus V_1$, the set

$$\mathrm{Hom}(V \otimes V, V) = (\mathrm{Hom}(V \otimes V, V))_0 \oplus (\mathrm{Hom}(V \otimes V, V))_1$$

is a vector superspace of dimension $m^3 + 3mn^2$. This space has a structure of the affine variety $\mathbb{C}^{m^3 + 3mn^2}$. If we fix a basis $\{e_1, \dots, e_m, f_1, \dots, f_n\}$ of $V$, then any $\mu \in \mathrm{Hom}(V \otimes V, V)$ is determined by $m^3 + 3mn^2$ structure constants $\alpha_{i,j}^k, \beta_{i,j}^q, \gamma_{i,j}^q, \delta_{p,q}^k \in \mathbb{C}$ such that

$$\mu(e_i \otimes e_j) = \sum_{k=1}^m \alpha_{i,j}^k e_k, \quad \mu(e_i \otimes f_p) = \sum_{q=1}^n \beta_{i,p}^q f_q, \quad \mu(f_p \otimes e_i) = \sum_{q=1}^n \gamma_{p,i}^q f_q, \quad \mu(f_p \otimes f_q) = \sum_{k=1}^m \delta_{p,q}^k e_k.$$

A subset $\mathbb{L}(T)$ of $\mathrm{Hom}(V \otimes V, V)$ is *Zariski-closed* if it can be defined by a set of polynomial equations $T$ in the variables $\alpha_{i,j}^k, \beta_{i,p}^q, \gamma_{p,i}^q, \delta_{p,q}^k$ $(1 \le i,j,k \le m,\ 1 \le p,q \le n)$.

Let $\mathscr{S}^{m,n}$ be the set of all superalgebras of dimension $(m,n)$ defined by the family of polinomial super-identities $T$, understood as a subset $\mathbb{L}(T)$ of an affine variety $\mathrm{Hom}(V \otimes V, V)$. Then one can see that $\mathscr{S}^{m,n}$ is a Zariski-closed subset of the variety $\mathrm{Hom}(V \otimes V, V)$. The group $G = (\mathrm{Aut}\,V)_0 \simeq \mathrm{GL}(V_0) \oplus \mathrm{GL}(V_1)$ acts on $\mathscr{S}^{m,n}$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathbb{L}(T)$ and $g \in G$.

Thus, $\mathscr{S}^{m,n}$ is decomposed into $G$-orbits that correspond to the isomorphism classes of superalgebras. The dimension of the orbit is found via

$$\dim O(\mu) = n^2 - \dim \mathfrak{Der}(J), \tag{1}$$

where $\mathfrak{Der}(J)$ denotes the Lie algebra of derivations of $J$ corresponding to $\mu$.

Denote by $O(\mu)$ the orbit of $\mu \in \mathbb{L}(T)$ under the action of $G$ and by $\overline{O(\mu)}$ the Zariski closure of $O(\mu)$. Let $J, J' \in \mathscr{S}^{m,n}$ and $\lambda, \mu \in \mathbb{L}(T)$ represent $J$ and $J'$, respectively. We say that $\lambda$ degenerates to $\mu$ and write $\lambda \to \mu$ if $\mu \in \overline{O(\lambda)}$. Note that in this case we have $\overline{O(\mu)} \subset \overline{O(\lambda)}$. Hence, the definition of a degeneration does not depend on the choice of $\mu$ and $\lambda$, and we right indistinctly $J \to J'$ instead of $\lambda \to \mu$ and $O(J)$ instead of $O(\lambda)$. If $J \not\cong J'$, then the assertion $J \to J'$ is called a *proper degeneration*. We write $J \not\to J'$ if $J' \notin \overline{O(J)}$.

Let $J$ be represented by $\lambda \in \mathbb{L}(T)$. Then $J$ is *rigid* in $\mathbb{L}(T)$ if $O(\lambda)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called *irreducible component*. In particular, $J$ is rigid in $\mathscr{S}^{m,n}$ iff $\overline{O(\lambda)}$ is an irreducible component of $\mathbb{L}(T)$. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. We denote by $\mathrm{Rig}(\mathscr{S}^{m,n})$ the set of rigid superalgebras in $\mathscr{S}^{m,n}$.

**Remark 5.** *There is no degeneration between two superalgebras of types $(n-i,i)$ and $(n-j,j)$ if $i \neq j$.*

This statement follows from the fact that superalgebras of different types correspond to affine varieties of different dimensions. Suppose that superalgebras of types $(n-i,i)$ and $(n-j,j)$ correspond to the same variety $\mathbb{C}^{p^3+3pq^2}$ when $i \neq j$. In that case we have the following equality:

$$(n-i)^3 + 3(n-i)i^2 = (n-j)^3 + 3(n-j)j^2.$$

Therefore, we obtain

$$(i-j)(3n^2 - 6n(i+j) + 4(i^2 + ij + j^2)) = 0.$$

Since $i \neq j$, the second bracket, which contains quadratic term with respect to $n$, must be zero. However, this requires

$$36(i+j)^2 - 48(i^2 + ij + j^2) = 24ij - 12(i^2 + j^2) \geq 0,$$

leading to

$$(i-j)^2 \leq 0,$$

which cannot hold unless $i = j$.

1.3. **Principal notation.** Let $\mathcal{JS}^{m,n}$ be the set of all Jordan superalgebras of dimension $(m,n)$. Let $J$ be a Jordan superalgebra with a fixed basis $\{e_1,\ldots,e_m,f_1,\ldots f_n\}$, defined by

$$e_ie_j = \sum_{k=1}^m \alpha_{ij}^k e_k, \quad e_if_j = \sum_{k=1}^n \beta_{ij}^k f_k, \quad f_if_j = \sum_{k=1}^m \gamma_{ij}^k e_k.$$

In the sequel, we use the following notation:

(1) $\mathfrak{a}(J)$ is the Jordan superalgebra with the same underlying vector superspace as $J$ and defined by $f_if_j = \sum_{k=1}^n \gamma_{ij}^k e_k$.

(2) $J^1 = J$, $J^r = J^{r-1}J + J^{r-2}J^2 + \cdots + JJ^{r-1}$, and in every case $J^r = (J^r)_0 \oplus (J^r)_1$.

(3) $c_{i,j} = \dfrac{\text{tr}(L(x)^i) \cdot \text{tr}(L(y)^j)}{\text{tr}(L(x)^i \cdot L(y)^j)}$ is the Burde invariant, where $L(x)$ is the left multiplication. This invariant $c_{i,j}$ is defined as a quotient of two polynomials in the structure constants of $J$, for all $x,y \in J$ such that both polynomials are not zero and $c_{i,j}$ is independent of the choice of $x,y$.

1.4. **Methods.** First of all, if $J \to J'$ and $J \not\cong J'$, then $\dim \text{Aut}(J) < \dim \text{Aut}(J')$, where $\text{Aut}(J)$ is the space of automorphisms of $J$. Secondly, if $J \to J'$ and $J' \to J''$, then $J \to J''$. If there is no $J'$ such that $J \to J'$ and $J' \to J''$ are proper degenerations, then the assertion $J \to J''$ is called a *primary degeneration*. If $\dim \text{Aut}(J) < \dim \text{Aut}(J'')$ and there are no $J'$ and $J'''$ such that $J' \to J$, $J'' \to J'''$, $J' \not\to J'''$ and one of the assertions $J' \to J$ and $J'' \to J'''$ is a proper degeneration, then the assertion $J \not\to J''$ is called a *primary non-degeneration*. It suffices to prove only primary degenerations and non-degenerations to describe degenerations in the variety under consideration. It is easy to see that any superalgebra degenerates to the superalgebra with zero multiplication. From now on we use this fact without mentioning it.

Let us describe the methods for proving primary non-degenerations. The main tool for this is the following lemma [4].

**Lemma 6.** *If $J \to J'$, then the following hold:*

*(1)* $\dim(J^r)_i \geq \dim(J'^r)_i$, *for $i \in \mathbb{Z}_2$;*

(2) $(J)_0 \to (J')_0$;

(3) $\mathfrak{a}(J) \to \mathfrak{a}(J')$;

(4) *If the Burde invariant exists for $J$ and $J'$, then both superalgebras have the same Burde invariant;*

(5) *If $J$ is associative, then $J'$ must be associative. In fact, if $J$ satisfies a P.I. then $J'$ must satisfy the same P.I.*

In the cases where all of these criteria can't be applied to prove $J \not\to J'$, we define $\mathscr{R}$ by a set of polynomial equations and give a basis of $V$, in which the structure constants of $\lambda$ give a solution to all these equations. Further on, we omit the verification of the fact that $\mathscr{R}$ is stable under the action of the subgroup of upper triangular matrices and of the fact that $\mu \notin \mathscr{R}$ for any choice of a basis of $V$. These verifications can be done by direct calculations.

**Degenerations of Graded algebras.** Let $G$ be a trivial group and let $\mathcal{V}(\mathscr{F})$ be a variety of algebras defined by a family of polynomial identities $\mathscr{F}$. It is important to notice that degeneration on the $G$-graded variety $G\mathcal{V}(\mathscr{F})$ is a more restrictive notion than degeneration on the variety $\mathcal{V}(\mathscr{F})$, In fact, consider $A, A' \in G\mathcal{V}(\mathscr{F})$ such that $A, A' \in \mathcal{V}(\mathscr{F})$, a degeneration between the algebras $A$ and $A'$ may not give rise to a degeneration between the $G$-graded algebras $A$ and $A'$, since the matrices describing the basis changes in $G\mathcal{V}(\mathscr{F})$ must preserve the $G$-graduation. Hence, we have the following result.

**Lemma 7.** *Let $A, A' \in G\mathcal{V}(\mathscr{F}) \cap \mathcal{V}(\mathscr{F})$. If $A \not\to A'$ as algebras, then $A \not\to A'$ as $G$-graded algebras.*

Additionally, we need the following results from [15]

**Theorem 8.** *The graph of primary degenerations for two-dimensional Jordan algebras has the following form:*
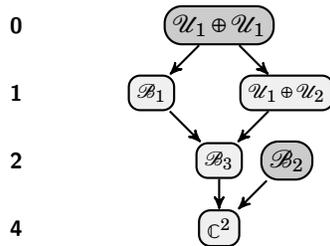


**Theorem 9.** *The graph of primary degenerations for three-dimensional Jordan algebras has the following form:*

The diagram at the top of the page (Hasse-type diagram with levels 0, 1, 2, 3, 4, 5, 6, 9 on the left):

- Level 0: $\mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{U}_1$
- Level 1: $\mathscr{B}_1 \oplus \mathscr{U}_1$, $\mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2$, $\mathscr{T}_5$
- Level 2: $\mathscr{T}_1$, $\mathscr{B}_1 \oplus \mathscr{U}_2$, $\mathscr{B}_3 \oplus \mathscr{U}_1$, $\mathscr{T}_{10}$, $\mathscr{T}_8$, $\mathscr{B}_2 \oplus \mathscr{U}_1$, $\mathscr{T}_9$
- Level 3: $\mathscr{T}_3$, $\mathscr{B}_2 \oplus \mathscr{U}_2$, $\mathscr{T}_6$
- Level 4: $\mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{U}_2$, $\mathscr{T}_4$, $\mathscr{T}_2$
- Level 5: $\mathscr{B}_3 \oplus \mathscr{U}_2$
- Level 6: $\mathscr{T}_7$
- Level 9: $\mathbb{C}^3$
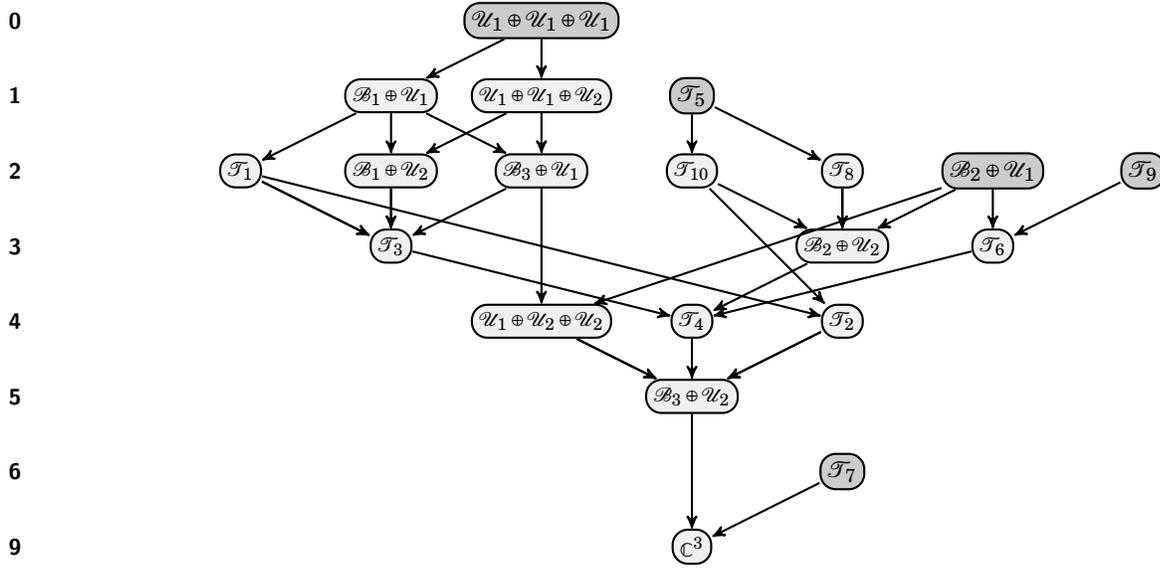
## 2. Algebraic classification of four dimensional Jordan superalgebras

All four dimensional indecomposable Jordan superalgebras are obtained in [18] while classifying low-dimensional commutative power-associative superalgebras. Below we present our classification of all four dimensional Jordan superalgebras. Dimensions of orbits in the theorems below are calculated using the formula (1).

**Theorem 10.** *Up to isomorphism there are 19 Jordan superalgebras of the type* (1,3), *which are presented below with some additional information:*

| № | Orbit | Multiplication rules | Decomposition |
|---|---|---|---|
| $\mathbf{J_1}$ | 6 | $f_1 f_2 = e$ | $\mathscr{S}_2^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_2}$ | 6 | $e f_1 = f_2$ | $\mathscr{S}_3^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_3}$ | 9 | $e f_1 = f_2,\ f_1 f_2 = e$ | $\mathscr{S}_1^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_4}$ | 9 | $e f_1 = f_2,\ f_1 f_3 = e$ | *Indecomposable* |
| $\mathbf{J_5}$ | 12 | $e f_1 = f_2,\ f_2 f_3 = e$ | *Indecomposable* |
| $\mathbf{J_6}$ | 11 | $e f_1 = f_2,\ e f_2 = f_3$ | *Indecomposable* |
| $\mathbf{J_7}$ | 7 | $e^2 = e$ | $\mathscr{U}_1 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_8}$ | 10 | $e^2 = e,\ e f_3 = \frac{1}{2} f_3$ | $\mathscr{S}_1^2 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_9}$ | 10 | $e^2 = e,\ e f_3 = f_3$ | $\mathscr{S}_2^2 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{10}}$ | 9 | $e^2 = e,\ e f_2 = \frac{1}{2} f_2,\ e f_3 = \frac{1}{2} f_3$ | $\mathscr{S}_5^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{11}}$ | 10 | $e^2 = e,\ e f_2 = \frac{1}{2} f_2,\ e f_3 = \frac{1}{2} f_3,\ f_2 f_3 = e$ | $\mathscr{S}_7^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{12}}$ | 12 | $e^2 = e,\ e f_2 = \frac{1}{2} f_2,\ e f_3 = f_3$ | $\mathscr{S}_4^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{13}}$ | 11 | $e^2 = e,\ e f_2 = f_2,\ e f_3 = f_3$ | $\mathscr{S}_6^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{14}}$ | 12 | $e^2 = e,\ e f_2 = f_2,\ e f_3 = f_3,\ f_2 f_3 = e$ | $\mathscr{S}_8^3 \oplus \mathscr{S}_1^1$ |
| $\mathbf{J_{15}}$ | 4 | $e^2 = e,\ e f_1 = \frac{1}{2} f_1,\ e f_2 = \frac{1}{2} f_2,\ e f_3 = \frac{1}{2} f_3$ | *Indecomposable* |
| $\mathbf{J_{16}}$ | 9 | $e^2 = e,\ e f_1 = \frac{1}{2} f_1,\ e f_2 = \frac{1}{2} f_2,\ e f_3 = f_3$ | *Indecomposable* |

| $\mathbf{J_{17}}$ | 10 | $e^2 = e,\ ef_1 = \frac{1}{2}f_1,\ ef_2 = f_2,\ ef_3 = f_3$ | Indecomposable |
|---|---|---|---|
| $\mathbf{J_{18}}$ | 7 | $e^2 = e,\ ef_1 = f_1,\ ef_2 = f_2,\ ef_3 = f_3$ | Indecomposable |
| $\mathbf{J_{19}}$ | 10 | $e^2 = e,\ ef_1 = f_1,\ ef_2 = f_2,\ ef_3 = f_3,\ f_1f_2 = e$ | Indecomposable |

*Proof.* As $\mathfrak{J}_0$ is a Jordan algebra, we have subcases $\mathfrak{J}_0 \cong \mathscr{U}_1$ and $\mathfrak{J}_0 \cong \mathscr{U}_2$.

Then we have the following multiplications for $e \in J_0$, $f_1, f_2, f_3 \in J_1$

$$
\begin{aligned}
ef_1 &= \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3, & f_1f_2 &= \xi_1 e, \\
ef_2 &= \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, & f_1f_3 &= \xi_2 e, \\
ef_3 &= \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3, & f_2f_3 &= \xi_3 e.
\end{aligned}
$$

The linear operator $L_x : \mathfrak{J} \to \mathfrak{J}, x \in \mathfrak{J}$ such that $L_x(y) = xy$ is called a left multiplication operator. It is obvious that:

$$
x \in \mathfrak{J}_0,\ L_x : \mathfrak{J}_0 \to \mathfrak{J}_0,\ L_x : \mathfrak{J}_1 \to \mathfrak{J}_1,
$$
$$
x \in \mathfrak{J}_1,\ L_x : \mathfrak{J}_0 \to \mathfrak{J}_1,\ L_x : \mathfrak{J}_1 \to \mathfrak{J}_0.
$$

For the action of the operator $L_e$ on $\mathfrak{J}_1$ we can write the following matrix

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}.
$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_e$ has one of the following forms:

$$
\begin{pmatrix}
\mu_1 & 0 & 0 \\
0 & \mu_2 & 0 \\
0 & 0 & \mu_3
\end{pmatrix}, \quad
\begin{pmatrix}
\mu_1 & 1 & 0 \\
0 & \mu_1 & 0 \\
0 & 0 & \mu_3
\end{pmatrix}, \quad
\begin{pmatrix}
\mu_1 & 1 & 0 \\
0 & \mu_1 & 1 \\
0 & 0 & \mu_1
\end{pmatrix}.
$$

Let $\mathfrak{J}_0 \cong \mathscr{U}_2$.

**i)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$
ef_1 = \mu_1 f_1,\quad ef_2 = \mu_2 f_2,\quad ef_3 = \mu_3 f_3,\quad f_1f_2 = \xi_1 e,\quad f_1f_3 = \xi_2 e,\quad f_2f_3 = \xi_3 e.
$$

In this case, from $J(e,e,e,f_1) = 0$, $J(e,e,e,f_2) = 0$, and $J(e,e,e,f_3) = 0$ we get $\mu_i = 0$, $i = \overline{1,3}$.

If $(\xi_1, \xi_2, \xi_3) \neq (0,0,0)$, then by changing $e' = \xi_1 e$, $f_3' = \xi_1 f_3 - \xi_2 f_2 + \xi_3 f_1$ we obtian the Jordan superalgebra:

$$
e'^2 = 0,\quad f_1f_2 = e'.
$$

We denote this superalgebra by $\mathbf{J_1}$.

**ii)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$
ef_1 = \mu_1 f_1 + f_2,\quad ef_2 = \mu_1 f_2,\quad ef_3 = \mu_3 f_3,\quad f_1f_2 = \xi_1 e,\quad f_1f_3 = \xi_2 e,\quad f_2f_3 = \xi_3 e.
$$

In this case, from $J(e,e,e,f_2) = 0$ and $J(e,e,e,f_3) = 0$ we get $2\mu_1^3 = 0$, $2\mu_3^3 = 0$, respectively. So $\mu_1 = \mu_3 = 0$. When $(\xi_1, \xi_2, \xi_3) = (0,0,0)$ we get the superalgebra

$$ef_1 = f_2,$$

which we denote by $\mathbf{J_2}$. Hence, if $(\xi_1, \xi_2, \xi_3) \neq (0,0,0)$, then we can change the basis as follows:

$$
\begin{aligned}
f_1' &= a_1 f_1 + a_2 f_2 + a_3 f_3, & f_1' f_2' &= a_1^2 \xi_1 e - a_1 a_3 \xi_3 e = \xi_1' e, \\
f_2' &= a_1 f_2, & f_1' f_3' &= a_1 b_1 \xi_1 e + a_1 b_2 \xi_2 e + a_2 b_2 \xi_3 e - a_3 b_1 \xi_3 e = \xi_2' e, \\
f_3' &= b_1 f_2 + b_2 f_3, & f_2' f_3' &= a_1 b_2 \xi_3 e.
\end{aligned}
$$

We proceed as follows:

    a) If $\xi_3 = 0$ then $\xi_1' = a_1^2 \xi_1$, $\xi_2' = a_1 b_1 \xi_1 + a_1 b_2 \xi_2$.

        a.1) If $\xi_1 \neq 0$ then by choosing $b_1 = -\frac{b_2 \xi_2}{\xi_1}$, $a_1 = \frac{1}{\xi_1}$ we get $\xi_2' = 0$ and $\xi_1' = 1$ which gives the superalgebra $\mathbf{J_3}$:

$$e^2 = 0,\ ef_1' = f_2',\ f_1' f_2' = e.$$

        a.2) If $\xi_1 = 0$ then $\xi_2' = a_1 b_2 \xi_2$. In this case by choosing $a_1 b_2 = \frac{1}{\xi_2}$ we get the superalgebra $\mathbf{J_4}$:

$$e^2 = 0,\ ef_1' = f_2',\ f_1' f_3' = e.$$

    b) If $\xi_3 \neq 0$ then by choosing $a_3 = \frac{a_1 \xi_1}{\xi_3}$, $b_2 = \frac{1}{a_1 \xi_3}$, $a_2 = -\frac{a_1 \xi_2}{\xi_3}$ and taking $b_2 = 1$ for simplicity we get the superalgebra $\mathbf{J_5}$: $e^2 = 0$, $ef_1' = f_2'$, $f_2' f_3' = e$.

**iii)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1 + f_2,\quad ef_2 = \mu_1 f_2 + f_3,\quad ef_3 = \mu_1 f_3,\quad f_1 f_2 = \xi_1 e,\quad f_1 f_3 = \xi_2 e,\quad f_2 f_3 = \xi_3 e.$$

In this case from $J(e,e,e,f_1) = 0$ we get $\mu_1 = 0$. Moreover, from $J(e,e,f_1,f_2) = 0$ and $J(e,f_1,f_1,f_2) = 0$ we get $\xi_3 = 0$ and $\xi_2 = \xi_1 = 0$, respectively. As a result, we have the superalgebra $\mathbf{J_6}$:

$$ef_1 = f_2,\ ef_2 = f_3.$$

<u>Let $\mathfrak{J}_0 \cong \mathcal{U}_1$.</u>

**i)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1,\quad ef_2 = \mu_2 f_2,\quad ef_3 = \mu_3 f_3,\quad f_1 f_2 = \xi_1 e,\quad f_1 f_3 = \xi_2 e,\quad f_2 f_3 = \xi_3 e.$$

From $J(e,e,e,f_1) = 0$, $J(e,e,e,f_2) = 0$, and $J(e,e,e,f_3) = 0$ we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0,\quad i = \overline{1,3}.$$

Up to permutation of $f_1, f_2$ and $f_3$ we have ten possibilities:
$(\mu_1, \mu_2, \mu_3) \in \{(0,0,0),(0,0,\frac{1}{2}),(0,0,1),(0,\frac{1}{2},\frac{1}{2}),(0,\frac{1}{2},1),(0,1,1),(\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},1),(\frac{1}{2},1,1),(1,1,1)\}$.

    1. $(\mu_1, \mu_2, \mu_3) = (0,0,0)$. In this case, we have the following results

$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0,\quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0,\quad J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$

    Hence, the obtained superalgebra is $\mathbf{J_7}$.

2. $(\mu_1, \mu_2, \mu_3) = (0, 0, \frac{1}{2})$. In this case, we have the following results

$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_3,f_1) = 0 \Rightarrow \xi_2 = 0, \quad J(e,e,f_3,f_2) = 0 \Rightarrow \xi_3 = 0.$$

Hence, the the obtained superalgebra is $\mathbf{J_8}$.

3. $(\mu_1, \mu_2, \mu_3) = (0, 0, 1)$. In this case, we have the following results

$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0, \quad J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$

Hence, the the obtained superalgebra is $\mathbf{J_9}$.

4. $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{2}, \frac{1}{2})$. In this case, we have the following results

$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$

Hence, the obtained superalgebras are $\mathbf{J_{10}}$ and $\mathbf{J_{11}}$.

5. $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{2}, 1)$ In this case, we have the following results

$$J(e,e,f_2,f_1) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0, \quad J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$

Hence, the obtained superalgebra is $\mathbf{J_{12}}$.

6. $(\mu_1, \mu_2, \mu_3) = (0, 1, 1)$. In this case, we have the following results

$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$

Hence, the obtained superalgebras are $\mathbf{J_{13}}$ and $\mathbf{J_{14}}$.

7. $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{2}, \frac{1}{2})$. In this case, we have the following results

$$J(f_1,f_2,e,f_3) = 0 \Rightarrow \xi_1 = \xi_2 = \xi_3 = 0.$$

Hence, the obtained superalgebra is $\mathbf{J_{15}}$.

8. $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, \frac{1}{2}, 1)$. In this case, we have the following results

$$J(e,f_1,f_2,f_3) = 0 \Rightarrow \xi_1 = \xi_3 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$

Hence, the obtained superalgebra is $\mathbf{J_{16}}$.

9. $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}, 1, 1)$. In this case, we have the following results

$$J(e,f_1,f_2,f_3) = 0 \Rightarrow \xi_1 = \xi_3 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$

Hence, the obtained superalgebra is $\mathbf{J_{17}}$.

10. $(\mu_1, \mu_2, \mu_3) = (1, 1, 1)$. In this case, the multiplication rules in the obtained superalgebra are $e^2 = e$, $ef_1 = f_1$, $ef_2 = f_2$, $ef_3 = f_3$, $f_1 f_2 = \xi_1 e$, $f_1 f_3 = \xi_2 e$, $f_2 f_3 = \xi_3 e$. When $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$ we get the superalgebra $\mathbf{J_{18}}$.

However, when $(\xi_1, \xi_2, \xi_3) \neq (0, 0, 0)$ we can assume that $\xi_1 \neq 0$ and by changing the basis as follows

$$f_1' = \frac{1}{\xi_1} f_1 + f_2, \quad f_2' = f_2, \quad f_3' = \xi_1 f_3 - \xi_2 f_2 + \xi_3 f_1,$$

we get the superalgebra $\mathbf{J_{19}}$.

**ii)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1 + f_2, \quad ef_2 = \mu_1 f_2, \quad ef_3 = \mu_3 f_3, \quad f_1 f_2 = \xi_1 e, \quad f_1 f_3 = \xi_2 e, \quad f_2 f_3 = \xi_3 e.$$

However, the following two equations

$$J(e,e,e,f_1) = 0 \Rightarrow 1 - 6\mu_1 + 6\mu_1^2 = 0, \quad J(e,e,e,f_2) = 0 \Rightarrow (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0,$$

which can not be satisfied at the same time, give a contradiction, thereby no superalgebras can be found in this subcase.

**iii)** Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1 + f_2, \quad ef_2 = \mu_1 f_2 + f_3, \quad ef_3 = \mu_1 f_3, \quad f_1 f_2 = \xi_1 e, \quad f_1 f_3 = \xi_2 e, \quad f_2 f_3 = \xi_3 e.$$

However, the following two equations

$$J(e,e,e,f_2) = 0 \;\Rightarrow\; 1 - 6\mu_1 + 6\mu_1^2 = 0, \quad J(e,e,e,f_3) = 0 \;\Rightarrow\; (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0,$$

which can not be satisfied at the same time, give a contradiction, thereby no superalgebras can be found in this subcase. $\qquad\square$

**Theorem 11.** *Up to isomorphism there are 71 Jordan superalgebras of type $(2,2)$, which are presented below with some additional information:*

| № | Orbit | Multiplication rules | Decomposition |
|---|---|---|---|
| $\mathscr{J}_1$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_2$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_2 = f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_2^2 \oplus S_1^1$ |
| $\mathscr{J}_3$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_2 = \frac{1}{2} f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_1^2 \oplus S_1^1$ |
| $\mathscr{J}_4$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_1 = f_1,\ e_2 f_2 = f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_6^3$ |
| $\mathscr{J}_5$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_1 = f_1,\ e_2 f_2 = \frac{1}{2} f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_4^3$ |
| $\mathscr{J}_6$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_1 = f_1,\ e_2 f_2 = f_2,\ f_1 f_2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_8^3$ |
| $\mathscr{J}_7$ | 10 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_1 = \frac{1}{2} f_1,\ e_2 f_2 = \frac{1}{2} f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_5^3$ |
| $\mathscr{J}_8$ | 11 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_2 f_1 = \frac{1}{2} f_1,\ e_2 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_7^3$ |
| $\mathscr{J}_9$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_2 = \frac{1}{2} f_2$ | $\mathscr{S}_{13}^3 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{10}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_1$ | $\mathscr{S}_1^2 \oplus \mathscr{S}_2^2$ |
| $\mathscr{J}_{11}$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_1,\ e_2 f_2 = \frac{1}{2} f_2$ | *Indecomposable* |
| $\mathscr{J}_{12}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = \frac{1}{2} f_1$ | $\mathscr{S}_1^2 \oplus \mathscr{S}_1^2$ |
| $\mathscr{J}_{13}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = \frac{1}{2} f_1,\ e_2 f_2 = \frac{1}{2} f_2$ | *Indecomposable* |
| $\mathscr{J}_{14}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_2 = f_2,\ e_2 f_1 = f_1$ | $\mathscr{S}_2^2 \oplus \mathscr{S}_2^2$ |
| $\mathscr{J}_{15}$ | 10 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2, e_2 f_1 = \frac{1}{2} f_1,\ e_2 f_2 = \frac{1}{2} f_2$ | *Indecomposable* |
| $\mathscr{J}_{16}^{\mathbf{t}}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,$ $e_2 f_1 = \frac{1}{2} f_1,\ e_2 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_1 + t e_2$ | *Indecomposable* |
| $\mathscr{J}_{17}$ | 6 | $f_1 f_2 = e_1$ | $\mathscr{U}_2 \oplus \mathscr{S}_2^3$ |
| $\mathscr{J}_{18}$ | 7 | $e_2 f_1 = f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_3^3$ |
| $\mathscr{J}_{19}$ | 9 | $e_2 f_1 = f_2,\ f_1 f_2 = e_1$ | *Indecomposable* |
| $\mathscr{J}_{20}$ | 10 | $e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_1^3$ |
| $\mathscr{J}_{21}$ | 7 | $e_1^2 = e_1$ | $\mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{22}$ | 10 | $e_1^2 = e_1,\ f_1 f_2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_2^3$ |
| $\mathscr{J}_{23}$ | 11 | $e_1^2 = e_1,\ e_2 f_1 = f_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_3^3$ |

| | | | |
|---|---|---|---|
| $\mathscr{J}_{24}$ | 12 | $e_1^2 = e_1,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{S}_1^3$ |
| $\mathscr{J}_{25}$ | 10 | $e_1^2 = e_1,\ e_1 f_2 = \frac{1}{2} f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_1^2 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{26}$ | 10 | $e_1^2 = e_1,\ e_1 f_2 = f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_2^2 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{27}$ | 9 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_5^3$ |
| $\mathscr{J}_{28}$ | 10 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{29}$ | 10 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_1$ | $\mathscr{U}_2 \oplus \mathscr{S}_7^3$ |
| $\mathscr{J}_{30}$ | 11 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_1 + e_2$ | Indecomposable |
| $\mathscr{J}_{31}$ | 11 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{32}$ | 12 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{33}$ | 12 | $e_1^2 = e_1,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_4^3$ |
| $\mathscr{J}_{34}$ | 11 | $e_1^2 = e_1,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2$ | $\mathscr{U}_2 \oplus \mathscr{S}_6^3$ |
| $\mathscr{J}_{35}$ | 12 | $e_1^2 = e_1,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_1$ | $\mathscr{U}_2 \oplus \mathscr{S}_8^3$ |
| $\mathscr{J}_{36}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2$ | $\mathscr{B}_1 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{37}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_2 = \frac{1}{2} f_2$ | $\mathscr{S}_9^3 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{38}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_2 = f_2$ | $\mathscr{S}_{10}^3 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{39}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2$ | Indecomposable |
| $\mathscr{J}_{40}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{41}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{42}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{43}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2$ | Indecomposable |
| $\mathscr{J}_{44}$ | 7 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2$ | Indecomposable |
| $\mathscr{J}_{45}$ | 8 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{46}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_1$ | Indecomposable |
| $\mathscr{J}_{47}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_1 + e_2$ | Indecomposable |
| $\mathscr{J}_{48}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{49}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{50}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2$ | $\mathscr{B}_2 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{51}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2$ | $\mathscr{S}_{11}^3 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{52}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{53}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_2 = f_1$ | Indecomposable |
| $\mathscr{J}_{54}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_2 = f_1,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{55}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{56}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \frac{1}{2} f_2,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{57}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = f_2$ | $\mathscr{S}_{12}^3 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{58}$ | 4 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = \frac{1}{2} f_2$ | Indecomposable |
| $\mathscr{J}_{59}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2$ | Indecomposable |
| $\mathscr{J}_{60}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{61}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2,\ e_2 f_2 = f_1$ | Indecomposable |

| | | | |
|---|---|---|---|
| $\mathscr{J}_{62}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2,\ e_2 f_2 = f_1,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{63}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{64}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = \frac{1}{2} f_1,\ e_1 f_2 = f_2,\ e_2 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{65}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2$ | Indecomposable |
| $\mathscr{J}_{66}$ | 6 | $e_1^2 = e_2$ | $\mathscr{B}_3 \oplus \mathscr{S}_1^1 \oplus \mathscr{S}_1^1$ |
| $\mathscr{J}_{67}$ | 7 | $e_1^2 = e_2,\ f_1 f_2 = e_2$ | Indecomposable |
| $\mathscr{J}_{68}$ | 10 | $e_1^2 = e_2,\ f_1 f_2 = e_1$ | Indecomposable |
| $\mathscr{J}_{69}$ | 11 | $e_1^2 = e_2,\ e_2 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{70}$ | 8 | $e_1^2 = e_2,\ e_1 f_1 = f_2$ | Indecomposable |
| $\mathscr{J}_{71}$ | 10 | $e_1^2 = e_2,\ e_1 f_1 = f_2,\ f_1 f_2 = e_2$ | Indecomposable |

*Proof.* Let $\mathfrak{J}_0 \cong \mathscr{U}_1 \oplus \mathscr{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F} e_1 + \mathbb{F} e_2) + (\mathbb{F} f_1 + \mathbb{F} f_2)$ with multiplication rules

$$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2,\ e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,\ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,\ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ have one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},\quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f_1 = \mu_1 f_1,\ e_1 f_2 = \mu_2 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,\ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,\ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_1 - 1)\mu_1 (2\mu_1 - 1) = 0,\quad i = \overline{1,2}.$$

Up to permutation of $f_1$ and $f_2$ we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0,0), (0, \tfrac{1}{2}), (0,1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1,1)\}.$$

    1. $(\mu_1, \mu_2) = (0,0)$ In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ have one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix},\quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have $\xi_1 = 0$, $(\tau_1 - 1)\tau_1(2\tau_1 - 1) = 0$, $(\tau_2 - 1)\tau_2(2\tau_2 - 1) = 0$ from $J(e_1, e_1, f_1, f_2) = 0$, $J(e_2, e_2, e_2, f_1) = 0$ and $J(e_2, e_2, e_2, f_2) = 0$, respectively.

- If $\tau_1 = 0$ then from
$$J(e_2, e_2, f_1, f_2) = 0 \;\Rightarrow\; \xi_2(2\tau_2 - 1) = 0, \quad J(e_2, e_2, f_2, f_1) = 0 \;\Rightarrow\; \xi_2(\tau_2 - 1) = 0.$$
we get $\xi_2 = 0$ and $\tau_2 \in 0, 1, \frac{1}{2}$, which gives superalgebras $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$.

- If $\tau_1 = 1$ then we have
$$J(e_2, e_2, f_2, f_1) = 0 \;\Rightarrow\; \xi_2(\tau_2 - 1) = 0.$$
So, either $\xi_2 = 0$ and we have Jordan superalgebras $\mathcal{J}_4$ and $\mathcal{J}_5$ with $\tau_2 \in \{1, \frac{1}{2}\}$, or $\xi_2 \neq 0$ and $\tau_2 = 1$ which gives us the Jordan superalgebra $\mathcal{J}_6$.

- If $\tau_1 = \frac{1}{2}$ then from
$$J(e_2, e_2, f_1, f_2) = 0 \;\Rightarrow\; \xi_2(2\tau_2 - 1) = 0.$$
So, either $\xi_2 = 0$ and we have a Jordan superalgebra $\mathcal{J}_7$ with $\tau_2 = \frac{1}{2}$ ( $\tau_2 \in \{0, 1\}$ repeat the previous superalgebras), or $\xi_2 \neq 0$ and $\tau_2 = \frac{1}{2}$ which gives us the Jordan superalgebra $\mathcal{J}_8$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:
$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (1 - 6\tau_1 + 6\tau_1^2) = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \;\Rightarrow\; (\tau_1 - 1)\tau_1(2\tau_1 - 1) = 0.$$

This contradiction implies that no superalgebras can be found in this case.

2. $(\mu_1, \mu_2) = (0, \frac{1}{2})$. In this case we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0, \quad J(e_1, e_1, f_2, e_2) = 0 \;\Rightarrow\; \beta_3 = 0,$$
$$J(e_1, e_1, f_2, f_1) = 0 \;\Rightarrow\; \xi_1 = \xi_2 = 0, \quad J(e_1, e_2, e_2, f_2) = 0 \;\Rightarrow\; \beta_4(2\beta_4 - 1) = 0,$$
$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Hence, in this case, we obtain 6 Jordan superalgebras with $\beta_4 \in \{0, \frac{1}{2}\}$ and $\beta_1 \in \{0, 1, \frac{1}{2}\}$. Among them we have $\mathcal{J}_9, \mathcal{J}_{10}, \mathcal{J}_{11}, \mathcal{J}_{12}, \mathcal{J}_{13}$, while $\beta_1 = \beta_4 = 0$ gives a superalgebra isomorphic to previously obtained one.

3. $(\mu_1, \mu_2) = (0, 1)$ In this case we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0, \quad J(e_1, e_1, e_2, f_2) = 0 \;\Rightarrow\; \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \;\Rightarrow\; \xi_1 = 0, \quad J(e_1, e_1, f_2, f_1) = 0 \;\Rightarrow\; \xi_2 = 0,$$
$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Hence, in this case, we obtain 3 Jordan superalgebras with $\beta_1 \in \{0, 1, \frac{1}{2}\}$. Only $\beta_1 = 1$ gives a new superalgebra $\mathcal{J}_{14}$.

4. $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:
$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:
$$J(e_1, e_2, e_2, f_1) = 0 \;\Rightarrow\; \tau_1(2\tau_1 - 1) = 0, \quad J(e_1, e_2, e_2, f_4) = 0 \;\Rightarrow\; \tau_2(2\tau_2 - 1) = 0.$$

- If $(\tau_1, \tau_2) = (\frac{1}{2}, \frac{1}{2})$ then we get the well-known one parametric family of four-dimensional Jordan superalgebras, which we denoted as $\mathcal{J}_{16}^t$.

- If $(\tau_1, \tau_2) = (0, \frac{1}{2})$ then from $J(e_1, e_2, f_1, f_2) = 0$ we get $\xi_1 = \xi_2 = 0$ which gives a Jordan superalgebra isomorphic to $\mathscr{J}_{13}$.
- If $(\tau_1, \tau_2) = (0, 0)$ then from $J(e_1, e_2, f_1, f_2) = 0$ we get $\xi_2 = 0$ and hence the following Jordan superalgebra: $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 f_1 = \frac{1}{2} f_1$, $e_1 f_2 = \frac{1}{2} f_2$, $f_1 f_2 = \xi_1 e_1$. Though only $\xi_1 = 0$ gives us a new superalgebra $\mathscr{J}_{15}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_1, e_2, e_2, f_1) = 0 \Rightarrow 2\tau_1 - \frac{1}{2} = 0, \quad J(e_1, e_2, e_2, f_4) = 0 \Rightarrow \tau_1(2\tau_1 - 1) = 0.$$

This is a contradiction.

5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case we have the following results

$$J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0, \quad J(e_1, e_1, f_1, e_2) = 0 \Rightarrow \beta_2 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0, \quad J(e_1, e_2, e_2, f_3) = 0 \Rightarrow \beta_1(2\beta_1 - 1) = 0.$$

Hence, we have two Jordan superalgebras with $\beta_1 \in \{0, \frac{1}{2}\}$, which are isomorphic to $\mathscr{J}_5$ and $\mathscr{J}_{11}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \quad J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \tau_2 = 0, \quad J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_2 = 0.$$

Hence, we have the following Jordan superalgebra: $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 f_1 = f_1$, $e_1 f_2 = f_2$, $f_1 f_2 = \xi_1 e_1$ which repeats $\mathscr{J}_2$ and $\mathscr{J}_6$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow f_2 = 0.$$

Thus, there are no superalgebras in this case.

**ii)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 f_1 = \mu_1 f_1 + f_2, \quad e_1 f_2 = \mu_1 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \quad e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$1 - 6\mu_1 + 6\mu_1^2 = 0, \quad (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0.$$

Evidently, this is a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{U}_2 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F} e_1 + \mathbb{F} e_2) + (\mathbb{F} f_1 + \mathbb{F} f_2)$ with multiplication rules

$$e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \quad e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2, \quad e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \quad e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$e_1 f_1 = \mu_1 f_1, \quad e_1 f_2 = \mu_2 f_2, \quad e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \quad e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Using Jordan super identity we get the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \implies \mu_1 = 0, \quad J(e_1, e_1, e_1, f_2) = 0 \implies \mu_2 = 0.$$

Thus, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \implies \tau_1 = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \implies \tau_2 = 0.$$

Hence, we have the Jordan superalgebra: $f_1 f_2 = \xi_1 e_1 + \xi_2 e_2$.

- If $(\xi_1, \xi_2) = (0, 0)$ the superalgebra is trivial.
- If $(\xi_1, \xi_2) \neq (0, 0)$ then by changing the basis as follows

$$e'_1 = \xi_1 e_1 + \xi_2 e_2, \ e'_2 = \begin{cases} e_2, & \text{if } \xi_1 \neq 0, \\ e_1, & \text{if } \xi_1 = 0. \end{cases}$$

we get the superalgebra $\mathscr{J}_{17}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \implies \tau_1 = 0.$$

Hence, we have the Jordan superalgebra: $e_2 f_1 = f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2$.

- If $(\xi_1, \xi_2) = (0, 0)$, we have the superalgebra $\mathscr{J}_{18}$.
- If $\xi_2 = 0, \xi_1 \neq 0$, then by changing $f'_1 = \frac{1}{\sqrt{\xi_1}} f_1, \ f'_2 = \frac{1}{\sqrt{\xi_1}} f_2$ we get the superalgebra $\mathscr{J}_{19}$.
- If $\xi_2 \neq 0$, then changing $f'_1 = \frac{1}{\sqrt{\xi_2}} f_1, \ f'_2 = \frac{1}{\sqrt{\xi_2}} f_2, \ \frac{\xi_1}{\xi_2} = t$ we have

$$e_2 f'_1 = f'_2, \ f'_1 f'_2 = t e_1 + e_2.$$

Further, by changing $e'_2 = t e_1 + e_2$ we obtain the superalgebra $\mathscr{J}_{20}$.

Let $\mathfrak{J}_0 \cong \mathcal{U}_1 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_1, \quad e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \quad e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2, \quad e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,$$
$$e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \quad e_1 f_1 = \mu_1 f_1, \quad e_1 f_2 = \mu_2 f_2, \quad e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \quad e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1,2}.$$

Up to permutation of $f_1$ and $f_2$ we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0,0), (0,\tfrac{1}{2}), (0,1), (\tfrac{1}{2},\tfrac{1}{2}), (\tfrac{1}{2},1), (1,1)\}.$$

1. $(\mu_1, \mu_2) = (0,0)$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e_2, e_2, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \Rightarrow \tau_2 = 0.$$

So we have the superalgebras $\mathscr{J}_{21}$ and $\mathscr{J}_{22}$. When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \Rightarrow \tau_1 = 0.$$

Which gives us the superalgebras $\mathscr{J}_{23}$ and $\mathscr{J}_{24}$.

2. $(\mu_1, \mu_2) = (0, \tfrac{1}{2})$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_2 = 0, \quad J(e_1, e_1, f_2, e_2) = 0 \Rightarrow \beta_3 = 0,$$
$$J(e_1, e_1, f_2, f_1) = 0 \Rightarrow \xi_1 = \xi_2 = 0, \quad J(e_1, e_2, e_2, f_2) = 0 \Rightarrow \beta_4 = 0,$$
$$J(e_2, e_2, e_2, f_1) = 0 \Rightarrow \beta_1 = 0.$$

Hence, we obtain the superalgebra $\mathscr{J}_{25}$.

3. $(\mu_1, \mu_2) = (0, 1)$. In this case, we have the following results

$$J(e_1, e_2, e_1, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0, \quad J(e_1, f_1, e_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0,$$
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_2 = 0, \quad J(e_2, e_2, e_2, f_1) = 0 \Rightarrow \beta_1 = 0.$$

Hence, we get the superalgebra $\mathscr{J}_{26}$.

4. $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_2, e_2, f_1) = 0 \implies \tau_1 = 0, \quad J(e_1, e_2, e_2, f_2) = 0 \implies \tau_2 = 0.$$

So we have the following superalgebra:

$$e_1^2 = e_1, \quad e_1 f_1 = \tfrac{1}{2} f_1, \quad e_1 f_2 = \tfrac{1}{2} f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2.$$

- If $\xi_1 = \xi_2 = 0$ then we have the superalgebra $\mathscr{J}_{27}$.
- If $\xi_1 = 0, \xi_2 \neq 0$ then by the change $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathscr{J}_{28}$.
- If $\xi_1 \neq 0, \xi_2 = 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ we get the superalgebra $\mathscr{J}_{29}$.
- If $\xi_1 \neq 0, \xi_2 \neq 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ and $e_2' = \frac{\xi_2}{\xi_1} e_2$ we obtain the superalgebra $\mathscr{J}_{30}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_2, e_2, f_1) = 0 \implies \tau_1 = 0, \quad J(e_1, e_2, f_1, f_1) = 0 \implies \xi_1 = 0.$$

So we have the superalgebras $\mathscr{J}_{31}$ and $\mathscr{J}_{32}$.

5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_2) = 0 \implies \beta_3 = \beta_4 = 0, \quad J(e_1, e_1, f_1, e_2) = 0 \implies \beta_2 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \implies \xi_1 = \xi_2 = 0, \quad J(e_1, e_2, e_2, f_1) = 0 \implies \beta_1 = 0.$$

Hence, the obtained superalgebra is $\mathscr{J}_{33}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \implies \tau_1 = 0, \quad J(e_1, e_1, e_2, f_2) = 0 \implies \tau_2 = 0, \quad J(e_1, e_1, f_1, f_2) = 0 \implies \xi_2 = 0.$$

So we have superalgebras $\mathscr{J}_{34}$ and $\mathscr{J}_{35}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_1, e_2, f_1) = 0 \implies 1 = 0.$$

Which is, obviously, a contradiction.

Let $\mathfrak{J}_0 \cong \mathscr{B}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F} e_1 + \mathbb{F} e_2) + (\mathbb{F} f_1 + \mathbb{F} f_2)$ with multiplication rules

$$e_1^2 = e_1, \quad e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \quad e_1 e_2 = e_2, \quad e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2, \quad e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,$$
$$e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_1 f_1 = \mu_1 f_1, \ e_1 e_2 = e_2, \ e_1 f_2 = \mu_2 f_2, \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,$$
$$e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0, \quad i = \overline{1,2}.$$

Up to permutation of $f_1$ and $f_2$ we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0,0), (0, \tfrac{1}{2}), (0,1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1,1)\}.$$

1. $(\mu_1, \mu_2) = (0,0)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \ \Rightarrow \ \beta_1 = \beta_2 = 0, \ \ J(e_1, e_1, e_2, f_2) = 0 \ \Rightarrow \ \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathscr{J}_{36}$.

2. $(\mu_1, \mu_2) = (0, \tfrac{1}{2})$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \ \Rightarrow \ \beta_1 = \beta_2 = 0, \ \ J(e_1, e_1, f_2, e_2) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1, e_2, e_2, f_2) = 0 \ \Rightarrow \ \beta_4 = 0, \ \ J(e_1, e_1, f_2, f_1) = 0 \ \Rightarrow \ \xi_1 = \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathscr{J}_{37}$.

3. $(\mu_1, \mu_2) = (0,1)$. In this case, we have the following results
$$J(e_1, e_2, e_1, f_1) = 0 \ \Rightarrow \ \beta_1 = \beta_2 = 0, \ \ J(e_1, e_1, e_2, f_2) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = \xi_2 = 0, \ \ J(f_2, e_2, e_2, e_2) = 0 \ \Rightarrow \ \beta_4 = 0.$$
Hence, the obtained superalgebra is $\mathscr{J}_{38}$.

4. $(\mu_1, \mu_2) = (\tfrac{1}{2}, \tfrac{1}{2})$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ have one of the following forms:
$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_2, e_2, f_1) = 0 \ \Rightarrow \ \tau_1 = 0, \ \ J(e_1, e_2, e_2, f_2) = 0 \ \Rightarrow \ \tau_2 = 0, \ \ J(e_1, e_2, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = 0.$$

So we have the superalgebras $\mathscr{J}_{39}$ and $\mathscr{J}_{40}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_2, e_2, f_1) = 0 \ \Rightarrow \ \tau_1 = 0, \ \ J(e_1, e_2, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = 0.$$

which gives us the superalgebras $\mathcal{J}_{41}$ and $\mathcal{J}_{42}$.

5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_2) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_1, e_1, f_1, e_2) = 0 \ \Rightarrow \ \beta_2 = 0,$$

$$J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = \xi_2 = 0, \ J(e_1, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_1 = 0, \ J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow \ \beta_4 = 0.$$

Hence, the obtained superalgebra is $\mathcal{J}_{43}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we can consider the action of $e_2$. By using a simple change of basis, the matrix of $L_{e_2}$ have one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \tau_1 = 0, \ \ J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow \ \tau_2 = 0.$$

So we have the following superalgebra:

$$e_1^2 = e_1, \ \ e_1 f_1 = f_1, \ \ e_1 e_2 = e_2, \ \ e_1 f_2 = f_2, \ \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,$$

- If $(\xi_1, \xi_2) = (0, 0)$ we have the superalgebra $\mathcal{J}_{44}$.
- If $(\xi_1, \xi_2) \neq (0, 0)$
  - (a) When $\xi_1 = 0$, by changing $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathcal{J}_{45}$.
  - (b) When $\xi_1 \neq 0$, by changing $f_1' = \frac{1}{\xi_1} f_1$ and denoting $\frac{\xi_2}{\xi_1} = t$ we can write

$$e_1^2 = e_1, \ \ e_1 e_2 = e_2, \ \ e_1 f_1 = f_1, \ \ e_1 f_2 = f_2, \ \ f_1 f_2 = e_1 + t e_2.$$

In this case, we get $\mathcal{J}_{46}$ when $t = 0$ and $\mathcal{J}_{47}$ when $t \neq 0$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow \ \tau_1 = 0, \ \ J(e_2, e_2, f_1, f_1) = 0 \ \Rightarrow \ \xi_1 = 0.$$

So we have the superalgebras $\mathcal{J}_{48}$ and $\mathcal{J}_{49}$.

**ii)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_1 e_2 = e_2, \ e_1 f_1 = \mu_1 f_1 + f_2, \ e_1 f_2 = \mu_1 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

However, from

$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ 1 - 6\mu_1 + 6\mu_1^2 = 0, \ \ J(e_1, e_1, e_1, f_2) = 0 \ \Rightarrow \ (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0.$$

we get a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{B}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_1, \ e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \ e_1 e_2 = \frac{1}{2} e_2, \ e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_1 f_1 = \mu_1 f_1, \ e_1 e_2 = \frac{1}{2} e_2, \ e_1 f_2 = \mu_2 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0, \quad i = \overline{1,2}.$$

Up to permutation of $f_1$ and $f_2$ we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0,0), (0, \tfrac{1}{2}), (0,1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1,1)\}.$$

1. $(\mu_1, \mu_2) = (0,0)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_1 = \beta_2 = 0, \ J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathscr{J}_{50}$.

2. $(\mu_1, \mu_2) = (0, \tfrac{1}{2})$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_1 = 0, \ J(e_1, e_1, f_2, f_1) = 0 \Rightarrow \xi_1 = 0, \ J(e_1, e_2, e_1, f_2) = 0 \Rightarrow \beta_4 = 0,$$
$$J(e_1, e_2, e_2, f_1) = 0 \Rightarrow \beta_2 \beta_3 = 0.$$
We consider the following subcases:
   (a) $\beta_2 = 0$. If $\beta_3 = 0$ then we take the superalgebra $\mathscr{J}_{51}$ when $\xi_2 = 0$, and when $\xi_2 \neq 0$ we can change $f_1' = \frac{1}{\xi_2} f_1$ and thereby obtain superalgebra $\mathscr{J}_{52}$.
   If $\beta_3 \neq 0$, we get the superalgebra $\mathscr{J}_{53}$ when $\xi_2 = 0$, and when $\xi_2 \neq 0$ then we change the basis by taking $f_1' = \sqrt{\frac{\beta_3}{\xi_2}} f_1$, $f_2' = \frac{1}{\sqrt{\beta_3 \xi_2}} f_2$, which gives us the superalgebra $\mathscr{J}_{54}$.
   (b) $\beta_3 = 0$, $\beta_2 \neq 0$. In this subcase we obtain superalgebras $\mathscr{J}_{55}$ (when $\xi_2 = 0$) and $\mathscr{J}_{56}$ (when $\xi_2 \neq 0$).

3. $(\mu_1, \mu_2) = (0,1)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_1 = \beta_2 = 0, \ J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0.$$
This gives us the Jordan superalgebra $\mathscr{J}_{57}$.

4. $(\mu_1, \mu_2) = (\tfrac{1}{2}, \tfrac{1}{2})$. In this case, we have the following results
$$J(e_1, e_2, e_1, f_1) = 0 \Rightarrow \beta_1 = \beta_2 = 0, \ J(e_1, e_2, e_1, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, f_1, e_1, f_2) = 0 \Rightarrow \xi_2 = 0, \ J(e_2, f_1, e_1, f_2) = 0 \Rightarrow \xi_1 = 0.$$
This gives us the Jordan superalgebra $\mathscr{J}_{58}$.

5. $(\mu_1, \mu_2) = (\tfrac{1}{2}, 1)$. In this case, we have the following results

$$J(e_1,e_1,e_2,f_2) = 0 \Rightarrow \beta_4 = 0, \quad J(e_1,e_1,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e_1,e_2,e_1,f_1) = 0 \Rightarrow \beta_1 = 0,$$
$$J(e_1,e_2,e_2,f_2) = 0 \Rightarrow \beta_2\beta_3 = 0.$$

This case, being similar to the second one, gives us the Jordan superalgebras $\mathscr{J}_{59} - \mathscr{J}_{64}$.

6. $(\mu_1, \mu_2) = (1,1)$. In this case, we have the following results

$$J(e_1,e_1,e_2,f_1) = 0 \Rightarrow \beta_1 = \beta_2 = 0, \quad J(e_1,e_1,e_2,f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1,e_1,f_1,f_2) = 0 \Rightarrow \xi_2 = 0, \quad J(e_2,f_1,e_1,f_2) = 0 \Rightarrow \xi_1 = 0.$$

Here we obtain the Jordan superalgebra $\mathscr{J}_{65}$.

**ii)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_1 f_1 = \mu_1 f_1 + f_2, \ e_1 e_2 = \frac{1}{2} e_2, \ e_1 f_2 = \mu_1 f_2, \ e_2^2 = 0,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

However, from the following

$$J(e_1,e_1,e_1,f_1) = 0 \Rightarrow 1 - 6\mu_1 + 6\mu_1^2 = 0, \quad J(e_1,e_1,e_1,f_2) = 0 \Rightarrow (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0,$$

we get a contradiction.

Let $\mathfrak{J}_0 \cong \mathscr{B}_3$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_2, \ e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \ e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator $L_{e_1}$ on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of $L_{e_1}$ has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \ \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

**i)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_2, \ e_1 f_1 = \mu_1 f_1, \ e_1 f_2 = \mu_2 f_2, \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Let's assume that $\mu_1 \neq 0$. Then, from

$$J(e_1,e_1,e_1,f_1) = 0 \Rightarrow \beta_2 = 0, \quad J(e_2,e_2,e_2,f_1) = 0 \Rightarrow \beta_1 = 0, \quad J(e_1,e_1,e_1,f_1) = 0 \Rightarrow \mu_1 = 0,$$

results, we get a contradiction. So $\mu_1 = 0$.

Further, let's assume that $\mu_2 \neq 0$. Then, from

$$J(e_1,e_1,e_1,f_2) = 0 \Rightarrow \beta_3 = 0, \quad J(e_1,e_1,e_2,f_1) = 0 \Rightarrow \beta_1 = 0,$$

$$J(e_2,e_2,e_2,f_2) = 0 \Rightarrow \beta_4 = 0, \quad J(e_1,e_1,e_1,f_2) = 0 \Rightarrow \mu_2 = 0.$$

results we get a contradiction again. So $\mu_2 = 0$.

With obtained results we can consider the action of the operator $L_{e_2}$ on $\mathfrak{J}_1$ which can be written in one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

**i.1)** $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ gives the following results:

$$J(e_1,e_1,e_2,f_1) = 0 \Rightarrow \tau_1 = 0, \quad J(e_1,e_1,e_2,f_2) = 0 \Rightarrow \tau_2 = 0,$$

by which we obtain the following superalgebra:

$$e_1^2 = e_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2.$$

   a) If $\xi_1 = 0, \xi_2 = 0$ we have the superalgebra $\mathscr{J}_{66}$.
   b) If $\xi_1 = 0, \xi_2 \neq 0$, then by changing $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathscr{J}_{67}$.
   c) If $\xi_1 \neq 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ and $e_1' = e_1 + \frac{\xi_2}{\xi_1} e_2$ we obtain the superalgebra $\mathscr{J}_{68}$.

**i.2)** $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ gives the following results:

$$J(e_1,e_1,e_2,f_1) = 0 \Rightarrow \tau_1 = 0, \quad J(e_1,e_1,f_1,f_1) = 0 \Rightarrow \xi_1 = \xi_2 = 0,$$

by which we obtain $\mathscr{J}_{69}$.

**ii)** Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_2, \ e_1 f_1 = \mu_1 f_1 + f_2, \ e_1 f_2 = \mu_1 f_2, \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Then we have the following results:

$$J(e_1,e_1,f_1,e_1) = 0 \Rightarrow \beta_3 = 0, \ J(e_2,e_2,e_2,f_2) = 0 \Rightarrow \beta_4 = 0, \ J(e_1,e_1,e_1,f_2) = 0 \Rightarrow \mu_1 = 0,$$
$$J(e_1,e_1,f_1,e_1) = 0 \Rightarrow \beta_1 = 0, \ J(e_1,e_1,f_1,f_1) = 0 \Rightarrow \beta_2 \xi_1 = 0, \ \beta_2 \xi_2 = 2\xi_1.$$

If $\beta_2 = 0$ then we have $\xi_1 = 0$ and get the superalgebras $\mathscr{J}_{70}$ and $\mathscr{J}_{71}$.
If $\beta_2 \neq 0$ then we have $\xi_1 = \xi_2 = 0$. Then by changing the basis as $e_1' = e_1 - \frac{1}{\beta_2} e_2$, $f_2' = \beta_2 f_2$ we obtain the superalgebra

$$e_1'^2 = e_2, \ e_2 f_1 = f_2'.$$

However, this superalgebra is isomorphic to $\mathscr{J}_{69}$. $\qquad\square$

**Theorem 12.** *Up to isomorphism there are 59 Jordan superalgebras of type $(3,1)$, which are presented below with some additional information:*

| № | Orbit | Multiplication rules | Decomposition |
|---|---|---|---|
| $\mathfrak{J}_1$ | 15 | $e_1^2 = e_1, \ e_2^2 = e_2, \ e_3^2 = e_3$ | $\mathcal{U}_1 \oplus \mathcal{U}_1 \oplus \mathcal{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_2$ | 14 | $e_1^2 = e_1, \ e_2^2 = e_2, \ e_3^2 = e_3, \ e_1 f = f$ | $\mathscr{S}_2^2 \oplus \mathcal{U}_1 \oplus \mathcal{U}_1$ |
| $\mathfrak{J}_3$ | 14 | $e_1^2 = e_1, \ e_2^2 = e_2, \ e_3^2 = e_3, \ e_1 f = \frac{1}{2} f$ | $\mathscr{S}_1^2 \oplus \mathcal{U}_1 \oplus \mathcal{U}_1$ |
| $\mathfrak{J}_4$ | 14 | $e_1^2 = e_1, \ e_2^2 = e_2, \ e_3^2 = e_3, \ e_1 f = \frac{1}{2} f, \ e_2 f = \frac{1}{2} f$ | $\mathscr{S}_{13}^3 \oplus \mathcal{U}_1$ |

| $\mathfrak{J}_5$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2$ | $\mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
|---|---|---|---|
| $\mathfrak{J}_6$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f = f$ | $\mathscr{S}_2^2 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_7$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_1^2 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_8$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 f = \frac{1}{2}f,\ e_2 f = \frac{1}{2}f$ | $\mathscr{S}_{13}^3 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_9$ | 7 | $e_1^2 = e_1$ | $\mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{10}$ | 10 | $e_1^2 = e_1,\ e_1 f = f$ | $\mathscr{S}_2^2 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{11}$ | 11 | $e_1^2 = e_1,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_1^2 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{12}$ | 14 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3$ | $\mathscr{B}_1 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{13}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3,\ e_3 f = f$ | $\mathscr{B}_1 \oplus \mathscr{S}_2^2$ |
| $\mathfrak{J}_{14}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3,\ e_3 f = \frac{1}{2}f$ | $\mathscr{B}_1 \oplus \mathscr{S}_2^1$ |
| $\mathfrak{J}_{15}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3,\ e_1 f = f$ | $\mathscr{S}_{10}^3 \oplus \mathscr{U}_1$ |
| $\mathfrak{J}_{16}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_9^3 \oplus \mathscr{U}_1$ |
| $\mathfrak{J}_{17}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_3^2 = e_3,\ e_1 f = \frac{1}{2}f,\ e_3 f = \frac{1}{2}f$ | Indecomposable |
| $\mathfrak{J}_{18}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2$ | $\mathscr{B}_1 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{19}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f = f$ | $\mathscr{S}_{10}^3 \oplus \mathscr{U}_2^1$ |
| $\mathfrak{J}_{20}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_9^3 \oplus \mathscr{U}_2^1$ |
| $\mathfrak{J}_{21}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3$ | $\mathscr{B}_2 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{22}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3,\ e_3 f = f$ | $\mathscr{B}_2 \oplus \mathscr{S}_2^2$ |
| $\mathfrak{J}_{23}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3,\ e_3 f = \frac{1}{2}f$ | $\mathscr{B}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{24}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3,\ e_1 f = f$ | $\mathscr{S}_{12}^3 \oplus \mathscr{U}_1$ |
| $\mathfrak{J}_{25}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_{11}^3 \oplus \mathscr{U}_1$ |
| $\mathfrak{J}_{26}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_3^2 = e_3,\ e_1 f = \frac{1}{2}f,\ e_3 f = \frac{1}{2}f$ | Indecomposable |
| $\mathfrak{J}_{27}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2$ | $\mathscr{B}_2 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{28}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_1 f = f$ | $\mathscr{S}_{12}^3 \oplus \mathscr{U}_2$ |
| $\mathfrak{J}_{29}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2}e_2,\ e_1 f = \frac{1}{2}f$ | $\mathscr{S}_{11}^3 \oplus \mathscr{U}_2^1$ |
| $\mathfrak{J}_{30}$ | 11 | $e_1^2 = e_2,\ e_3^2 = e_3$ | $\mathscr{B}_3 \oplus \mathscr{U}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{31}$ | 12 | $e_1^2 = e_2,\ e_3^2 = e_3,\ e_3 f = f$ | $\mathscr{S}_2^2 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2$ |
| $\mathfrak{J}_{32}$ | 12 | $e_1^2 = e_2,\ e_3^2 = e_3,\ e_3 f = \frac{1}{2}f$ | $\mathscr{S}_1^2 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2$ |
| $\mathfrak{J}_{33}$ | 6 | $e_1^2 = e_2$ | $\mathscr{B}_3 \oplus \mathscr{U}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{34}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_2^2 = e_3$ | $\mathscr{T}_1 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{35}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_2^2 = e_3,\ e_1 f = f$ | Indecomposable |
| $\mathfrak{J}_{36}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_2^2 = e_3,\ e_1 f = \frac{1}{2}f$ | Indecomposable |
| $\mathfrak{J}_{37}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3$ | $\mathscr{T}_2 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{38}$ | 7 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_1 f = f$ | Indecomposable |
| $\mathfrak{J}_{39}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 e_3 = e_3,\ e_1 f = \frac{1}{2}f$ | Indecomposable |
| $\mathfrak{J}_{40}$ | 10 | $e_1^2 = e_2,\ e_1 e_2 = e_3$ | $\mathscr{T}_3 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{41}$ | 8 | $e_1^2 = e_2,\ e_1 e_3 = e_2$ | $\mathscr{T}_4 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{42}$ | 14 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_1 + e_2,\ e_1 e_3 = \frac{1}{2}e_3,\ e_2 e_3 = \frac{1}{2}e_3$ | $\mathscr{T}_5 \oplus \mathscr{S}_1^1$ |

| $\mathfrak{J}_{43}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_1 + e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_2 e_3 = \frac{1}{2} e_3,$ $e_1 f = \frac{1}{2} f,\ e_2 f = \frac{1}{2} f$ | *Indecomposable* |
|---|---|---|---|
| $\mathfrak{J}_{44}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = e_3$ | $\mathscr{T}_6 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{45}$ | 10 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = e_3,\ e_1 f = f$ | *Indecomposable* |
| $\mathfrak{J}_{46}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = e_3,\ e_1 f = \frac{1}{2} f$ | *Indecomposable* |
| $\mathfrak{J}_{47}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = \frac{1}{2} e_3$ | $\mathscr{T}_7 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{48}$ | 9 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_1 f = f$ | *Indecomposable* |
| $\mathfrak{J}_{49}$ | 4 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_1 f = \frac{1}{2} f$ | *Indecomposable* |
| $\mathfrak{J}_{50}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3$ | $\mathscr{T}_8 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{51}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3,\ e_1 f = f$ | *Indecomposable* |
| $\mathfrak{J}_{52}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3,\ e_1 f = \frac{1}{2} f$ | *Indecomposable* |
| $\mathfrak{J}_{53}$ | 13 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3,\ e_1 e_3 = e_3$ | $\mathscr{T}_9 \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{54}$ | 12 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3,\ e_1 e_3 = e_3,\ e_1 f = f$ | *Indecomposable* |
| $\mathfrak{J}_{55}$ | 11 | $e_1^2 = e_1,\ e_1 e_2 = \frac{1}{2} e_2,\ e_2^2 = e_3,\ e_1 e_3 = e_3,\ e_1 f = \frac{1}{2} f$ | *Indecomposable* |
| $\mathfrak{J}_{56}$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_2 e_3 = \frac{1}{2} e_3$ | $\mathscr{T}_{10} \oplus \mathscr{S}_1^1$ |
| $\mathfrak{J}_{57}$ | 13 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_2 e_3 = \frac{1}{2} e_3,\ e_1 f = f$ | *Indecomposable* |
| $\mathfrak{J}_{58}$ | 12 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_2 e_3 = \frac{1}{2} e_3,\ e_1 f = \frac{1}{2} f$ | *Indecomposable* |
| $\mathfrak{J}_{59}$ | 10 | $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1 e_3 = \frac{1}{2} e_3,\ e_2 e_3 = \frac{1}{2} e_3,\ e_1 f = \frac{1}{2} f,\ e_2 f = \frac{1}{2} f$ | *Indecomposable* |

*Proof.* Let $\mathfrak{J}_0 \cong \mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F} e_1 + \mathbb{F} e_2 + \mathbb{F} e_3) + \mathbb{F} f_1$ with multiplication rules

$$e_1^2 = e_1,\ \ e_3^2 = e_3,\ \ e_2^2 = e_2,\ \ e_i f_1 = \beta_i f_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow\ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,\ \ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0,$$

$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow\ (\beta_3 - 1)\beta_3(2\beta_3 - 1) = 0,\ \ J(e_1, e_2, e_3, f_1) = 0 \ \Rightarrow\ \beta_1 \beta_2 \beta_3 = 0.$$

- If $\beta_1 = 1$ then from $J(e_1, e_1, e_2, f_1) = 0$ and $J(e_1, e_1, e_3, f_1) = 0$ we get $\beta_2 = 0$ and $\beta_3 = 0$, respectively, which gives us the superalgebra $\mathfrak{J}_2$.
- If $\beta_1 = 0$ then
  - a) When $\beta_2 = 0$, from $J(e_3, e_3, e_3, f_1) = 0$ we get three superalgebras with $\beta_3 \in \{0, 1, \frac{1}{2}\}$. While $\beta_3 = 1$ gives a superalgebra that is isomorphic to $\mathfrak{J}_2$, from $\beta_3 \in \{0, \frac{1}{2}\}$ we obtain superalgebras $\mathfrak{J}_1$ and $\mathfrak{J}_3$.
  - b) When $\beta_2 = 1$, from $J(e_2, e_2, e_3, f_1) = 0$ we get $\beta_3 = 0$, which gives $\mathfrak{J}_2$.
  - c) When $\beta_2 = \frac{1}{2}$, from $J(e_2, e_3, e_3, f_1) = 0$ we get $\beta_3 \in \{0, \frac{1}{2}\}$, which gives superalgebras isomorphic to $\mathfrak{J}_3$ and $\mathfrak{J}_4$.
- If $\beta_1 = \frac{1}{2}$ from $J(e_1, e_2, e_2, f_1) = 0$ we get either $\beta_2 = 0$ or $\beta_2 = \frac{1}{2}$. When the former occurs, we have two superalgebras with $\beta_3 \in \{0, \frac{1}{2}\}$, and when the latter does we have $\beta_3 = 0$. However, all superalgebras obtained here are isomorphic to those of previous steps.

Let $\mathfrak{J}_0 \cong \mathscr{U}_1 \oplus \mathscr{U}_1 \oplus \mathscr{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F} e_1 + \mathbb{F} e_2 + \mathbb{F} e_3) + \mathbb{F} f_1$ with multiplication rules

$$e_1^2 = e_1 \quad e_2^2 = e_2 \quad e_i f_1 = \beta_i f_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \; \Rightarrow \; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$

$$J(e_2, e_2, e_2, f_1) = 0 \; \Rightarrow \; (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0, \;\; J(e_3, e_3, e_3, f_1) = 0 \; \Rightarrow \; \beta_3 = 0.$$

- If $\beta_1 = 1$ then from $J(e_1, e_1, e_2, f_1) = 0$ we get $\beta_2 = 0$, which gives $\mathfrak{J}_6$.
- If $\beta_1 = 0$ then from $J(e_2, e_2, e_2, f_1) = 0$ we get three superalgebras with $\beta_2 \in \{0, 1, \frac{1}{2}\}$. New superalgebras here are $\mathfrak{J}_5$ and $\mathfrak{J}_7$.
- If $\beta_1 = \frac{1}{2}$ from $J(e_1, e_2, e_2, f_1) = 0$ we get either $\beta_2 = 0$ or $\beta_2 = \frac{1}{2}$. The only new superlgebra here is $\mathfrak{J}_8$.

Let $\mathfrak{J}_0 \cong \mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \quad e_i f_1 = \beta_i f_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \; \Rightarrow \; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$

$$J(e_2, e_2, e_2, f_1) = 0 \; \Rightarrow \; \beta_2 = 0, \;\; J(e_3, e_3, e_3, f_1) = 0 \; \Rightarrow \; \beta_3 = 0.$$

In this case we have three superalgebras with $\beta_1 \in \{0, 1, \frac{1}{2}\}$, which give us $\mathfrak{J}_9, \mathfrak{J}_{10}$ and $\mathfrak{J}_{11}$.

Let $\mathfrak{J}_0 \cong \mathscr{U}_2 \oplus \mathscr{U}_2 \oplus \mathscr{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_i f_1 = \beta_i f_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \; \Rightarrow \; \beta_1 = 0, \;\; J(e_2, e_2, e_2, f_1) = 0 \; \Rightarrow \; \beta_2 = 0, \;\; J(e_3, e_3, e_3, f_1) = 0 \; \Rightarrow \; \beta_3 = 0.$$

So in this case we obtain a trivial superalgebra.

Let $\mathfrak{J}_0 \cong \mathscr{B}_1 \oplus \mathscr{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_3^2 = e_3, \;\; e_1 e_2 = e_2, \;\; e_i f_1 = \beta_i f_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2, e_2, e_2, f_1) = 0 \; \Rightarrow \; \beta_2 = 0, \;\; J(e_1, e_1, e_1, f_1) = 0 \; \Rightarrow \; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$
$$J(e_3, e_3, e_3, f_1) = 0 \; \Rightarrow \; (\beta_3 - 1)\beta_3(2\beta_3 - 1) = 0, \;\; J(e_1, e_1, e_3, f_1) = 0 \; \Rightarrow \; \beta_1\beta_3(2\beta_1 - 1) = 0,$$
$$J(e_1, e_3, e_3, f_1) = 0 \; \Rightarrow \; \beta_1\beta_3(2\beta_3 - 1) = 0.$$

- If $\beta_1 = 0$, then $\beta_3 \in \{0, 1, \frac{1}{2}\}$, which gives $\mathfrak{J}_{12}$, $\mathfrak{J}_{13}$ and $\mathfrak{J}_{14}$.
- If $\beta_1 = 1$, then $\beta_3 = 0$, which gives us the superalgebra $\mathscr{J}_{15}$.
- If $\beta_1 = \frac{1}{2}$, then $\beta_3 \in \{0, \frac{1}{2}\}$, which gives us $\mathfrak{J}_{16}$ and $\mathfrak{J}_{17}$.

Let $\mathfrak{J}_0 \cong \mathscr{B}_1 \oplus \mathscr{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_1 e_2 = e_2, \;\; e_i f_1 = \beta_i f_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \;\Rightarrow\; \beta_2=0, \quad J(f_1,f_1,f_1,e_3)=0 \;\Rightarrow\; \beta_3=0,$$
$$J(e_1,e_1,e_1,f_1)=0 \;\Rightarrow\; (\beta_1-1)\beta_1(2\beta_1-1)=0.$$

So, we have three superalgebras with $\beta_1 \in \{0,1,\frac{1}{2}\}$, which give us $\mathfrak{J}_{18}$, $\mathfrak{J}_{19}$ and $\mathfrak{J}_{20}$.

$\underline{\text{Let } \mathfrak{J}_0 \cong \mathcal{B}_2 \oplus \mathcal{U}_1.}$ Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_3^2 = e_3, \;\; e_1e_2 = \tfrac{1}{2}e_2, \;\; e_if_1 = \beta_if_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \;\Rightarrow\; \beta_2=0, \quad J(e_1,e_1,e_1,f_1)=0 \;\Rightarrow\; (\beta_1-1)\beta_1(2\beta_1-1)=0,$$
$$J(e_3,e_3,e_3,f_1)=0 \;\Rightarrow\; (\beta_3-1)\beta_3(2\beta_3-1)=0, \quad J(e_1,e_1,e_3,f_1)=0 \;\Rightarrow\; \beta_1\beta_3(2\beta_1-1)=0,$$
$$J(e_1,e_3,e_3,f_1)=0 \;\Rightarrow\; \beta_1\beta_3(2\beta_3-1)=0.$$

- If $\beta_1 = 0$, then $\beta_3 \in \{0,1,\frac{1}{2}\}$ which gives us $\mathfrak{J}_{21}$, $\mathfrak{J}_{22}$ and $\mathfrak{J}_{23}$.
- If $\beta_1 = 1$, then $\beta_3 = 0$, which gives us the superalgebra $\mathfrak{J}_{24}$.
- If $\beta_1 = \frac{1}{2}$, then $\beta_3 \in \{0,\frac{1}{2}\}$, which gives $\mathfrak{J}_{25}$ and $\mathfrak{J}_{26}$.

$\underline{\text{Let } \mathfrak{J}_0 \cong \mathcal{B}_2 \oplus \mathcal{U}_2.}$ Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_1e_2 = \tfrac{1}{2}e_2 \;\; e_if_1 = \beta_if_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \;\Rightarrow\; \beta_2=0, \quad J(e_3,e_3,e_3,f_1)=0 \;\Rightarrow\; \beta_3=0,$$
$$J(e_1,e_1,e_1,f_1)=0 \;\Rightarrow\; (\beta_1-1)\beta_1(2\beta_1-1)=0.$$

So we have superalgebras $\mathfrak{J}_{27}$, $\mathfrak{J}_{28}$ and $\mathfrak{J}_{29}$ from $\beta_1 \in \{0,1,\frac{1}{2}\}$.

$\underline{\text{Let } \mathfrak{J}_0 \cong \mathcal{B}_3 \oplus \mathcal{U}_1.}$ Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \;\; e_3^2 = e_3 \;\; e_if_1 = \beta_if_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \;\Rightarrow\; \beta_2=0, \quad J(e_1,e_1,e_1,f_1)=0 \;\Rightarrow\; \beta_1=0,$$
$$J(e_3,e_3,e_3,f_1)=0 \;\Rightarrow\; (\beta_3-1)\beta_3(2\beta_3-1)=0.$$

So we have Jordan superalgebras $\mathfrak{J}_{30}$, $\mathfrak{J}_{31}$ and $\mathfrak{J}_{32}$ from $\beta_3 \in \{0,1,\frac{1}{2}\}$.

$\underline{\text{Let } \mathfrak{J}_0 \cong \mathcal{B}_3 \oplus \mathcal{U}_2.}$ Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \;\; e_if_1 = \beta_if_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \;\Rightarrow\; \beta_2=0, \quad J(e_1,e_1,e_1,f_1)=0 \;\Rightarrow\; \beta_1=0, \; J(e_3,e_3,e_3,f_1)=0 \;\Rightarrow\; \beta_3=0.$$

So the superalgebra in this case is $\mathfrak{J}_{33}$.

$\underline{\text{Let } \mathfrak{J}_0 \cong \mathcal{T}_1.}$ Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_2^2 = e_3, \;\; e_1e_2 = e_2, \;\; e_1e_3 = e_3, \;\; e_if_1 = \beta_if_1, \; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3,e_3,e_3,f_1)=0 \Rightarrow \beta_3=0, \quad J(e_2,e_2,e_2,f_1)=0 \Rightarrow \beta_2=0,$$
$$J(e_1,e_1,e_1,f_1)=0 \Rightarrow (\beta_1-1)\beta_1(2\beta_1-1)=0.$$

Thus, from $\beta_1 \in \{0,1,\frac{1}{2}\}$, we get $\mathfrak{J}_{34}$, $\mathfrak{J}_{35}$ and $\mathfrak{J}_{36}$.

Let $\underline{\mathfrak{J}_0 \cong \mathscr{T}_2}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \quad e_1e_3 = e_3, \quad e_1e_2 = e_2, \quad e_if_1 = \beta_if_1, \quad i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \Rightarrow \beta_2=0, \quad J(e_3,e_3,e_3,f_1)=0 \Rightarrow \beta_3=0,$$
$$J(e_1,e_1,e_1,f_1)=0 \Rightarrow (\beta_1-1)\beta_1(2\beta_1-1)=0.$$

This we obtain $\mathfrak{J}_{37}$, $\mathfrak{J}_{38}$ and $\mathfrak{J}_{39}$ from $\beta_1 \in \{0,1,\frac{1}{2}\}$.

Let $\underline{\mathfrak{J}_0 \cong \mathscr{T}_3}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_if_1 = \beta_if_1, \quad i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1)=0 \Rightarrow \beta_2=0, \quad J(e_3,e_3,e_3,f_1)=0 \Rightarrow \beta_3=0, \quad J(e_1,e_1,e_1,f_1)=0 \Rightarrow \beta_1=0.$$

Thus we have $\mathfrak{J}_{40}$.

Let $\underline{\mathfrak{J}_0 \cong \mathscr{T}_4}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_1e_3 = e_2, \quad e_if_1 = \beta_if_1, \quad i = \overline{1,3}.$$

Using the Jacobi super identity we obtain only the following result:

$$J(e_2,e_2,e_2,f_1)=0 \Rightarrow \beta_2=0, \quad J(e_3,e_3,e_3,f_1)=0 \Rightarrow \beta_3=0, \quad J(e_1,e_1,e_1,f_1)=0 \Rightarrow \beta_1=0.$$

Thus we obtain $\mathfrak{J}_{41}$.

Let $\underline{\mathfrak{J}_0 \cong \mathscr{T}_5}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1 \quad e_3^2 = e_1+e_2, \quad e_1e_3 = \tfrac{1}{2}e_3 \quad e_2^2 = e_2, \quad e_2e_3 = \tfrac{1}{2}e_3 \quad e_if_1 = \beta_if_1, \quad i = \overline{1,3}.$$

Using the Jacobi super identity we obtain only the following result:

$$J(e_1,e_1,e_1,f_1)=0 \Rightarrow (\beta_1-1)\beta_1(2\beta_1-1)=0.$$

If $\beta_1 = 0$ then from

$$J(e_1,e_1,e_3,f_1)=0 \Rightarrow \beta_3=0, \quad J(e_1,f_1,e_3,e_3)=0 \Rightarrow \beta_2=0,$$

we get $\mathfrak{J}_{42}$.

If $\beta_1 = 1$ then from

$$J(e_1,e_1,e_2,f_1)=0 \Rightarrow \beta_2=0, \quad J(e_1,e_1,e_3,f_1)=0 \Rightarrow \beta_3=0,$$
$$J(e_1,f_1,e_3,e_3)=0 \Rightarrow -\tfrac{1}{2}(1+\beta_2-2\beta_3^2)=0.$$

we get a contradiction. So there is no superalgebra in this case.

If $\beta_1 = \frac{1}{2}$ then from

$$J(e_1,e_3,e_1,f_1)=0 \Rightarrow \beta_3=0, \quad J(e_3,e_3,e_1,f_1)=0 \Rightarrow \beta_2=\tfrac{1}{2}.$$

we have $\mathfrak{J}_{43.}$

Let $\mathfrak{J}_0 \cong \mathcal{T}_6$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (e_1\mathbb{F} + e_2\mathbb{F} + e_3\mathbb{F}) + f_1\mathbb{F}$ with multiplication rules

$$e_1^2 = e_1, \ e_1e_2 = \tfrac{1}{2}e_2, \ e_1e_3 = e_3, \ e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1-1)\beta_1(2\beta_1-1) = 0, \ \ J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0,$$
$$J(f_1,e_3,e_3,e_3) = 0 \ \Rightarrow \ \beta_3 = 0.$$

Thus we get $\mathfrak{J}_{44}$, $\mathfrak{J}_{45}$ and $\mathfrak{J}_{46}$ from $\beta_1 = 0, \beta_1 = 1$ and $\beta_1 = \tfrac{1}{2}$, respectively.

Let $\mathfrak{J}_0 \cong \mathcal{T}_7$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ e_1e_2 = \tfrac{1}{2}e_2, \ e_1e_3 = \tfrac{1}{2}e_3, \ e_if_1 = \beta_if_1 \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0,$$
$$J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1-1)\beta_1(2\beta_1-1) = 0.$$

So, we get $\mathfrak{J}_{47}$, $\mathfrak{J}_{48}$ and $\mathfrak{J}_{49}$ superalgebras from $\beta_1 = 0, \beta_1 = 1$ and $\beta_1 = \tfrac{1}{2}$, respectively.

Let $\mathfrak{J}_0 \cong \mathcal{T}_8$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ e_1e_2 = \tfrac{1}{2}e_2, \ e_2^2 = e_3, \ e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0,$$
$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1-1)\beta_1(2\beta_1-1) = 0.$$

Thus we get $\mathfrak{J}_{50}$, $\mathfrak{J}_{51}$ and $\mathfrak{J}_{52}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_9$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ \ e_1e_2 = \tfrac{1}{2}e_2, \ \ e_1e_3 = e_3, \ \ e_2^2 = e_3, \ \ e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0,$$
$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1-1)\beta_1(2\beta_1-1) = 0.$$

Thus we obtain $\mathfrak{J}_{53}$, $\mathfrak{J}_{54}$ and $\mathfrak{J}_{55}$

Let $\mathfrak{J}_0 \cong \mathcal{T}_{10}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ e_1e_3 = \tfrac{1}{2}e_3, \ e_2^2 = e_2, \ e_2e_3 = \tfrac{1}{2}e_3, \ e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1-1)\beta_1(2\beta_1-1) = 0.$$

If $\beta_1 = 0$ then

$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0.$$

So we have three superalgebras isomorphic to $\mathfrak{J}_{56}$, $\mathfrak{J}_{57}$ and $\mathfrak{J}_{58}$.

If $\beta_1 = 1$ then $J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0$ gives us $\mathfrak{J}_{57}$.

If $\beta_1 = \frac{1}{2}$ then $J(e_1, e_2, e_2, f_1) = 0 \;\Rightarrow\; \frac{1}{2}\beta_2(2\beta_2 - 1) = 0$. So we have $\mathfrak{J}_{58}$ and $\mathfrak{J}_{59}$. $\qquad\square$

## 3. IRREDUCIBLE COMPONENTS

**Theorem 13.** *The variety of 4-dimensional Jordan superalgebras of type $(1,3)$ has dimension 12 and it has 11 irreducible components defined by*

$$\mathscr{C}_1 = \overline{\{\mathbf{J}_5\}}, \;\; \mathscr{C}_2 = \overline{\{\mathbf{J}_7\}}, \;\; \mathscr{C}_3 = \overline{\{\mathbf{J}_8\}}, \;\; \mathscr{C}_4 = \overline{\{\mathbf{J}_9\}}, \;\; \mathscr{C}_5 = \overline{\{\mathbf{J}_{11}\}},$$
$$\mathscr{C}_6 = \overline{\{\mathbf{J}_{12}\}}, \;\; \mathscr{C}_7 = \overline{\{\mathbf{J}_{14}\}}, \;\; \mathscr{C}_8 = \overline{\{\mathbf{J}_{15}\}}, \;\; \mathscr{C}_9 = \overline{\{\mathbf{J}_{16}\}}, \;\; \mathscr{C}_{10} = \overline{\{\mathbf{J}_{17}\}}, \;\; \mathscr{C}_{11} = \overline{\{\mathbf{J}_{19}\}},$$

*In particular, all of them are rigid superalgebras.*

*Proof.* After carefully checking the dimensions of orbit closures of the more important for us superalgebras, we have

$$
\begin{aligned}
\dim \mathscr{O}(\mathbf{J}_5) = \dim \mathscr{O}(\mathbf{J}_{12}) = \dim \mathscr{O}(\mathbf{J}_{14}) &= 12, \\
\dim \mathscr{O}(\mathbf{J}_8) = \dim \mathscr{O}(\mathbf{J}_9) = \dim \mathscr{O}(\mathbf{J}_{11}) = \dim \mathscr{O}(\mathbf{J}_{17}) = \dim \mathscr{O}(\mathbf{J}_{19}) &= 10, \\
\dim \mathscr{O}(\mathbf{J}_{16}) &= 9, \\
\dim \mathscr{O}(\mathbf{J}_7) &= 7, \\
\dim \mathscr{O}(\mathbf{J}_{15}) &= 4.
\end{aligned}
$$

If $E^t_{f_1}, E^t_{f_2}, E^t_{f_3}, E^t_e$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by $\mathbf{A} \xrightarrow{(E^t_{f_1}, E^t_{f_2}, E^t_{f_3}, E^t_e)} \mathbf{B}$.

| $\mathbf{J}_3$ | $\xrightarrow{(tf_1, f_2+f_3+te, f_3+te, te)}$ | $\mathbf{J}_1$ | $\mathbf{J}_5$ | $\xrightarrow{(tf_1, tf_2, f_3, e)}$ | $\mathbf{J}_2$ |
|---|---|---|---|---|---|
| $\mathbf{J}_5$ | $\xrightarrow{(f_1-f_3, f_2, tf_3-te, e)}$ | $\mathbf{J}_3$ | $\mathbf{J}_5$ | $\xrightarrow{(tf_1+f_2, tf_2, f_3, e)}$ | $\mathbf{J}_4$ |
| $\mathbf{J}_{12}$ | $\xrightarrow{(f_1+2f_2+2f_3, tf_2+2tf_3, t^2f_3, te)}$ | $\mathbf{J}_6$ | $\mathbf{J}_{14}$ | $\xrightarrow{(f_1, f_2, tf_3, e)}$ | $\mathbf{J}_{10}$ |
| $\mathbf{J}_{14}$ | $\xrightarrow{(f_1, f_2, tf_3, e)}$ | $\mathbf{J}_{13}$ | $\mathbf{J}_{19}$ | $\xrightarrow{(tf_1, f_2, f_3, e)}$ | $\mathbf{J}_{18}$ |

Below we list all the important reasons for necessary non-degenerations.

| Non-degenerations reasons | | |
|---|---|---|
| $\mathbf{J}_5$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_8, \mathbf{J}_9, \mathbf{J}_{11}, \mathbf{J}_{15}, \mathbf{J}_{16}, \mathbf{J}_{17}, \mathbf{J}_{19}$ | $\mathscr{R} = \Big\{$ According to Lemma 6 (2) $\Big\}$ |
| $\mathbf{J}_{12}$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_8, \mathbf{J}_9, \mathbf{J}_{11}, \mathbf{J}_{15}, \mathbf{J}_{16}, \mathbf{J}_{17}, \mathbf{J}_{19}$ | $\mathscr{R} = \Big\{\, JJ \subset \{e, f_2, f_3\},\; c^4_{44} = 2c^2_{24},\; c^4_{44} = c^3_{34} \,\Big\}$ |
| $\mathbf{J}_{14}$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_8, \mathbf{J}_9, \mathbf{J}_{11}, \mathbf{J}_{15}, \mathbf{J}_{16}, \mathbf{J}_{17}, \mathbf{J}_{19}$ | $\mathscr{R} = \Big\{\, JJ \subset \{e, f_2, f_3\},\; c^4_{44} = c^3_{34},\; c^2_{42} = c^3_{34} \,\Big\}$ |
| $\mathbf{J}_8$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}, \mathbf{J}_{16}$ | $\mathscr{R} = \Big\{\, c^1_{ij} = 0,\; 2c^3_{34} = c^4_{44} \,\Big\}$ |
| $\mathbf{J}_9$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}, \mathbf{J}_{16}$ | $\mathscr{R} = \Big\{\, c^1_{ij} = 0,\; c^3_{34} = c^4_{44} \,\Big\}$ |
| $\mathbf{J}_{11}$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}, \mathbf{J}_{16}$ | $\mathscr{R} = \Big\{\, c^1_{ij} = 0,\; 2c^3_{34} = c^4_{44} \,\Big\}$ |
| $\mathbf{J}_{17}$ $\not\rightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}, \mathbf{J}_{16}$ | $\mathscr{R} = \Big\{\, A_2A_2 \subset A_2,\; 2c^1_{14} = c^4_{44},\; c^2_{24} = c^4_{44},\; c^3_{34} = c^4_{44} \,\Big\}$ |

| $\mathbf{J}_{19}$ | $\nrightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}, \mathbf{J}_{16}$ | $\mathscr{R} = \left\{ \ A_2 A_2 \subset A_2, \ c_{14}^1 = c_{44}^4, \ c_{24}^2 = c_{44}^4, \ c_{34}^3 = c_{44}^4 \ \right\}$ |
|---|---|---|---|
| $\mathbf{J}_{16}$ | $\nrightarrow$ | $\mathbf{J}_7, \mathbf{J}_{15}$ | $\mathscr{R} = \left\{ \ A_2 A_2 \subset A_2, \ 2c_{14}^1 = c_{44}^4, \ 2c_{24}^2 = c_{44}^4, \ c_{34}^3 = c_{44}^4 \ \right\}$ |
| $\mathbf{J}_7$ | $\nrightarrow$ | $\mathbf{J}_{15}$ | $\mathscr{R} = \left\{ \ JJ \subset \{e\} \ \right\}$ |

Here $c_{ij}^k$ coefficients are structural constants in the $x_1 = f_1$, $x_2 = f_2$, $x_3 = f_3$, $x_4 = e$ basis.

$\square$

**Theorem 14.** *The variety of 4-dimensional Jordan superalgebras of type $(2,2)$ has dimension* $13$ *and it has* $25$ *irreducible components defined by*

$$\mathscr{C}_1 = \overline{\{\mathscr{J}_1\}}, \ \ \mathscr{C}_2 = \overline{\{\mathscr{J}_2\}}, \ \ \mathscr{C}_3 = \overline{\{\mathscr{J}_3\}}, \ \ \mathscr{C}_4 = \overline{\{\mathscr{J}_5\}}, \ \ \mathscr{C}_5 = \overline{\{\mathscr{J}_6\}},$$

$$\mathscr{C}_6 = \overline{\{\mathscr{J}_8\}}, \ \ \mathscr{C}_7 = \overline{\{\mathscr{J}_9\}}, \ \ \mathscr{C}_8 = \overline{\{\mathscr{J}_{10}\}}, \ \ \mathscr{C}_9 = \overline{\{\mathscr{J}_{11}\}}, \ \ \mathscr{C}_{10} = \overline{\{\mathscr{J}_{12}\}}, \ \ \mathscr{C}_{11} = \overline{\{\mathscr{J}_{13}\}}, \ \ \mathscr{C}_{12} = \overline{\{\mathscr{J}_{14}\}}, \ \ \mathscr{C}_{13} = \overline{\{\mathscr{J}_{16}^t\}},$$

$$\mathscr{C}_{14} = \overline{\{\mathscr{J}_{24}\}}, \ \ \mathscr{C}_{15} = \overline{\{\mathscr{J}_{32}\}}, \ \ \mathscr{C}_{16} = \overline{\{\mathscr{J}_{42}\}}, \ \ \mathscr{C}_{17} = \overline{\{\mathscr{J}_{49}\}}, \ \ \mathscr{C}_{18} = \overline{\{\mathscr{J}_{50}\}}, \ \ \mathscr{C}_{19} = \overline{\{\mathscr{J}_{54}\}}, \ \ \mathscr{C}_{20} = \overline{\{\mathscr{J}_{56}\}},$$

$$\mathscr{C}_{21} = \overline{\{\mathscr{J}_{57}\}}, \ \ \mathscr{C}_{22} = \overline{\{\mathscr{J}_{58}\}}, \ \ \mathscr{C}_{23} = \overline{\{\mathscr{J}_{62}\}}, \ \ \mathscr{C}_{24} = \overline{\{\mathscr{J}_{64}\}}, \ \ \mathscr{C}_{25} = \overline{\{\mathscr{J}_{65}\}}.$$

*In particular,* $24$ *of them are rigid superalgebras.*

*Proof.* After carefully checking the dimensions of orbit closures of the more important for us super-algebras, we have

$$\dim \mathscr{O}(\mathscr{J}_2) = \dim \mathscr{O}(\mathscr{J}_3) = \dim \mathscr{O}(\mathscr{J}_5) = \dim \mathscr{O}(\mathscr{J}_6) \ = \ 13,$$

$$\dim \mathscr{O}(\mathscr{J}_9) = \dim \mathscr{O}(\mathscr{J}_{11}) = \dim \mathscr{O}(\mathscr{J}_{56}) \ = \ 13,$$

$$\dim \mathscr{O}(\mathscr{J}_1) = \dim \mathscr{O}(\mathscr{J}_{10}) = \dim \mathscr{O}(\mathscr{J}_{12}) = \dim \mathscr{O}(\mathscr{J}_{13}) = \dim \mathscr{O}(\mathscr{J}_{14}) = \dim \mathscr{O}(\mathscr{J}_{16}^t) \ = \ 12,$$

$$\dim \mathscr{O}(\mathscr{J}_{24}) = \dim \mathscr{O}(\mathscr{J}_{32}) = \dim \mathscr{O}(\mathscr{J}_{42}) = \dim \mathscr{O}(\mathscr{J}_{54}) = \dim \mathscr{O}(\mathscr{J}_{57}) = \dim \mathscr{O}(\mathscr{J}_{64}) \ = \ 12,$$

$$\dim \mathscr{O}(\mathscr{J}_8) = \dim \mathscr{O}(\mathscr{J}_{49}) = \dim \mathscr{O}(\mathscr{J}_{62}) \ = \ 11,$$

$$\dim \mathscr{O}(\mathscr{J}_{50}) = \dim \mathscr{O}(\mathscr{J}_{65}) \ = \ 10,$$

$$\dim \mathscr{O}(\mathscr{J}_{58}) \ = \ 4.$$

If $E_{e_1}^t, E_{e_2}^t, E_{f_1}^t, E_{f_2}^t$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by

$$\mathbf{A} \xrightarrow{\ (E_{e_1}^t, E_{e_2}^t, E_{f_1}^t, E_{f_2}^t)\ } \mathbf{B}.$$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\mathscr{J}_8$ | $\xrightarrow{(e_1, e_2, tf_1, f_2)}$ | $\mathscr{J}_7$ | $\mathscr{J}_{16}^0$ | $\xrightarrow{(e_1, e_2, f_1, tf_2)}$ | $\mathscr{J}_{15}$ | $\mathscr{J}_{68}$ | $\xrightarrow{(e_1, \frac{1}{t}e_2, f_1, f_2)}$ | $\mathscr{J}_{17}$ |
| $\mathscr{J}_{23}$ | $\xrightarrow{(te_1, e_2, f_1, f_2)}$ | $\mathscr{J}_{18}$ | $\mathscr{J}_{24}$ | $\xrightarrow{(te_1, e_2, f_1, f_2)}$ | $\mathscr{J}_{20}$ | $\mathscr{J}_{22}$ | $\xrightarrow{(e_1, e_2, tf_1, f_2)}$ | $\mathscr{J}_{21}$ |
| $\mathscr{J}_{24}$ | $\xrightarrow{(e_1, te_2, tf_1, f_2)}$ | $\mathscr{J}_{22}$ | $\mathscr{J}_{24}$ | $\xrightarrow{(e_1, e_2, tf_1, tf_2)}$ | $\mathscr{J}_{23}$ | $\mathscr{J}_{13}$ | $\xrightarrow{(e_1, te_2, f_1, f_2)}$ | $\mathscr{J}_{25}$ |
| $\mathscr{J}_{14}$ | $\xrightarrow{(e_1, te_2, f_1, f_2)}$ | $\mathscr{J}_{26}$ | $\mathscr{J}_{30}$ | $\xrightarrow{(e_1, te_2, f_1, tf_2)}$ | $\mathscr{J}_{28}$ | $\mathscr{J}_{30}$ | $\xrightarrow{(e_1, \frac{1}{t}e_2, f_1, f_2)}$ | $\mathscr{J}_{29}$ |
| $\mathscr{J}_{16}^t$ | $\xrightarrow{(e_1, te_2, f_1, f_2)}$ | $\mathscr{J}_{30}$ | $\mathscr{J}_{32}$ | $\xrightarrow{(e_1, \frac{1}{t}e_2, tf_1, f_2)}$ | $\mathscr{J}_{31}$ | $\mathscr{J}_{35}$ | $\xrightarrow{(e_1, e_2, f_1, f_2)}$ | $\mathscr{J}_{34}$ |
| $\mathscr{J}_1$ | $\xrightarrow{(e_1 + e_2, te_2, f_1, f_2)}$ | $\mathscr{J}_{36}$ | $\mathscr{J}_{40}$ | $\xrightarrow{(e_1, e_2, f_1, f_2)}$ | $\mathscr{J}_{39}$ | $\mathscr{J}_{42}$ | $\xrightarrow{(e_1, te_2, tf_1, f_2)}$ | $\mathscr{J}_{40}$ |
| $\mathscr{J}_{42}$ | $\xrightarrow{(e_1, e_2, tf_1, tf_2)}$ | $\mathscr{J}_{41}$ | $\mathscr{J}_5$ | $\xrightarrow{(e_1 + e_2, te_2, f_2, f_1)}$ | $\mathscr{J}_{43}$ | $\mathscr{J}_{45}$ | $\xrightarrow{(e_1, e_2, tf_1, f_2)}$ | $\mathscr{J}_{44}$ |
| $\mathscr{J}_{47}$ | $\xrightarrow{(e_1, te_2, f_1, tf_2)}$ | $\mathscr{J}_{45}$ | $\mathscr{J}_{47}$ | $\xrightarrow{(e_1, \frac{1}{t}e_2, f_1, f_2)}$ | $\mathscr{J}_{46}$ | $\mathscr{J}_{49}$ | $\xrightarrow{(e_1, e_2, tf_1, f_2)}$ | $\mathscr{J}_{48}$ |
| $\mathscr{J}_{52}$ | $\xrightarrow{(e_1, e_2, f_1, tf_2)}$ | $\mathscr{J}_{51}$ | $\mathscr{J}_{54}$ | $\xrightarrow{(e_1, e_2, tf_1, tf_2)}$ | $\mathscr{J}_{53}$ | $\mathscr{J}_{56}$ | $\xrightarrow{(e_1, e_2, tf_1, tf_2)}$ | $\mathscr{J}_{55}$ |

| | | | | | |
|---|---|---|---|---|---|
| $\mathscr{J}_{60}$ | $\xrightarrow{(e_1,e_2,f_1,tf_2)}$ $\mathscr{J}_{59}$ | $\mathscr{J}_{64}$ | $\xrightarrow{(e_1,te_2,tf_1,f_2)}$ $\mathscr{J}_{60}$ | $\mathscr{J}_{62}$ | $\xrightarrow{(e_1,e_2,tf_1,tf_2)}$ $\mathscr{J}_{61}$ |
| $\mathscr{J}_{64}$ | $\xrightarrow{(e_1,e_2,tf_1,tf_2)}$ $\mathscr{J}_{63}$ | $\mathscr{J}_{67}$ | $\xrightarrow{(e_1,e_2,f_1,f_2)}$ $\mathscr{J}_{66}$ | $\mathscr{J}_{71}$ | $\xrightarrow{(e_1,e_2,tf_1,tf_2)}$ $\mathscr{J}_{70}$ |

| | | |
|---|---|---|
| $\mathscr{J}_6$ | $\xrightarrow{(e_1,e_2+tf_1+t^2f_2,f_1+tf_2,tf_2)}$ | $\mathscr{J}_4$ |
| $\mathscr{J}_{42}$ | $\xrightarrow{(t^{\frac{2}{3}}e_2-\frac{t^{\frac{13}{12}}}{2}f_2,te_1+\frac{2+t}{2}e_2+t^{\frac{7}{12}}f_2,t^{\frac{1}{3}}f_1,t^{\frac{1}{3}}f_2)}$ | $\mathscr{J}_{19}$ |
| $\mathscr{J}_{28}$ | $\xrightarrow{(e_1+f_1+f_2,\frac{1}{t}e_2,f_1+f_2,f_2)}$ | $\mathscr{J}_{27}$ |
| $\mathscr{J}_5$ | $\xrightarrow{(\frac{t}{3}e_1+(1+\frac{2t}{3})e_2,\sqrt{t}e_1+\sqrt{t}e_2,\sqrt{t}f_1+(2t-3)f_2,f_1+4\sqrt{t}f_2)}$ | $\mathscr{J}_{33}$ |
| $\mathscr{J}_6$ | $\xrightarrow{(-te_1+(1+t)e_2,\sqrt{t}e_1+\sqrt{t}e_2,f_1,(1+2t)f_2)}$ | $\mathscr{J}_{35}$ |
| $\mathscr{J}_3$ | $\xrightarrow{((1-t)e_1+(1+t)e_2,\frac{(t-1)\sqrt{t}}{1+t}e_1+\sqrt{t}e_2,f_1,f_2)}$ | $\mathscr{J}_{37}$ |
| $\mathscr{J}_2$ | $\xrightarrow{((1+2t-t^2)e_1+(1+t^2)e_2,\frac{\sqrt{t}(-1-2t+t^2)}{1+t^2}e_1+\sqrt{t}e_2,f_1,f_2)}$ | $\mathscr{J}_{38}$ |
| $\mathscr{J}_{16}^{\mathbf{1+t}}$ | $\xrightarrow{e_1+e_2+t^2f_2,\ te_2,\ \frac{1}{t}f_1,\ tf_2}$ | $\mathscr{J}_{47}$ |
| $\mathscr{J}_{54}$ | $\xrightarrow{(e_1,te_2+f_1+t^3f_2,\frac{1}{t}f_1,t^2f_2)}$ | $\mathscr{J}_{52}$ |
| $\mathscr{J}_{68}$ | $\xrightarrow{(te_1+e_2,t^2e_2,-t^3f_1,f_2)}$ | $\mathscr{J}_{67}$ |
| $\mathscr{J}_{16}^{-1}$ | $\xrightarrow{(te_1-te_2,\frac{2t^2}{1+t}e_2,f_1,tf_2)}$ | $\mathscr{J}_{68}$ |
| $\mathscr{J}_9$ | $\xrightarrow{(-\frac{t}{\sqrt{2}}e_1+\frac{t}{\sqrt{2}}e_2,t^2e_2,f_1+2f_2,t^2f_2)}$ | $\mathscr{J}_{69}$ |
| $\mathscr{J}_{24}$ | $\xrightarrow{(-t^2e_1+e_2+t^2f_1,t^2e_2+2t^2f_2,tf_1,tf_2)}$ | $\mathscr{J}_{71}$ |

Below we list all the important reasons for necessary non-degenerations. All other nondegenerations which are not in this table, can be inferred from Theorem 8 and Lemma 6(2). Since the even parts of $\mathscr{J}_{56},\mathscr{J}_{54},\mathscr{J}_{57},\mathscr{J}_{64},\mathscr{J}_{50},\mathscr{J}_{65},\mathscr{J}_{58}$ superalgebras coincide with $\mathscr{B}_2$, we conclude that these superalgebras do not degenerate to others whose even part is not isomorphic to $\mathscr{B}_2$ and vice versa.

| Non-degenerations reasons | | | |
|---|---|---|---|
| $\mathscr{J}_{56}$ | $\nrightarrow$ | $\mathscr{J}_{50},\mathscr{J}_{54},\mathscr{J}_{57},$ $\mathscr{J}_{58},\mathscr{J}_{62},\mathscr{J}_{64},\mathscr{J}_{65}$ | $\mathscr{R}=\left\{\begin{array}{l} c_{41}^4-c_{12}^2-c_{13}^3=0, c_{24}^3c_{43}^2-(c_{24}^2)^2=0, \\ c_{22}^2=0,\ c_{11}^1-c_{12}^2-c_{13}^3-c_{14}^4=0 \end{array}\right\}$ |
| $\mathscr{J}_{54}$ | $\nrightarrow$ | $\mathscr{J}_{50},\mathscr{J}_{58},\mathscr{J}_{62},\mathscr{J}_{65}$ | $\mathscr{R}=\left\{\ c_{12}^1=0,\ c_{11}^1-c_{12}^2-c_{13}^3-c_{14}^4=0\ \right\}$ |
| $\mathscr{J}_{57}$ | $\nrightarrow$ | $\mathscr{J}_{50},\mathscr{J}_{58},\mathscr{J}_{62},\mathscr{J}_{65}$ | $\mathscr{R}=\left\{\ c_{12}^1=0,\ c_{13}^3=0,\ c_{11}^1-c_{14}^4=0\ \right\}$ |
| $\mathscr{J}_{64}$ | $\nrightarrow$ | $\mathscr{J}_{50},\mathscr{J}_{58},\mathscr{J}_{62},\mathscr{J}_{65}$ | $\mathscr{R}=\left\{\begin{array}{l} c_{12}^1=0,\ 2c_{11}^1-c_{12}^2-c_{13}^3-c_{14}^4=0, \\ c_{13}^3+c_{31}^3=c_{11}^1,\ c_{34}^2c_{24}^3+(c_{24}^2)^2=0 \end{array}\right\}$ |
| $\mathscr{J}_{62}$ | $\nrightarrow$ | $\mathscr{J}_{50},\mathscr{J}_{58},\mathscr{J}_{65}$ | $\mathscr{R}=\left\{\ c_{12}^1=0,\ 2c_{11}^1-c_{12}^2-c_{13}^3-c_{14}^4=0\ \right\}$ |
| $\mathscr{J}_{50}$ | $\nrightarrow$ | $\mathscr{J}_{58}$ | $\mathscr{R}=\left\{\ c_{12}^1=0,\ c_{13}^3=0\ \right\}$ |

| | | | |
|---|---|---|---|
| $\mathscr{I}_{65}$ | $\nrightarrow$ | $\mathscr{I}_{58}$ | $\mathscr{R} = \left\{\ c_{12}^1 = 0,\ c_{11}^1 - c_{13}^3 = 0\ \right\}$ |
| $\mathscr{I}_2$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\begin{array}{l} A_1 A_3 \subset A_4,\ c_{12}^2 = c_{14}^4,\ c_{24}^4 = c_{22}^2, \\ c_{11}^2 c_{23}^4 = c_{13}^4 (c_{14}^4 - c_{11}^1) \end{array}\right\}$ |
| $\mathscr{I}_3$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\begin{array}{l} A_1 A_3 \subset A_4,\ c_{12}^2 = 2c_{14}^4,\ 2c_{24}^4 = c_{22}^2, \\ c_{11}^2 c_{23}^4 = c_{13}^4 (2c_{14}^4 - c_{11}^1) \end{array}\right\}$ |
| $\mathscr{I}_5$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\begin{array}{l} 2c_{14}^4 = c_{12}^2,\ 2c_{24}^4 = c_{22}^2,\ c_{12}^1 = 0,\ c_{14}^3 = 0,\ c_{34}^2 = 0, \\ c_{13}^3 = c_{12}^2,\ c_{23}^3 = c_{22}^2,\ c_{22}^1 = 0,\ c_{24}^3 = 0,\ c_{34}^1 = 0, \\ c_{22}^2 c_{11}^2 + (c_{11}^1 - c_{12}^2)c_{12}^2 = 0,\ c_{11}^2 c_{23}^4 = c_{13}^4 (2c_{14}^4 - c_{11}^1) \end{array}\right\}$ |
| $\mathscr{I}_6$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\begin{array}{l} c_{22}^1 = 0,\ c_{12}^1 = 0,\ c_{34}^1 = 0,\ c_{24}^4 = 2c_{22}^2 - c_{23}^3 \\ c_{23}^4 + c_{32}^4 = 0,\ c_{24}^3 + c_{42}^3 = 0,\ c_{34}^2 c_{23}^4 = (c_{34}^4)^2, \\ c_{22}^2 (c_{34}^2)^2 c_{11}^2 + (c_{14}^4 c_{34}^2 - c_{14}^2 c_{34}^4)(c_{11}^1 c_{34}^2 - c_{14}^4 c_{34}^2 + c_{14}^2 c_{34}^4) = 0 \end{array}\right\}$ |
| $\mathscr{I}_9$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ A_1 A_3 \subset A_4,\ c_{12}^1 = 0,\ 2c_{24}^4 = c_{22}^2,\ 2c_{14}^4 = c_{11}^1 + c_{12}^2\ \right\}$ |
| $\mathscr{I}_{11}$ | $\nrightarrow$ | $\mathscr{I}_1, \mathscr{I}_8, \mathscr{I}_{10},$ $\mathscr{I}_{12}, \mathscr{I}_{13}, \mathscr{I}_{14},$ $\mathscr{I}_{16}, \mathscr{I}_{24}, \mathscr{I}_{32},$ $\mathscr{I}_{42}, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\begin{array}{l} c_{12}^1 = 0,\ c_{14}^3 = 0,\ c_{34}^1 = 0,\ c_{23}^3 = c_{22}^2,\ 2c_{14}^4 = c_{11}^1 + c_{12}^2, \\ c_{22}^1 = 0,\ c_{24}^3 = 0,\ c_{34}^2 = 0,\ c_{13}^3 = c_{12}^2,\ 2c_{24}^4 = c_{22}^2 \end{array}\right\}$ |
| $\mathscr{I}_1$ | $\nrightarrow$ | $\mathscr{I}_8, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ A_1 A_3 = 0\ \right\}$ |
| $\mathscr{I}_{10}$ | $\nrightarrow$ | $\mathscr{I}_8, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ c_{22}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0,\ 2c_{23}^3 = c_{22}^2\ \right\}$ |
| $\mathscr{I}_{12}$ | $\nrightarrow$ | $\mathscr{I}_8, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ c_{22}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0\ \right\}$ |
| $\mathscr{I}_{13}$ | $\nrightarrow$ | $\mathscr{I}_8, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ c_{12}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0,\ c_{22}^1 = 0,\ 2c_{13}^3 = c_{12}^2,\ 2c_{14}^4 = c_{11}^1 + c_{12}^2\ \right\}$ |
| $\mathscr{I}_{14}$ | $\nrightarrow$ | $\mathscr{I}_8, \mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ c_{22}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0,\ c_{23}^3 = c_{22}^2\ \right\}$ |
| $\mathscr{I}_{16}$ | $\nrightarrow$ | $\mathscr{I}_8$ | $\mathscr{R} = \left\{\ c_{23}^3 + c_{24}^4 = c_{22}^2 + c_{12}^1,\ c_{11}^1 + c_{12}^2 + c_{14}^4 = c_{13}^3 + 2c_{31}^3\ \right\}$ |
| $\mathscr{I}_{16}$ | $\nrightarrow$ | $\mathscr{I}_{49}$ | $\mathscr{R} = \left\{\ c_{22}^1 = 0,\ c_{12}^1 + c_{21}^1 = 0,\ c_{23}^4 + c_{32}^4 = 0,\ c_{24}^4 + c_{42}^3 = 0\ \right\}$ |

| $\mathscr{J}_{42} \nrightarrow \mathscr{J}_{49}$ | $\mathscr{R} = \left\{ \; c_{22}^1 = 0, \; c_{22}^2 = 0, \; c_{14}^2 c_{24}^2 = -c_{14}^3 c_{34}^2, \; c_{31}^3 = c_{11}^1 - c_{13}^3 \right\}$ |
|---|---|

Here $c_{ij}^k$ coefficients are structural constants in the $x_1 = e_1$, $x_2 = e_2$, $x_3 = f_1$, $x_4 = f_2$ basis.

$\square$

**Theorem 15.** *The variety of $4$-dimensional Jordan superalgebras of type $(3,1)$ has dimension $15$ and it has $21$ irreducible components defined by*

$$\mathscr{C}_1 = \overline{\{\mathfrak{J}_1\}}, \;\; \mathscr{C}_2 = \overline{\{\mathfrak{J}_2\}}, \;\; \mathscr{C}_3 = \overline{\{\mathfrak{J}_3\}}, \;\; \mathscr{C}_4 = \overline{\{\mathfrak{J}_4\}}, \;\; \mathscr{C}_5 = \overline{\{\mathfrak{J}_{42}\}},$$

$$\mathscr{C}_6 = \overline{\{\mathfrak{J}_{21}\}}, \;\; \mathscr{C}_7 = \overline{\{\mathfrak{J}_{51}\}}, \;\; \mathscr{C}_8 = \overline{\{\mathfrak{J}_{53}\}}, \;\; \mathscr{C}_9 = \overline{\{\mathfrak{J}_{57}\}}, \;\; \mathscr{C}_{10} = \overline{\{\mathfrak{J}_{22}\}}, \;\; \mathscr{C}_{11} = \overline{\{\mathfrak{J}_{23}\}}, \;\; \mathscr{C}_{12} = \overline{\{\mathfrak{J}_{26}\}}, \;\; \mathscr{C}_{13} = \overline{\{\mathfrak{J}_{43}\}},$$

$$\mathscr{C}_{14} = \overline{\{\mathfrak{J}_{54}\}}, \;\; \mathscr{C}_{15} = \overline{\{\mathfrak{J}_{58}\}}, \;\; \mathscr{C}_{16} = \overline{\{\mathfrak{J}_{55}\}}, \;\; \mathscr{C}_{17} = \overline{\{\mathfrak{J}_{47}\}}, \;\; \mathscr{C}_{18} = \overline{\{\mathfrak{J}_{48}\}}, \;\; \mathscr{C}_{19} = \overline{\{\mathfrak{J}_{49}\}}, \;\; \mathscr{C}_{20} = \overline{\{\mathfrak{J}_{24}\}}, \;\; \mathscr{C}_{21} = \overline{\{\mathfrak{J}_{25}\}},$$

*In particular, all of them are rigid superalgebras.*

*Proof.* After carefully checking the dimensions of orbit closures of the more important for us superalgebras, we have

$$
\begin{aligned}
\dim \mathscr{O}(\mathfrak{J}_1) &= 15, \\
\dim \mathscr{O}(\mathfrak{J}_2) = \dim \mathscr{O}(\mathfrak{J}_3) = \dim \mathscr{O}(\mathfrak{J}_4) = \dim \mathscr{O}(\mathfrak{J}_{42}) &= 14, \\
\dim \mathscr{O}(\mathfrak{J}_{21}) = \dim \mathscr{O}(\mathfrak{J}_{51}) = \dim \mathscr{O}(\mathfrak{J}_{53}) = \dim \mathscr{O}(\mathfrak{J}_{24}) = \dim \mathscr{O}(\mathfrak{J}_{57}) &= 13, \\
\dim \mathscr{O}(\mathfrak{J}_{22}) = \dim \mathscr{O}(\mathfrak{J}_{23}) = \dim \mathscr{O}(\mathfrak{J}_{26}) = \dim \mathscr{O}(\mathfrak{J}_{43}) = \dim \mathscr{O}(\mathfrak{J}_{54}) = \dim \mathscr{O}(\mathfrak{J}_{58}) &= 12, \\
\dim \mathscr{O}(\mathfrak{J}_{55}) &= 11, \\
\dim \mathscr{O}(\mathfrak{J}_{25}) &= 10, \\
\dim \mathscr{O}(\mathfrak{J}_{47}) = \dim \mathscr{O}(\mathfrak{J}_{48}) &= 9, \\
\dim \mathscr{O}(\mathfrak{J}_{49}) &= 4.
\end{aligned}
$$

If $E_{e_1}^t, E_{e_2}^t, E_{e_3}^t, E_f^t$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by $\mathbf{A} \xrightarrow{(E_{f_1}^t, E_{f_2}^t, E_{f_3}^t, E_e^t)} \mathbf{B}$.

| $\mathfrak{J}_1$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_5$ | $\mathfrak{J}_2$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_6$ |
|---|---|---|---|---|---|
| $\mathfrak{J}_3$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_7$ | $\mathfrak{J}_4$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_8$ |
| $\mathfrak{J}_1$ | $\xrightarrow{(e_1, te_2, te_3, f)}$ | $\mathfrak{J}_9$ | $\mathfrak{J}_2$ | $\xrightarrow{(e_1, te_2, te_3, f)}$ | $\mathfrak{J}_{10}$ |
| $\mathfrak{J}_3$ | $\xrightarrow{(e_1, te_2, te_3, f)}$ | $\mathfrak{J}_{11}$ | $\mathfrak{J}_1$ | $\xrightarrow{(e_1 + e_2, te_2, e_3, f)}$ | $\mathfrak{J}_{12}$ |
| $\mathfrak{J}_2$ | $\xrightarrow{(e_2 + e_3, te_2, e_1, f)}$ | $\mathfrak{J}_{13}$ | $\mathfrak{J}_3$ | $\xrightarrow{(e_2 + e_3, te_2, e_1, f)}$ | $\mathfrak{J}_{14}$ |
| $\mathfrak{J}_2$ | $\xrightarrow{(e_1 + e_2, te_2, e_3, f)}$ | $\mathfrak{J}_{15}$ | $\mathfrak{J}_3$ | $\xrightarrow{(e_1 + e_2, te_2, e_3, f)}$ | $\mathfrak{J}_{16}$ |
| $\mathfrak{J}_4$ | $\xrightarrow{(e_2 + e_3, te_2, e_1, f)}$ | $\mathfrak{J}_{17}$ | $\mathfrak{J}_1$ | $\xrightarrow{(e_1 + e_2, te_2, te_3, f)}$ | $\mathfrak{J}_{18}$ |
| $\mathfrak{J}_2$ | $\xrightarrow{(e_1 + e_2, te_2, te_3, f)}$ | $\mathfrak{J}_{19}$ | $\mathfrak{J}_3$ | $\xrightarrow{(e_1 + e_2, te_2, te_3, f)}$ | $\mathfrak{J}_{20}$ |
| $\mathfrak{J}_{21}$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_{27}$ | $\mathfrak{J}_{51}$ | $\xrightarrow{(e_1, te_2, e_3, f)}$ | $\mathfrak{J}_{28}$ |
| $\mathfrak{J}_{26}$ | $\xrightarrow{(e_1, e_2, te_3, f)}$ | $\mathfrak{J}_{29}$ | $\mathfrak{J}_1$ | $\xrightarrow{((t - t^2)e_1 + te_2, t^3 e_2, e_3, f)}$ | $\mathfrak{J}_{30}$ |
| $\mathfrak{J}_2$ | $\xrightarrow{(te_2 + t^2 e_3, t^2 e_2 + t^4 e_3, e_1, f)}$ | $\mathfrak{J}_{31}$ | $\mathfrak{J}_3$ | $\xrightarrow{(te_2 + t^2 e_3, t^2 e_2 + t^4 e_3, e_1, f)}$ | $\mathfrak{J}_{32}$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathfrak{J}_1$ | $\xrightarrow{((t-t^2)e_1+te_2,t^3e_2,te_3,f)}$ | $\mathfrak{J}_{33}$ | $\mathfrak{J}_1$ | $\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$ | $\mathfrak{J}_{34}$ |
| $\mathfrak{J}_2$ | $\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$ | $\mathfrak{J}_{35}$ | $\mathfrak{J}_3$ | $\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$ | $\mathfrak{J}_{36}$ |
| $\mathfrak{J}_1$ | $\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$ | $\mathfrak{J}_{37}$ | $\mathfrak{J}_2$ | $\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$ | $\mathfrak{J}_{38}$ |
| $\mathfrak{J}_3$ | $\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$ | $\mathfrak{J}_{39}$ | $\mathfrak{J}_{34}$ | $\xrightarrow{(te_1+e_2,te_2+e_3,te_3,f)}$ | $\mathfrak{J}_{40}$ |
| $\mathfrak{J}_{40}$ | $\xrightarrow{(te_1+\frac{1+t}{2}e_2,te_3,e_2+e_3,f)}$ | $\mathfrak{J}_{41}$ | $\mathfrak{J}_{21}$ | $\xrightarrow{(e_1+e_3,e_2,te_3,f)}$ | $\mathfrak{J}_{44}$ |
| $\mathfrak{J}_{22}$ | $\xrightarrow{(e_1+e_3,e_2,te_3,f)}$ | $\mathfrak{J}_{45}$ | $\mathfrak{J}_{23}$ | $\xrightarrow{(e_1+e_3,e_2,te_3,f)}$ | $\mathfrak{J}_{46}$ |
| $\mathfrak{J}_{42}$ | $\xrightarrow{(e_1,te_3,t^2e_2,f)}$ | $\mathfrak{J}_{50}$ | $\mathfrak{J}_{43}$ | $\xrightarrow{(e_1,te_3,t^2e_2,f)}$ | $\mathfrak{J}_{52}$ |
| $\mathfrak{J}_{42}$ | $\xrightarrow{(e_1,e_2,te_3,f)}$ | $\mathfrak{J}_{56}$ | $\mathfrak{J}_{43}$ | $\xrightarrow{(e_1,e_2,te_3,f)}$ | $\mathfrak{J}_{59}$ |

Below we list all the important reasons for necessary non-degenerations. All other nondegenerations which are not in this table, can be inferred from Theorem 9 and Lemma 6(2).

| | | | |
|---|---|---|---|
| | | Non-degenerations reasons | |
| $\mathfrak{J}_1$ | $\not\to$ | $\mathfrak{J}_2,\mathfrak{J}_3,\mathfrak{J}_4$ | $\mathscr{R} = \left\{ A_1A_4 = 0 \right\}$ |
| $\mathfrak{J}_{42}$ | $\not\to$ | $\mathfrak{J}_{43},\mathfrak{J}_{51},\mathfrak{J}_{57},\mathfrak{J}_{58}$ | $\mathscr{R} = \left\{ A_1A_4 = 0 \right\}$ |
| $\mathfrak{J}_{21}$ | $\not\to$ | $\mathfrak{J}_{22},\mathfrak{J}_{23},\mathfrak{J}_{25},\mathfrak{J}_{26}$ | $\mathscr{R} = \left\{ A_1A_4 = 0 \right\}$ |
| $\mathfrak{J}_{24}$ | $\not\to$ | $\mathfrak{J}_{22},\mathfrak{J}_{23},\mathfrak{J}_{25},\mathfrak{J}_{26}$ | $\mathscr{R} = \left\{ A_2A_2 \subset A_3,\ A_2A_4 = 0,\ c_{11}^1 = c_{14}^4 \right\}$ |
| $\mathfrak{J}_{47}$ | $\not\to$ | $\mathfrak{J}_{49}$ | $\mathscr{R} = \left\{ A_1A_4 = 0 \right\}$ |
| $\mathfrak{J}_{53}$ | $\not\to$ | $\mathfrak{J}_{54},\ \mathfrak{J}_{55}$ | $\mathscr{R} = \left\{ A_1A_4 = 0 \right\}$ |
| $\mathfrak{J}_{48}$ | $\not\to$ | $\mathfrak{J}_{49}$ | $\mathscr{R} = \left\{ c_{12}^1 = 0,\ c_{13}^1 = 0,\ c_{11}^1 = c_{14}^4 \right\}$ |
| $\mathfrak{J}_{54}$ | $\not\to$ | $\mathfrak{J}_{55}$ | $\mathscr{R} = \left\{ c_{34}^4 = 0,\ c_{24}^4 = 0,\ c_{11}^1 = c_{14}^4 \right\}$ |
| $\mathfrak{J}_{57}$ | $\not\to$ | $\mathfrak{J}_{58}$ | $\mathscr{R} = \left\{ c_{34}^4 = 0,\ c_{24}^4 = 0,\ c_{11}^1 = c_{14}^4 \right\}$ |
| $\mathfrak{J}_{22}$ | $\not\to$ | $\mathfrak{J}_{25}$ | $\mathscr{R} = \left\{ c_{22}^1 = 0,\ c_{22}^2 = 0,\ c_{33}^1 = 0,\ c_{33}^2 = 0,\ c_{33}^3 = c_{34}^4 \right\}$ |
| $\mathfrak{J}_{23}$ | $\not\to$ | $\mathfrak{J}_{25}$ | $\mathscr{R} = \left\{ c_{22}^2 = 0,\ c_{23}^1 = 0,\ c_{23}^2 = 0,\ c_{33}^2 = 0,\ c_{24}^4 = \frac{1}{2}c_{23}^3,\ c_{34}^4 = \frac{1}{2}c_{33}^3 \right\}$ |
| $\mathfrak{J}_{26}$ | $\not\to$ | $\mathfrak{J}_{25}$ | $\mathscr{R} = \left\{ c_{22}^2 = 0,\ c_{23}^1 = 0,\ c_{23}^2 = 0,\ c_{33}^2 = 0,\ c_{24}^4 = \frac{1}{2}c_{23}^3,\ c_{34}^4 = \frac{1}{2}c_{33}^3 \right\}$ |

Here $c_{ij}^k$ coefficients are structural constants in the $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = f$ basis.

$\square$

## References

[1] Adashev J., Kaygorodov I., Khudoyberdiyev A., Sattarov A., The algebraic and geometric classification of nilpotent left-symmetric algebras, Journal of Geometry and Physics, 167 (2021), 104287.

[2] Adashev J., Kaygorodov I., Khudoyberdiyev A., Sattarov A., The algebraic and geometric classification of nilpotent right commutative algebras, Results in Mathematics, 76 (2021), 1, 24.

[3] Alvarez M.A., Hernández I., On degenerations of Lie superalgebras, Linear Multilinear Algebra 68 (2020), 1, 29–44.

[4] Alvarez M., Hernández I., Kaygorodov I., Degenerations of Jordan superalgebras, Bull. Malays. Math. Sci. Soc. 42 (2019), 6, 3289–3301.

[5] Ancochea Bermúdez J.M., Campoamor-Stursberg R., García Vergnolle L., Sánchez Hernández J., Contractions d´algebres de Jordan en dimension 2, Journal of Algebra, 319 (2008), 6, 2395–2409.

[6] Ancochea Bermúdez J.M., Fresán J., Margalef Bentabol J., Contractions of low-dimensional nilpotent Jordan algebras, Communications in Algebra, 39 (2011), 3, 1139–1151.

[7] Armour A., Zhang Yi., Geometric classification of 4-dimensional superalgebras, Chapter Algebra, Geometry and Mathematical Physics, 315 Springer Proceedings in Mathematics and Statistics, 85, (2014), 291–323.

[8] Ben Hassine A., Chtioui T., Elhamdadi M., Mabrouk S., Cohomology and deformations of left-symmetric Rinehart algebras, Communications in Mathematics, 32 (2024), 2, 127–152.

[9] Burde D., Steinhoff C., Classification of orbit closures of 4-dimensional complex Lie algebras, Journal of Algebra, 214 (1999), 2, 729–739.

[10] Cantarini N., Kac V., Classification of linearly compact simple Jordan and generalized Poisson superalgebras, Journal of Algebra, 313 (2007), 100–124

[11] Camacho L., Kaygorodov I., Lopatkin V., Salim M., The variety of dual Mock-Lie algebras, Communications in Mathematics, 28 (2020), 2, 161–178.

[12] Ehret Q., Makhlouf A., On classification and deformations of Lie-Rinehart superalgebras, Communications in Mathematics, 30 (2022), 2, 67–100.

[13] Eick B., Moede T., Computing subalgebras and $\mathbb{Z}_2$-gradings of simple Lie algebras over finite fields, Communications in Mathematics, 30 (2022), 2, 37–50.

[14] Fernández Ouaridi A., Kaygorodov I., Khrypchenko M., Volkov Yu., Degenerations of nilpotent algebras, Journal of Pure and Applied Algebra, 226 (2022), 3, 106850.

[15] Gorshkov I., Kaygorodov I., Popov, Yu.: Degenerations of Jordan algebras and "Marginal" algebras. Algebra Colloquium, 28 (2021), 2, 281–294.

[16] Grunewald F., O'Halloran J., Varieties of nilpotent Lie algebras of dimension less than six, Journal of Algebra, 112 (1988), 315–325.

[17] Grunewald F., O'Halloran J., A characterization of orbit closure and applications, Journal of Algebra, 116 (1988), 163–175.

[18] Hernández I., Rodrigues R., Oveimar Quintero Vanegas E., Low-dimensional commutative power-associative superalgebras, International Journal of Algebra and Computation, 31 (2021), 8, 1613–1632.

[19] Jumaniyozov D., Kaygorodov I., Khudoyberdiyev A., The algebraic and geometric classification of nilpotent noncommutative Jordan algebras, Journal of Algebra and its Applications, 20 (2021), 11, 2150202.

[20] Jumaniyozov D., Kaygorodov I., Khudoyberdiyev A., The geometric classification of nilpotent commutative $\mathfrak{CD}$-algebras, Bollettino dell'Unione Matematica Italiana, 15 (2022), 3, 465–481.

[21] Kadyrov S., Mashurov F., Unified computational approach to nilpotent algebra classification problems, Communications in Mathematics, 29 (2021), 2, 215–226.

[22] Kashuba I., Martin M. E. Deformations of Jordan algebras of dimension four, Journal of Algebra, 399 (2014), 277–289.

[23] Kaygorodov I., Non-associative algebraic structures: classification and structure, Communications in Mathematics, 32 (2024), 3, 1–62.

[24] Kaygorodov I., Khrypchenko M., Lopes S., The geometric classification of nilpotent algebras, Journal of Algebra, 633 (2023), 857–886.

[25] Kaygorodov I., Khrypchenko M., Páez-Guillán P., The geometric classification of non-associative algebras: a survey, Communications in Mathematics, 32 (2024), 2, 185–284.

[26] Lopes S., Noncommutative Algebra and Representation Theory: Symmetry, Structure & Invariants, Communications in Mathematics, 32 (2024), 3, 63–117.

[27] Kac V., Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, Communications in Algebra, 13 (1977), 5, 1375–1400.

[28] Kantor I.L., Connection between Poisson brackets and Jordan and Lie superalgebras, in: Lie Theory, Differential Equations and Representation Theory, Publications CRM, Montreal, 1990, pp. 213–225.

[29] Lopez-Diaz M. C., Shestakov I., Sverchkov S., On speciality of Bernstein Jordan algebras, Communications in Algebra, 28 (2000), 4375–4387.

[30] Martin M. E., Four-dimensional Jordan algebras, International Journal of Mathematics, Game Theory and Algebra, 20 (2013), 41–59.

[31] Martin M.E., Shestakov I., Classification of three-dimensional Jordan superalgebras, arXiv:1708.01963v1.

[32] Martinez C., Zelmanov E., Simple finite-dimensional Jordan superalgebras of prime characteristic, J. Algebra, 236 (2001), 2, 575–629.

[33] Martinez C., Zelmanov E., Simple Jordan superalgebras with semisimple even part, J. Algebra, 270 (2003), 2, 374–444.

[34] Mondal B., Saha R., Cohomology, deformations and extensions of Rota-Baxter Leibniz algebras, Communications in Mathematics, 30 (2022), 2, 93–117.

[35] Seeley C., Degenerations of 6-dimensional nilpotent Lie algebras over $\mathbb{C}$, Communications in Algebra, 18 (1990), 3493–3505.