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A VISCOPLASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

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Abstract. The aim of this paper is to present a new result in the study of a contact problem between a viscoplastic body and an obstacle, the so-called foundation. The process is supposed to be quasistatic and the contact is modelled with a version of Coulomb's law of dry friction, normal compliance and an ordinary differential equation which describes the adhesion effect. We derive a variational formulation for the model and under smallness assumption, we establish the existence of a weak solution to the problem. The proof is based on the Rothe time-discretization method, the Banach fixed point theorem and arguments of monotonicity, compactness and lower semicontinuity.

Keywords: viscoplastic materials, adhesion, quasistatic process, Coulomb's law of dry friction, normal compliance, Rothe method, lower semicontinuity, the Banach fixed point theorem, variational inequalities.

1. Introduction

Adhesive contact problems have recently received increased attention in the mathematical literature. An early attempt to study the contact with adhesion, was done in [10, 11]. The main new idea is the introduction of an internal variable $\beta \in [0, 1]$, which represents the intensity of the adhesion. When $\beta = 0$ there are no active bonds; when $\beta = 1$ there is total adhesion; when $0 < \beta < 1$ partial adhesion takes place. Recent modeling and analysis of contact problems with adhesion can be found in [5, 7, 17, 18, 22, 23, 24, 25] and references therein. In [5], a class of dynamic thermal contact problems, with nonclamped condition, friction, normal compliance and adhesion was analyzed for viscoelastic materials with long memory. The unilateral contact problem with local friction and adhesion was studied in

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[7]; an existence result, for small enough friction coefficients, was established. The dynamic contact problem with Tresca's friction law and adhesion, for viscoelastic materials with damage, was considered in [17] where an existence and uniqueness result was obtained. In [18] the quasistatic bilateral contact problem with a general nonlocal friction law and adhesion, for viscoelastic materials with long memory, was investigated and fully discrete scheme based on the finite element method was introduced to the case of Tresca's friction law.

The novelty, in this paper, consists in dealing with a quasistatic contact problem for viscoplastic materials with a constitutive law of the form

$$\dot{\sigma} = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon\,(u)),$$

such that the contact is modelled with normal compliance condition and both friction and adhesion are taken into account, which leads to a new and nonstandard mathematical problem. Here and below, the dot above a variable represents its derivative with respect to the time variable, u denotes the displacement field, σ represents the stress tensor, $\varepsilon\,(u)$ is the linearized strain tensor, $\mathcal{A}$ is a fourth order tensor which describes the elastic behaviour of the material and $\mathcal{B}$ is a constitutive function that describes its viscoplastic properties.

Our analysis is based on the application of the so-called Rothe time-discretization method. By using the backward Euler scheme, we transform the quasistatic contact problem into a sequence of elliptic quasi-variational inequalities for which we prove an existence and uniqueness result of the solution. Then, after obtaining the necessary estimates, we construct approximate solutions and prove that the limit of a subsequence of the solutions of the approximate problems is a solution of the continuous problem. We note that the Rothe method was first introduced in [21] and since then used to investigate various types of boundary value problems by many authors, see for instance [7, 16, 23, 25].

The rest of this paper is organized as follows. In Section 2 we present the notation and some preliminary material. Section 3 is dedicated to describe the frictional contact problem and after state the assumptions on the data we derive its variational formulation in Section 4. In Section 5 we establish the existence of a weak solution to the model.

2. NOTATION AND PRELIMINARIES

Here we introduce the notation we shall use and some preliminary materials. For further details we refer the reader to [8, 13, 20]. We use the notation $\mathbb{N}^*$ for the set of positive integers. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, (d=2, 3), and we define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = \sum_{i=1}^{d} u_i v_i, \ \ |u| = \sqrt[2]{u \cdot u},\ \forall u,\ v\ \in \mathbb{R}^d;$$

$$\sigma \cdot \zeta = \sum_{1 \le i,j \le d} \sigma_{ij}\ \zeta_{ij}\ ,\ |\sigma| = \sqrt[2]{\sigma \cdot \sigma},\ \forall \sigma,\ \zeta \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$, ($d$=2, 3), be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. Let $[0, T]$, $T > 0$ be the time interval of interest and let $x \in \bar{\Omega}$ and $t \in [0, T]$ be the spatial and time variables, respectively. We introduce the spaces

$$H = L^2(\Omega;\mathbb{R}^d),\ \mathcal{Q} = L^2(\Omega;\mathbb{S}^d),$$
$$H_1 = \{u \in H;\ \varepsilon(u) \in \mathcal{Q}\},$$
$$\mathcal{Q}_1 = \{\sigma \in \mathcal{Q};\ Div\sigma \in H\},$$

where $\varepsilon : H_1 \to \mathcal{Q}$ is the deformation operator, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)),\ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ 1 \le i,j \le d,\ \forall u \in H_1,$$

$Div : \mathcal{Q}_1 \to H$ is the divergence operator, defined by

$$Div\sigma = ((Div\sigma)_i)_{1\le i\le d} = \left(\sum_{j=1}^{d}\frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1\le i\le d},\ \forall \sigma \in \mathcal{Q}_1.$$

Note that H, $\mathcal{Q}$, H_1 and $\mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u,v)_H = \int_\Omega u\cdot v dx,\ (\sigma,\zeta)_{\mathcal{Q}} = \int_\Omega \sigma\cdot\zeta dx,$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{Q}},$$
$$(\sigma,\zeta)_{\mathcal{Q}_1} = (Div\sigma, Div\zeta)_H + (\sigma,\zeta)_{\mathcal{Q}},$$

where the associated norms are denoted by $\|.\|_H$, $\|.\|_{\mathcal{Q}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{Q}_1}$.

Let $\tilde{\gamma} : H_1 \to L^2(\Gamma;\mathbb{R}^d)$ be the trace map. We recall that $\tilde{\gamma}$ is a compact operator, i.e., for any bounded sequence $\{v_n\}$ in H_1, there is a subsequence of $\{v_n\}$ which is convergent in $L^2(\Gamma;\mathbb{R}^d)$. For every element $v \in H_1$ we use the notation v to denote the trace $\tilde{\gamma}(v)$ of v on Γ and for all $v \in H_1$ we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ

$$v_\nu = v\cdot\nu,\ v_\tau = v - v_\nu\nu \text{ on } \Gamma.$$

In a similar manner, the normal and the tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu\cdot\nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu\nu \text{ on } \Gamma,$$

moreover the following Green's formula holds

$$(Div\sigma, v)_H + (\sigma,\varepsilon(v))_{\mathcal{Q}} = \int_\Gamma \sigma\nu\cdot v da,\ \forall\ v \in H_1, \tag{1}$$

where da is the surface measure element. For every real Banach space $(X,\|.\|_X)$, we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to X and we use the standard notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [1,\infty]$ and $k \geqslant 1$.

Finally, we conclude this section with two Gronwall type inequalities. Other versions of Gronwall inequalities can be found for instance in [9] and references therein.

Lemma 1. *Assume that $\tilde{a}$ and $\tilde{b} : [0,T] \to \mathbb{R}$ are two functions in $L^1(0,T)$ satisfying*

$$\tilde{a}(t) \le \tilde{b}(t) + \alpha\int_0^t \tilde{a}(s)\,ds,\ a.e.\ t \in [0,T], \tag{2}$$

where α is a nonnegative constant. Then, it follows

$$\tilde{a}\,(t) \leq \tilde{b}\,(t) + \alpha \int_0^t e^{\alpha(t-s)} \tilde{b}\,(s)\,ds,\ a.e.\ t \in [0,T]\,. \tag{3}$$

Moreover, if we replace "a.e. $t \in [0,T]$" in (2) with "$\forall\ t \in [0,T]$", then (3) holds for all $t \in [0,T]$.

Proof. Use arguments similar to those in [9, proof of Proposition 2.1]. □

Lemma 2. *Let $T > 0$ be a constant. Let α_1 and α_2 be two nonnegative constants. Let $m \in \mathbb{N}^*$, let $\{w_i\}_{i=0}^m \subset \mathbb{R}$ be a nonnegative sequence, which satisfies*

$$w_{i+1} \leq \alpha_1 + \alpha_2 h \sum_{j=0}^{i} w_j,\ 0 \leq i \leq m-1, \tag{4}$$

where $h = \dfrac{T}{m}$. Then, it holds

$$w_{i+1} \leq (\alpha_1 + \alpha_2 T w_0)\, e^{\alpha_2 T},\ 0 \leq i \leq m-1. \tag{5}$$

Proof. We introduce the function $y : [0,T] \to \mathbb{R}$, defined by

$$y(0) = 0,\ y(t) = w_{i+1},\ \forall t \in (t_i, t_{i+1}]\,,\ t_i = ih,\ i = 0, ..., m-1. \tag{6}$$

Using (4) and (6), we get

$$\begin{aligned} y(t) &\leq \alpha_1 + \alpha_2 \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} y(s) ds + \alpha_2 h w_0 \\ &\leq (\alpha_1 + \alpha_2 h w_0) + \alpha_2 \int_0^t y(s) ds - \alpha_2 \int_{t_i}^t y(s) ds \\ &\leq (\alpha_1 + \alpha_2 h w_0) + \alpha_2 \int_0^t y(s) ds,\ t \in (t_i, t_{i+1}]\,, i \in \{0,...,m-1\}\,. \end{aligned}$$

We use now Lemma 1 to obtain

$$\begin{aligned} y(t) &\leq (\alpha_1 + \alpha_2 h w_0)\, e^{\alpha_2 t} \\ &\leq (\alpha_1 + \alpha_2 T w_0)\, e^{\alpha_2 T},\ t \in (t_i, t_{i+1}]\,,\ i \in \{0,...,m-1\}\,, \end{aligned}$$

and inequality (5) follows. □

3. Problem statement

The physical setting is as follows. A deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (with d=2, 3). The material's behavior is modelled with a rate-type viscoplastic constitutive law and the process is quasistatic in the time interval of interest $[0,T]$. We assume that the boundary Γ of the domain Ω is Lipschitz continuous, and it is divided into three disjoint measurable parts Γ_1, Γ_2, Γ_3, such that meas$(\Gamma_1) > 0$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω and surface tractions of density f_2 act on Γ_2. The body is supposed to be in adhesive contact over Γ_3 with a foundation and, moreover, both normal compliance and a version of Coulomb's law of dry friction are included. Under the above assumptions, the classical formulation of our problem is the following.

Problem 1. *Find a displacement field* $u : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\dot{\sigma} = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\sigma, \varepsilon(u)), \ \text{in } \Omega \times (0,T), \tag{7}$$

$$Div\sigma + f_0 = 0, \ \text{in } \Omega \times (0,T), \tag{8}$$

$$u = 0, \ \text{on } \Gamma_1 \times (0,T), \tag{9}$$

$$\sigma\nu = f_2, \ \text{on } \Gamma_2 \times (0,T), \tag{10}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu), \ \text{on } \Gamma_3 \times (0,T), \tag{11}$$

$$\begin{cases} |\sigma_\tau| \leqslant p_\tau(u_\nu), \\ |\sigma_\tau| < p_\tau(u_\nu) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = p_\tau(u_\nu) \Rightarrow \exists \lambda \geq 0 \\ \text{such that } \ \sigma_\tau = -\lambda \dot{u}_\tau, \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\dot{\beta} = H_{ad}(\beta, \theta_\beta, R_\nu(u_\nu)), \ \text{on } \Gamma_3 \times (0,T), \tag{13}$$

$$\beta(0) = \beta_0, \ \text{on } \Gamma_3, \tag{14}$$

$$u(0) = u_0, \ \sigma(0) = \sigma_0 \ \text{in } \Omega. \tag{15}$$

We assume, here and in the sequel, that the functions ($\mathcal{A}$; $\mathcal{B}$; p_ν; p_τ; H_{ad}; σ; u; β; θ_β) depend on the spatial variable x explicitly. For example, the complete writing of expression (7) should be

$$\dot{\sigma}(x,t) = \mathcal{A}(x)\varepsilon(\dot{u}(x,t)) + \mathcal{B}(x, \sigma(x,t), \varepsilon(u(x,t))), \ (x,t) \in \Omega \times (0,T).$$

We now briefly comment on the problem (7)-(15). Equations (7) represents the rate-type viscoplastic constitutive law. Analysis of various contact problems with constitutive laws of the form (7) can be found for instance in [1, 2, 6, 13] and the references therein. For mechanical interpretation of constitutive laws of the form (7) we refer to [4]. Equation (8) is the equilibrium equation posed on the domain Ω. Equations (9)-(10) are the displacement-traction boundary conditions where $\sigma\nu$ represents the Cauchy stress vector. As in [5], (11)-(12) characterize the contact boundary conditions. Equation (11) is the normal compliance condition in which the contribution of the adhesive to the normal traction is represented by $\gamma_\nu \beta^2 R_\nu(u_\nu)$, where γ_ν is the adhesion coefficient, u_ν is the normal displacement and $R_\nu : \mathbb{R} \to \mathbb{R}$ is a truncation function defined by

$$R_\nu(s) = \begin{cases} 0, \ \text{if } 0 \leq s, \\ -s, \ \text{if } -L \leq s \leq 0, \\ L, \ \text{if } s \leq -L, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it stretches without offering any additional resistance (see, e.g., [22, 24]). Clearly, the function R_ν satisfies

$$R_\nu(s)s \leqslant 0, \ |R_\nu(s)| \leqslant |s|, \ |R_\nu(s)| \leqslant L, \ \forall s \in \mathbb{R}, \tag{16}$$

$$|R_\nu(s_1) - R_\nu(s_2)| \leqslant |s_1 - s_2|, \ \forall s_1, s_2 \in \mathbb{R}. \tag{17}$$

Moreover, R_ν is a decreasing function, i.e.

$$(R_\nu(s_1) - R_\nu(s_2))\,(s_1 - s_2) \leqslant 0,\ \forall s_1, s_2 \in \mathbb{R}. \tag{18}$$

The normal compliance function p_ν is a nonnegative prescribed function which vanishes for negative arguments. An example of the normal compliance function p_ν is

$$p_\nu\,(r) = c_\nu\,(r)_+\,, \tag{19}$$

where $(r)_+$ denotes the positive part of r, that is $(r)_+ = \max\{r, 0\}$, c_ν is the surface stiffness coefficient, such that Signorini's nonpenetration condition is obtained in the limit $c_\nu \to \infty$. We note that an early attempt to study the contact problem with normal compliance was done in [14, 15]. Since then, the normal compliance law has been extensively employed as an approximation of the Signorini contact condition, see for instance [2, 23, 24] and references therein. The relations (12) represent a version of Coulomb's law of dry friction where p_τ is a prescribed nonnegative function, the so-called friction bound. A possible choice of the function p_τ is

$$p_\tau\,(r) = \mu p_\nu\,(r)\,, \tag{20}$$

where $\mu \geqslant 0$ is the coefficient of friction (see, e.g., [23]). Equation (13) describes the evolution of the bonding field where H_{ad} is a general function which may change sign. This condition implies that cycles of rebonding after debonding may take place. Moreover, the process depends on the bonding history, which we denote by

$$\theta_\beta\,(x,t) = \int_0^t \beta\,(x,s)\,ds,\ on\ \Gamma_3 \times (0,T). \tag{21}$$

An example of the adhesion rate function H_{ad} is

$$\dot{\beta} = H_{ad}\,(\beta, r) = -\left(\epsilon_\nu \beta r^2 - \epsilon_a\right)_+\,,\ on\ \Gamma_3 \times (0,T), \tag{22}$$

where ϵ_ν, ϵ_a are given positive material parameters. In (22), since $\dot{\beta} \leq 0$, the process is irreversible and once debonding occurs bonding cannot be reestablished. Another example, in which H_{ad} depends on all three variables is

$$H_{ad}\,(\beta, \theta_\beta, r) = -\gamma_1 \beta r^2 + \gamma_2 \frac{\beta_+\,(1-\beta)_+}{1 + d_*\,(\theta_\beta)^2},$$

where γ_1, γ_2 are given positive material parameters and $d_* > 0$ is the history weight factor, see [23, 24] for details. Finally, (14)-(15) are the initial conditions.

4. Assumptions and variational formulation

In order to obtain the variational formulation of the mechanical problem (7)-(15), we introduce the space V defined by

$$V = \{v \in H_1,\ v = 0\ on\ \Gamma_1\}.$$

Since $meas\,(\Gamma_1) > 0$, Korn's inequality holds

$$C_K\,\|v\|_{H_1} \leq \|\varepsilon\,(v)\,\|_{\mathcal{Q}},\ \forall v \in V, \tag{23}$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found, for instance, in [19, page 79]. Over the space V, we consider the inner product given by

$$(w,v)_V = (\varepsilon(w), \varepsilon(v))_{\mathcal{Q}},\ \forall w, v \in V,$$

and let $\|.\|_V$ be the associated norm. It follows from Korn's inequality (23) that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, (.,.)_V\ )$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leq c_0\|v\|_V,\ \forall v \in V. \tag{24}$$

In the study of the mechanical problem (7)-(15), we consider the following assumptions. We assume that $\mathcal{A} = (\mathcal{A}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is a bounded symmetric positive definite fourth order tensor, i.e.

$$\begin{cases} \text{(i) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}\varepsilon \cdot \varepsilon \geqslant m_{\mathcal{A}} \left|\varepsilon\right|^2 \ \ a.e.\ x \in \Omega,\ \forall \varepsilon \in \mathbb{S}^d; \\ \\ \text{(ii) } \mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij};\ \mathcal{A}_{ijkl} \in L^\infty(\Omega), \\ \\ \forall i,\ j,\ k,\ l \in \{1, ..., d\}\,. \end{cases} \tag{25}$$

We assume that $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(i) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \left|\mathcal{B}\left(x, \sigma_1, \varepsilon_1\right) - \mathcal{B}\left(x, \sigma_2, \varepsilon_2\right)\right| \leqslant L_{\mathcal{B}}\left(\left|\sigma_1 - \sigma_2\right| + \left|\varepsilon_1 - \varepsilon_2\right|\right) \\ \\ \ a.e.\ x \in \Omega,\ \forall \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d; \\ \\ \text{(ii) The mapping } x \mapsto \mathcal{B}\left(x, \sigma, \varepsilon\right) \text{ is Lebesgue measurable on } \Omega \\ \text{for any } \sigma,\ \varepsilon \in \mathbb{S}^d; \\ \\ \text{(iii) The mapping } x \mapsto \mathcal{B}\left(x, 0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}\right) \text{ belongs to } \mathcal{Q}. \end{cases} \tag{26}$$

We assume that the function $p_\alpha : \Gamma_3\ \times \mathbb{R} \to \mathbb{R}^+$, $(\alpha = \nu,\ \tau)$, satisfies

$$\begin{cases} \text{(i) There exists } L_\alpha > 0 \text{ such that} \\ \left|p_\alpha\left(x, r_1\right) - p_\alpha\left(x, r_2\right)\right| \leqslant L_\alpha \left|r_1 - r_2\right|, \\ \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3\ ; \\ \\ \text{(ii) } p_\alpha\left(x, r\right) = 0,\ \forall r \leq 0,\ a.e.\ x \in \Gamma_3; \\ \\ \text{(iii) The mapping } x \mapsto p_\alpha\left(x, r\right) \text{ is Lebesgue measurable on } \Gamma_3, \\ \forall r \in \mathbb{R}. \end{cases} \tag{27}$$

The adhesion rate function $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R}\times [-L, L] \to \mathbb{R}$ is assumed to satisfy

(28)
$$\begin{cases} \text{(i) There exists } L_{H_{ad}} > 0 \text{ such that} \\ |H_{ad}(x,\beta_1,z,r) - H_{ad}(x,\beta_2,z,r)| \leqslant L_{H_{ad}} |\beta_1-\beta_2|, \\ \ a.e. x \in \Gamma_3,\ \forall \beta_1,\ \beta_2,\ z \in \mathbb{R},\ \forall r \in [-L,L]; \\ \\ \text{(ii) } \forall b_1,\ b_2 \in \mathbb{R},\ \text{there exists } L_{b_1b_2} > 0 \text{ such that} \\ |H_{ad}(x,\beta_1,z_1,r_1) - H_{ad}(x,\beta_2,z_2,r_2)| \leqslant L_{b_1b_2}(|\beta_1-\beta_2| + |z_1-z_2| + |r_1-r_2|), \\ \forall \beta_1,\ \beta_2 \in [b_1,b_2],\ \forall z_1,\ z_2 \in \mathbb{R},\ \forall r_1,\ r_2 \in [-L,L],\ a.e.\ x \in \Gamma_3; \\ \\ \text{(iii) The mapping } x \mapsto H_{ad}(x,\beta,z,r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \forall \beta,\ z \in \mathbb{R},\ \forall r \in [-L,L]; \\ \\ \text{(iv) The mapping } (\beta,z,r) \mapsto H_{ad}(x,\beta,z,r) \text{ is continuous on } \mathbb{R}\times\mathbb{R}\times[-L,L], \\ a.e.\ x \in \Gamma_3; \\ \\ \text{(v) } H_{ad}(x,0,z,r) = 0,\ \forall z \in \mathbb{R},\ \forall r \in [-L,L],\ a.e. x \in \Gamma_3; \\ \\ \text{(vi) } H_{ad}(x,\beta,z,r) \geqslant 0,\ \forall \beta \leqslant 0,\ \forall z \in \mathbb{R},\ \forall r \in [-L,L],\ a.e.\ x \in \Gamma_3 \text{ and} \\ \quad H_{ad}(x,\beta,z,r) \leqslant 0,\ \forall \beta \geqslant 1,\ \forall z \in \mathbb{R},\ \forall r \in [-L,L],\ a.e.\ x \in \Gamma_3. \end{cases}$$

We assume that the adhesion coefficient $\gamma_\nu : \Gamma_3 \to \mathbb{R}^+$ satisfies

(29) $$\gamma_\nu \in L^\infty(\Gamma_3).$$

The densities of forces satisfy

(30) $$(i)\ \ f_0 \in W^{1,\infty}(0,T;H),\ (ii)\ \ f_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2;\mathbb{R}^d)).$$

Finally, we assume that the initial data satisfy

(31) $$\beta_0 \in L^\infty(\Gamma_3),\ 0 \leqslant \beta_0 \leqslant 1,\ a.e.\ x \in\ \Gamma_3,$$

(32) $$(i)\ \ u_0 \in V,\ (ii)\ \ \sigma_0 \in \mathcal{Q}.$$

In the sequel, we use the functionals $\psi : V \times V \longrightarrow \mathbb{R}$, $j_{ad} : L^\infty(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ and $\varphi : L^\infty(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ defined, respectively, by

(33) $$\psi(v,w) = \int_{\Gamma_3} p_\nu(v_\nu)\, w_\nu da + \int_{\Gamma_3} p_\tau(v_\nu)\, |w_\tau|\, da,$$

(34) $$j_{ad}(\beta,v,w) = -\int_{\Gamma_3} \gamma_\nu \beta^2 R_\nu(v_\nu)\, w_\nu da,$$

(35) $$\varphi(\beta,v,w) = j_{ad}(\beta,v,w) + \psi(v,w),$$

for all v, $w \in V$ and for all $\beta \in L^\infty(\Gamma_3)$. Using (24), (27) and (33), we deduce that the functional ψ satisfies the following

(36) $$\psi(g,v) - \psi(g,w) + \psi(z,w) - \psi(z,v) \leq c_0^2 (L_\tau + L_\nu) \|g-z\|_V \|v-w\|_V,$$

(37) $$\psi(g,-z) - \psi(g,w-z) \leq c_0^2 (L_\tau + L_\nu) \|g\|_V \|w\|_V,$$

(38) $$\psi(g,w) - \psi(z,w) \leq c_0^2 (L_\tau + L_\nu) \|g-z\|_V \|w\|_V,$$

(39) $$\psi(g,v) - \psi(g,w) \leqslant \psi(g,v-w),$$

$$|\psi(g,v)-\psi(g,w)| \leqslant c_0^2\,(L_\tau+L_\nu)\,\|g\|_V\,\|v-w\|_V\,, \tag{40}$$

$$\psi(g,w) \le c_0\,(L_\tau+L_\nu)\,\|g\|_V\,\|w\|_{L^2(\Gamma_3;\mathbb{R}^d)}\,, \tag{41}$$

for all $v, g, w,\ z \in V$. Also, using (16), (17), (18), (24), (29) and (34), we get the following properties: There exists $L_{ad} > 0$ such that

$$\left\{\begin{array}{c} |j_{ad}(\beta_1,v,w)-j_{ad}(\beta_2,v,w)| \\ \\ \leqslant L_{ad}\left(\|\beta_1\|_{L^\infty(\Gamma_3)}+\|\beta_2\|_{L^\infty(\Gamma_3)}\right)\|\beta_1-\beta_2\|_{L^2(\Gamma_3)}\,\|w\|_V\ \ , \end{array}\right. \tag{42}$$

$$|j_{ad}(\beta,g,w)| \leqslant L_{ad}\,\|\beta\|^2_{L^\infty(\Gamma_3)}\,\|g\|_V\,\|w\|_{L^2(\Gamma_3;\mathbb{R}^d)}\,, \tag{43}$$

$$|j_{ad}(\beta,v,g)-j_{ad}(\beta,w,g)| \leqslant L_{ad}\,\|\beta\|^2_{L^\infty(\Gamma_3)}\,\|v-w\|_V\,\|g\|_V\ \ , \tag{44}$$

$$|j_{ad}(\beta,g,v)-j_{ad}(\beta,g,w)| \leqslant L_{ad}\,\|\beta\|^2_{L^\infty(\Gamma_3)}\,\|g\|_V\,\|v-w\|_V\,, \tag{45}$$

$$j_{ad}(\beta,v,v-w)-j_{ad}(\beta,w,v-w) \ge 0, \tag{46}$$

$$j_{ad}(\beta,w,w) \ge 0, \tag{47}$$

for all $v,\ g, w\ \ \in V$ and for all $\beta_1,\ \beta_2,\ \beta \in L^\infty(\Gamma_3)$. It follows from (30) that the function $f : [0,T] \to V$ defined by

$$(f(t),w)_V = \int_\Omega f_0(t)\cdot w dx + \int_{\Gamma_2} f_2(t)\cdot w da,\ \forall w\ \in V, \tag{48}$$

has the following regularity

$$f \in W^{1,\infty}(0,T;V). \tag{49}$$

We turn now to derive a variational formulation of the mechanical problem (7)-(15). To do that, let us assume that (u,σ,β) are smooth functions satisfying (7)-(15). Let $w \in V$ and let $t \in [0,T]$. We use (8) and Green's formula (1), to obtain

$$(\sigma(t),\varepsilon(w))_{\mathcal{Q}} - \int_\Omega f_0(t)\cdot w dx = \int_\Gamma \sigma(t)\nu\cdot w da. \tag{50}$$

Moreover, since $w \in V$, it follows from (10)-(11), that

$$\begin{aligned} \int_\Gamma \sigma(t)\nu\cdot w da \ &=\ \int_{\Gamma_2}\sigma(t)\nu\cdot w da + \int_{\Gamma_3}\sigma_\nu(t)w_\nu da\ + \int_{\Gamma_3}\sigma_\tau(t)\cdot w_\tau da \\ &=\ \int_{\Gamma_2} f_2(t)\cdot w da - \int_{\Gamma_3} p_\nu\left(u_\nu(t)\right)w_\nu da \\ &\quad + \int_{\Gamma_3}\gamma_\nu(\beta(t))^2 R_\nu\left(u_\nu(t)\right)w_\nu da + \int_{\Gamma_3}\sigma_\tau(t)\cdot w_\tau da. \end{aligned} \tag{51}$$

Thus, (48), (50) and (51), lead us to

$$\left\{\begin{array}{c} \left(\sigma(t),\varepsilon(w)-\varepsilon(\dot u(t))\right)_{\mathcal{Q}} + \displaystyle\int_{\Gamma_3} p_\nu\left(u_\nu(t)\right)\left(w_\nu-\dot u_\nu(t)\right)da \\ -\displaystyle\int_{\Gamma_3}\gamma_\nu\left(\beta(t)\right)^2 R_\nu\left(u_\nu(t)\right)\left(w_\nu-\dot u_\nu(t)\right)da \\ -\displaystyle\int_{\Gamma_3}\sigma_\tau(t)\cdot\left(w_\tau-\dot u_\tau(t)\right)da = \left(f(t),w-\dot u(t)\right)_V\,. \end{array}\right. \tag{52}$$

On the other hand, using (12), we get

$$\begin{aligned}\int_{\Gamma_3} \sigma_\tau(t)\cdot \dot{u}_\tau(t)da &= -\int_{\Gamma_3} |\sigma_\tau(t)|\,|\dot{u}_\tau(t)|\,da\\ &= -\int_{\Gamma_3} p_\tau\left(u_\nu(t)\right)|\dot{u}_\tau(t)|\,da,\end{aligned}$$

which, together with the fact that

$$\begin{aligned}-\int_{\Gamma_3} \sigma_\tau(t)\cdot w_\tau da &\le \int_{\Gamma_3} |\sigma_\tau(t)|\,|w_\tau|\,da\\ &\le \int_{\Gamma_3} p_\tau\left(u_\nu(t)\right)|w_\tau|\,da,\end{aligned}$$

gives

$$\int_{\Gamma_3} p_\tau\left(u_\nu(t)\right)\left(|w_\tau| - |\dot{u}_\tau(t)|\right)da \ge -\int_{\Gamma_3} \sigma_\tau(t)\cdot\left(w_\tau - \dot{u}_\tau(t)\right)da. \tag{53}$$

Now, from (52) and (53), we find that

$$\left\{\begin{aligned}&\left(\sigma(t), \varepsilon\left(w\right) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{Q}} + \int_{\Gamma_3} p_\nu\left(u_\nu(t)\right)\left(w_\nu - \dot{u}_\nu(t)\right)da\\ &\qquad -\int_{\Gamma_3} \gamma_\nu\left(\beta(t)\right)^2 R_\nu\left(u_\nu(t)\right)\left(w_\nu - \dot{u}_\nu(t)\right)da\\ &+\int_{\Gamma_3} p_\tau\left(u_\nu(t)\right)\left(|w_\tau| - |\dot{u}_\tau(t)|\right)da \ge \left(f(t), w - \dot{u}(t)\right)_V.\end{aligned}\right. \tag{54}$$

Therefore, we use the fact that the operator $\mathcal{A}$ defined in (25) is a bounded linear operator, integrate (7) and (13) on $(0,t)$, use the initial conditions (14)-(15), combine (54) with the notation (33)-(35), to obtain the following variational formulation of the mechanical problem (7)-(15).

Problem 2. *Find a displacement field $u : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathcal{Q}$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\sigma(t) = \mathcal{A}\varepsilon(u\,(t)) + \int_0^t \mathcal{B}\left(\sigma(s), \varepsilon(u(s))\right)ds + \sigma_0 - \mathcal{A}\left(\varepsilon(u_0)\right),\ \forall t \in [0,T], \tag{55}$$

$$\left\{\begin{aligned}&\left(\sigma(t), \varepsilon\left(w - \dot{u}(t)\right)\right)_{\mathcal{Q}} + \varphi(\beta\,(t)\,, u\,(t)\,, w) - \varphi(\beta\,(t)\,, u\,(t)\,, \dot{u}(t))\\ &\ge \left(f(t), w - \dot{u}(t)\right)_V,\ \textit{for all } w \in V,\ \textit{for a.e. } t \in (0,T),\end{aligned}\right. \tag{56}$$

$$\beta(t) = \int_0^t H_{ad}\left(\beta\,(s)\,, \theta_\beta\,(s)\,, R_\nu\left(u_\nu\,(s)\right)\right)ds + \beta_0,\ \forall t \in [0,T], \tag{57}$$

$$u(0) = u_0. \tag{58}$$

Here θ_β is given by (21). To study Problem (55)-(58), we need the following additional compatibility condition on the initial data

$$\left(\sigma_0, \varepsilon(w)\right)_{\mathcal{Q}} + \varphi(\beta_0, u_0, w) \ge \left(f\,(0)\,, w\right)_V,\ \forall\, w \in V, \tag{59}$$

and we make the following smallness assumption

$$L_\tau + L_\nu < \frac{m_{\mathcal{A}}}{c_0^2}, \tag{60}$$

where c_0, $m_{\mathcal{A}}$ and L_α, $(\alpha = \nu,\, \tau)$ are given in (24), (25) and (27), respectively.

We end this section with few comments on the assumptions (59)-(60). First, we note that the assumptions (59)-(60) are made for mathematical reasons, otherwise the results obtained in Lemma 3 and Lemma 4, in the next Section, may be invalid. Physically, the compatibility condition (59) means that initially the state is in equilibrium and without it the inertial terms cannot be neglected and the problem becomes dynamic (see, [23]). Finally, in order to verify that the condition (60) is being met, we use specific mechanical problems. For example, if we consider the case in which the function p_ν is defined by (19) and the function p_τ is given by (20), then (27 (i)) holds if we take $L_\nu = c_\nu$ and $L_\tau = \mu c_\nu$ and therefore, the assumption (60) holds if $c_\nu(1+\mu) < \dfrac{m_{\mathcal{A}}}{c_0^2}$, which is satisfied if $c_\nu(1+\mu)$ is sufficiently small.

5. Existence of a weak solution

The following theorem is the main result of this paper.

Theorem 1. *Assume that (25)-(32) and (59)-(60) are fulfilled. Then, Problem (55)-(58) has at least a solution $\{u, \sigma, \beta\}$ which satisfies*

$$u \in W^{1,\infty}(0,T;V), \tag{61}$$

$$\sigma \in W^{1,\infty}(0,T;\mathcal{Q}_1), \tag{62}$$

$$\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)),\ 0 \leqslant \beta(t) \leqslant 1,\ a.e.\ x \in \Gamma_3,\ \forall t \in [0,T]. \tag{63}$$

We will divide the proof into several steps.

First step. For each $m \in \mathbb{N}^*$, we introduce a uniform partition of the time interval $[0,T]$, denoted by $t_i^m = ih_m,\ h_m = \dfrac{T}{m}$, $i = 0,...,m$. For a sequence $\{w_m^i\}_{i=0}^m$, we denote $\delta w_m^{i+1} = \dfrac{w_m^{i+1} - w_m^i}{h_m}$ and for a continuous function $g \in C([0,T];X)$ with values in a normed space X, we use the notation $g_i^m = g(t_i^m)$, $i = 0,...,m$. Using the Riesz representation theorem, we can introduce the operator $\mathcal{F} : V \to V$ defined by

$$(\mathcal{F}w, v)_V = (\mathcal{A}\varepsilon(w), \varepsilon(v))_{\mathcal{Q}} + (\sigma_0 - \mathcal{A}\varepsilon(u_0), \varepsilon(v))_{\mathcal{Q}},\ \forall v, w \in V. \tag{64}$$

It follows from (25) and (64), that the operator $\mathcal{F}$ satisfies

$$m_{\mathcal{A}} \|w_1 - w_2\|_V^2 \leqslant (\mathcal{F}w_1 - \mathcal{F}w_2, w_1 - w_2)_V,\ \forall w_1, w_2 \in V. \tag{65}$$

There exists $L_{\mathcal{A}} > 0$ such that

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\|_V \leqslant L_{\mathcal{A}} \|w_1 - w_2\|_V,\ \forall w_1, w_2 \in V. \tag{66}$$

We consider the following incremental problems $\mathcal{P}_m^{i+1}$, $i \in \{0\ ,...,\ m-1\}$.

Problem 3 $(\mathcal{P}_m^{i+1})$**.** *Find $u_m^{i+1} \in V$, such that*

$$\begin{cases} \left(\mathcal{F}u_m^{i+1}, w - \delta u_m^{i+1}\right)_V + \left(h_m \sum\limits_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right)\right), \varepsilon\left(w - \delta u_m^{i+1}\right)\right)_{\mathcal{Q}} \\ +\varphi(\beta_m^{i+1}, u_m^{i+1}, w) - \varphi(\beta_m^{i+1}, u_m^{i+1}, \delta u_m^{i+1}) \\ \geq \left(f_{i+1}^m, w - \delta u_m^{i+1}\right)_V,\ \textit{for all } w \in V, \end{cases} \tag{67}$$

where u_m^j *is the unique solution of problem* $\mathcal{P}_m^j$, $1 \leqslant j \leqslant i$,

$$\sigma_m^{j+1} = \mathcal{A}\varepsilon\left(u_m^{j+1}\right) + h_m \sum_{l=0}^{j} \mathcal{B}\left(\sigma_m^l, \varepsilon\left(u_m^l\right)\right) + \sigma_m^0 - \mathcal{A}\varepsilon\left(u_m^0\right),\ 0 \leqslant j \leqslant i, \tag{68}$$

$$\beta_m^{i+1} = h_m \sum_{j=0}^{i} H_{ad}\left(\beta_m^j,\ h_m \sum_{k=0}^{j} \beta_m^k,\ R_\nu\left(u_{m\nu}^j\right)\right) + \beta_m^0, \tag{69}$$

$$(i)\ u_m^0 = u_0,\ \ (ii)\ \sigma_m^0 = \sigma_0,\ \ (iii)\ \beta_m^0 = \beta_0, \tag{70}$$

$$f_0^m = f(0),\ f_{i+1}^m = f(t_{i+1}^m). \tag{71}$$

We notice that, from (28 (i)), (28 (v)) and (69), we obtain

$$\left|\beta_m^{i+1} - \beta_m^i\right| \leqslant L_{H_{ad}} h_m \left|\beta_m^i\right|,\ a.e. x \in \Gamma_3\ ,\ i \in \{0\ , ...,\ m-1\}\ ,$$

which implies that if $\beta_m^i \in L^\infty(\Gamma_3)$, then $\beta_m^{i+1} \in L^\infty(\Gamma_3)$, $i \in \{0\ , ...,\ m-1\}$. Now, by setting $w = \dfrac{v - u_m^i}{h_m}$ in (67), it follows that $\mathcal{P}_m^{i+1}$ is formally equivalent to the following problem.

Problem 4 ($\mathcal{Q}_m^{i+1}$). *Find a function* $u_m^{i+1} \in V$, *such that*

$$\left\{\begin{array}{l} \left(\mathcal{F}u_m^{i+1}, v - u_m^{i+1}\right)_V + \left(h_m \sum\limits_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right)\right), \varepsilon\left(v - u_m^{i+1}\right)\right)_{\mathcal{Q}} \\ +\varphi(\beta_m^{i+1}, u_m^{i+1}, v - u_m^i) - \varphi(\beta_m^{i+1}, u_m^{i+1}, u_m^{i+1} - u_m^i) \\ \geq \left(f_{i+1}^m, v - u_m^{i+1}\right)_V,\ \textit{for all } v \in\ V, \end{array}\right. \tag{72}$$

where $\left\{\sigma_m^i\right\}_{0\leqslant i\leqslant m}$, $\left\{\beta_m^i\right\}_{0\leqslant i\leqslant m}$, $\{f_i^m\}_{0\leqslant i\leqslant m}$ and u_m^0 are given by (68)-(71) and u_m^j is the unique solution of the problem $\mathcal{P}_m^j$, $1 \leqslant j \leqslant i$.

Lemma 3. *Problem* $\mathcal{P}_m^{i+1}$, $i \in \{0\ , ...,\ m-1\}$, *has a unique solution.*

Proof. Let $A : V \to V$ be the operator defined by

$$(Aw, v)_V = (\mathcal{F}w, v)_V + \left(h_m \sum_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right)\right), \varepsilon\left(v\right)\right)_{\mathcal{Q}} + j_{ad}(\beta_m^{i+1}, w, v), \tag{73}$$

for all w, $v \in V$. Using (65)-(66), (73), (44) and (46), we deduce that the operator A is strongly monotone and Lipschitz continuous on V. Let $g \in V$, it follows from (33) and (40), that the functional $\Theta : V \to \mathbb{R}$, defined by:

$$\Theta(v) = \psi(g, v - u_m^i),\ \forall v \in V,$$

is a proper convex and continuous functional on V. Hence, from a standard result on elliptic variational inequalities of second kind (see [13, p.60]), the following problem: Find $u_{mg}^{i+1} \in V$, such that

$$\left\{\begin{array}{l} \left(Au_{mg}^{i+1}, v - u_{mg}^{i+1}\right)_V + \psi(g, v - u_m^i) - \psi(g, u_{mg}^{i+1} - u_m^i) \\ \geq \left(f_{i+1}^m, v - u_{mg}^{i+1}\right)_V,\ \text{for all } v \in\ V, \end{array}\right. \tag{74}$$

has a unique solution $u_{mg}^{i+1} \in V$. To continue, we define the operator $\Psi : V \to V$, by

$$\Psi(g) = u_{mg}^{i+1},\ \forall g \in V. \tag{75}$$

Let g_1, $g_2 \in V$, using the notation $u_1 = u_{mg_1}^{i+1}$ and $u_2 = u_{mg_2}^{i+1}$, then by (74), we get

$$\begin{aligned}(Au_1 - Au_2, u_1 - u_2)_V \quad &\leq \quad \psi(g_1, u_2 - u_m^i) - \psi(g_1, u_1 - u_m^i) \\ &\qquad + \psi(g_2, u_1 - u_m^i) - \psi(g_2, u_2 - u_m^i),\end{aligned}$$

which together with (36), (46), (65) and (73), implies that

$$m_{\mathcal{A}} \|u_1 - u_2\|_V^2 \leqslant c_0^2 (L_\tau + L_\nu) \|g_1 - g_2\|_V \|u_1 - u_2\|_V ,$$

and using (75), we have

$$\|\Psi g_2 - \Psi g_1\|_V \leqslant \frac{c_0^2 (L_\tau + L_\nu)}{m_{\mathcal{A}}} \|g_1 - g_2\|_V .$$

This last inequality implies that if $L_\tau + L_\nu < \dfrac{m_{\mathcal{A}}}{c_0^2}$, then Ψ is a contraction function on the Banach space V. Therefore, Ψ has a unique fixed point $g^* \in V$. We have now all the ingredients to prove Lemma 3. Let g^* be the unique fixed point of Ψ defined by (75) and let $u_m^{i+1} = g^* = u_{mg^*}^{i+1}$ be the unique solution of the problem (74) for $g = g^*$ and keeping in mind (73) and (35), we deduce that u_m^{i+1} is a solution for the problem $\mathcal{Q}_m^{i+1}$ which is equivalent to $\mathcal{P}_m^{i+1}$. The uniqueness of the solution, now, is a consequence of the uniqueness of the fixed point of the operator Ψ and of the uniqueness of the solution of the problem (74). $\square$

In the rest of this paper, the same letter c will be used to denote different positive constants which do not depend on $m \in \mathbb{N}^*$ nor on $t \in (0, T)$.

Second step. In this step we have the following result.

Lemma 4. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$,*

$$\|\sigma_m^{i+1}\|_{\mathcal{Q}} + \|u_m^{i+1}\|_V + \|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leq c,\ i \in \{0, ..., m-1\}, \tag{76}$$

$$\|\delta u_m^{i+1}\|_V \leq c,\ i \in \{0, ..., m-1\}. \tag{77}$$

Proof. It follows from (31), (32) and (70), that there exists $c > 0$, such that

$$\|\sigma_m^0\|_{\mathcal{Q}} + \|u_m^0\|_V + \|\beta_m^0\|_{L^\infty(\Gamma_3)} \leq c, \tag{78}$$

for all $m \in \mathbb{N}^*$. Using (28 (i)), (28 (v)) and (69), we get

$$\|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leqslant c h_m \sum_{j=0}^{i} \|\beta_m^j\|_{L^\infty(\Gamma_3)} + \|\beta_m^0\|_{L^\infty(\Gamma_3)},\ i \in \{0, ..., m-1\}.$$

Therefore, applying Lemma 2, leads to

$$\|\beta_m^{i+1}\|_{L^\infty(\Gamma_3)} \leqslant c,\ i \in \{0, ..., m-1\}. \tag{79}$$

Taking $v = 0_V$ in (72) and keeping in mind (35), yields

$$\begin{aligned}\left(\mathcal{F}u_m^{i+1}, u_m^{i+1}\right)_V + j_{ad}(\beta_m^{i+1}, u_m^{i+1}, u_m^{i+1}) \quad &\leq \quad \psi(u_m^{i+1}, -u_m^i) - \psi(u_m^{i+1}, u_m^{i+1} - u_m^i) + \\ &- \left(h_m \sum_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right)\right), \varepsilon\left(u_m^{i+1}\right)\right)_{\mathcal{Q}} + \left(f_{i+1}^m, u_m^{i+1}\right)_V,\end{aligned}$$

and using (65), (47), (37), (26), we get

$$\begin{aligned} m_{\mathcal{A}} \left\|u_m^{i+1}\right\|_V^2 \quad &\leqslant \quad c_0^2 \left(L_\tau + L_\nu\right) \left\|u_m^{i+1}\right\|_V^2 + ch_m \sum_{j=0}^{i} \left(\left\|\sigma_m^j\right\|_{\mathcal{Q}} + \left\|u_m^j\right\|_V\right) \left\|u_m^{i+1}\right\|_V \\ &+ c \left\|\mathcal{B}\left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}\right)\right\|_{\mathcal{Q}} \left\|u_m^{i+1}\right\|_V + \left\|\mathcal{F}\left(0_V\right)\right\|_V \left\|u_m^{i+1}\right\|_V + \left\|f_{i+1}^m\right\|_V \left\|u_m^{i+1}\right\|_V. \end{aligned}$$

Hence, due to the assumption (60) and (49), we obtain

$$\left\|u_m^{i+1}\right\|_V \leqslant ch_m \sum_{j=0}^{i} \left(\left\|\sigma_m^j\right\|_{\mathcal{Q}} + \left\|u_m^j\right\|_V\right) + c. \tag{80}$$

On the other hand, from (25)-(26) and (68), we find that

$$\left\|\sigma_m^{i+1}\right\|_{\mathcal{Q}} \leqslant c \left\|u_m^{i+1}\right\|_V + ch_m \sum_{j=0}^{i} \left(\left\|\sigma_m^j\right\|_{\mathcal{Q}} + \left\|u_m^j\right\|_V + \left\|\mathcal{B}\left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d}\right)\right\|_{\mathcal{Q}}\right) + c,$$

this inequality combined with (80), gives

$$\left\|\sigma_m^{i+1}\right\|_{\mathcal{Q}} + \left\|u_m^{i+1}\right\|_V \leqslant ch_m \sum_{j=0}^{i} \left(\left\|\sigma_m^j\right\|_{\mathcal{Q}} + \left\|u_m^j\right\|_V\right) + c.$$

Now, applying Lemma 2, in the last inequality, yields

$$\left\|\sigma_m^{i+1}\right\|_{\mathcal{Q}} + \left\|u_m^{i+1}\right\|_V \leqslant c,\ i \in \ \{0, ...,\ m-1\},$$

and employing (79) we obtain (76). To continue, using (28 (i)), (28 (v)) and (69), one has

$$\left\|\beta_m^{i+1} - \beta_m^i\right\|_{L^\infty(\Gamma_3)} \leqslant ch_m \left\|\beta_m^i\right\|_{L^\infty(\Gamma_3)},\ i \in \ \{0, ...,\ m-1\},$$

and thanks to (78) and (79), we have

$$\left\|\delta \beta_m^{i+1}\right\|_{L^\infty(\Gamma_3)} \leqslant c,\ i \in \ \{0, ...,\ m-1\}. \tag{81}$$

Setting $v = u_m^0$ in (72) for i=0, and $w = u_m^1 - u_m^0$ in (59), adding the two inequalities, we obtain

$$\begin{cases} \left(\mathcal{F}u_m^1 - \mathcal{F}u_m^0, u_m^1 - u_m^0\right)_V \\ + j_{ad}(\beta_m^0, u_m^1, u_m^1 - u_m^0) - j_{ad}(\beta_m^0, u_m^0, u_m^1 - u_m^0) \\ \leqslant j_{ad}(\beta_m^0, u_m^1, u_m^1 - u_m^0) - j_{ad}(\beta_m^1, u_m^1, u_m^1 - u_m^0) \\ - \left(h_m \mathcal{B}\left(\sigma_m^0, \varepsilon\left(u_m^0\right)\right), \varepsilon\left(u_m^1 - u_m^0\right)\right)_{\mathcal{Q}} \\ + \psi(u_m^0, u_m^1 - u_m^0) - \psi(u_m^1, u_m^1 - u_m^0) + \left(f_1^m - f_0^m, u_m^1 - u_m^0\right)_V, \end{cases}$$

we use now (65), (46), (78), (79), (42), (26), (38), to see that

$$\begin{cases} m_{\mathcal{A}} \left\| u_m^1 - u_m^0 \right\|_V^2 \leqslant c \left\| \beta_m^1 - \beta_m^0 \right\|_{L^2(\Gamma_3)} \left\| u_m^1 - u_m^0 \right\|_V \\ +ch_m \left(\left\| \sigma_m^0 \right\|_{\mathcal{Q}} + \left\| u_m^0 \right\|_V \right) \left\| u_m^1 - u_m^0 \right\|_V \\ +ch_m \left\| \mathcal{B} \left(0_{\mathbb{S}^d}, 0_{\mathbb{S}^d} \right) \right\|_{\mathcal{Q}} \left\| u_m^1 - u_m^0 \right\|_V \\ +c_0^2 \left(L_\tau + L_\nu \right) \left\| u_m^1 - u_m^0 \right\|_V^2 + \left\| f_1^m - f_0^m \right\|_V \left\| u_m^1 - u_m^0 \right\|_V , \end{cases}$$

and thanks to (60), (49), (78) and (81), we get

$$\begin{aligned} \left\| \frac{u_m^1 - u_m^0}{h_m} \right\|_V &\leqslant c + c \left\| \frac{\beta_m^1 - \beta_m^0}{h_m} \right\|_{L^2(\Gamma_3)} + c \left\| \frac{f_1^m - f_0^m}{h_m} \right\|_V \\ &\leqslant c + c \left\| \dot{f} \right\|_{L^\infty(0,T;V)} . \end{aligned}$$

Thus, we have

$$\left\| \delta u_m^1 \right\|_V \leqslant c. \tag{82}$$

Taking $w = 0_V$ in problem $\mathcal{P}_m^{i+1}$, and $w = \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}$ in problem $\mathcal{P}_m^i$, adding the two inequalities, we obtain

$$\begin{cases} \left(\mathcal{F} u_m^{i+1} - \mathcal{F} u_m^i, \delta u_m^{i+1} \right)_V + j_{ad}(\beta_m^{i+1}, u_m^{i+1}, \delta u_m^{i+1}) - j_{ad}(\beta_m^{i+1}, u_m^i, \delta u_m^{i+1}) \\ \leqslant j_{ad}(\beta_m^i, u_m^i, \delta u_m^{i+1}) - j_{ad}(\beta_m^{i+1}, u_m^i, \delta u_m^{i+1}) + \\ - \left(h_m \mathcal{B} \left(\sigma_m^i, \varepsilon \left(u_m^i \right) \right), \varepsilon \left(\delta u_m^{i+1} \right) \right)_{\mathcal{Q}} \\ + \left(\psi(u_m^i, \dfrac{u_m^{i+1} - u_m^{i-1}}{h_m}) - \psi(u_m^i, \dfrac{u_m^i - u_m^{i-1}}{h_m}) \right) - \psi(u_m^{i+1}, \dfrac{u_m^{i+1} - u_m^i}{h_m}) \\ + \left(f_{i+1}^m - f_i^m, \delta u_m^{i+1} \right)_V . \end{cases}$$

It follows from (65), (42), (46), (79), (26), (38) and (39), that

$$\begin{cases} m_{\mathcal{A}} \left\| u_m^{i+1} - u_m^i \right\|_V^2 \leqslant c \left\| \beta_m^{i+1} - \beta_m^i \right\|_{L^2(\Gamma_3)} \left\| u_m^{i+1} - u_m^i \right\|_V \\ +ch_m \left(\left\| \sigma_m^i \right\|_{\mathcal{Q}} + \left\| u_m^i \right\|_V + c \right) \left\| u_m^{i+1} - u_m^i \right\|_V \\ +c_0^2 \left(L_\tau + L_\nu \right) \left\| u_m^{i+1} - u_m^i \right\|_V^2 + \left\| f_{i+1}^m - f_i^m \right\|_V \left\| u_m^{i+1} - u_m^i \right\|_V , \end{cases}$$

which, together with (60), (49) and (76), gives

$$\left\| \delta u_m^{i+1} \right\|_V \leqslant c \left\| \delta \beta_m^{i+1} \right\|_{L^2(\Gamma_3)} + c \ + c \left\| \dot{f} \right\|_{L^\infty(0,T;V)} . \tag{83}$$

Now, (77) is a consequence of (81), (82) and (83). □

Third step. In this step we construct an approximate solution for the problem (55)-(58). To this end, for each $m \in \mathbb{N}^*$, let u_m^j be the unique solution of the problem $\mathcal{P}_m^j$, $j = 1, ..., m$. We introduce the following functions $u_m : [0, T] \to V$,

$\tilde{u}_m : [0,T] \to V$, $\tilde{\sigma}_m : [0,T] \to \mathcal{Q}$, $\tilde{\beta}_m : [0,T] \to L^2(\Gamma_3)$, $\tilde{\theta}_m : [0,T] \to L^2(\Gamma_3)$, $\mathcal{B}_m : [0,T] \to \mathcal{Q}$ and $f_m : [0,T] \to V$ defined, respectively, by

$$u_m(0) = u_0,\ u_m(t) = u_m^i + (t - t_i^m)\,\delta u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{84}$$

$$\tilde{u}_m(0) = u_0,\ \tilde{u}_m(t) = u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{85}$$

$$\tilde{\sigma}_m(0) = \sigma_0,\ \tilde{\sigma}_m(t) = \sigma_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{86}$$

$$\tilde{\beta}_m(0) = \beta_0,\ \tilde{\beta}_m(t) = \beta_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{87}$$

Here $\left\{\sigma_m^i\right\}_{0\leqslant i\leqslant m}$, $\left\{\beta_m^i\right\}_{0\leqslant i\leqslant m}$ and u_m^0 are given by (68)-(70).

$$\tilde{\theta}_m(0) = 0,\ \tilde{\theta}_m(t) = h_m\sum_{j=0}^{i+1}\beta_m^j,\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{88}$$

$$\mathcal{B}_m(t) = \begin{cases} h_m \sum\limits_{j=0}^{i} \mathcal{B}\left(\sigma_m^j, \varepsilon\left(u_m^j\right)\right),\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1, \\ \qquad\qquad \mathcal{B}_m(0) = 0_{\mathcal{Q}}. \end{cases} \tag{89}$$

$$f_m(0) = f(0),\ f_m(t) = f_{i+1}^m,\ \forall t \in \left(t_i^m, t_{i+1}^m\right],\ i = 0,...,m-1. \tag{90}$$

From (84), the function u_m has a derivative function which is given by

$$\dot{u}_m(t) = \delta u_m^{i+1},\ \forall t \in \left(t_i^m, t_{i+1}^m\right),\ i = 0,...,m-1. \tag{91}$$

We have the following estimate results.

Lemma 5. *There exists $c > 0$, such that for all $m \in \mathbb{N}^*$,*

$$\begin{aligned}
&(92) & \|\tilde{\sigma}_m(t)\|_{\mathcal{Q}} + \|\tilde{u}_m(t)\|_V + \left\|\tilde{\beta}_m(t)\right\|_{L^\infty(\Gamma_3)} &\leq c,\ \forall t \in [0,T],\\
&(93) & \|u_m(t)\|_V &\leq c,\ \forall t \in [0,T],\\
&(94) & \|\dot{u}_m(t)\|_V &\leq c,\ a.e. t \in [0,T],\\
&(95) & \|\tilde{u}_m(t) - u_m(t)\|_V &\leq ch_m,\ \forall t \in [0,T],\\
&(96) & \|f_m(t) - f(t)\|_V &\leq ch_m,\ \forall t \in [0,T],\\
&(97) & \|u_m(t) - u_m(s)\|_V &\leq c\,|t-s|,\ \forall t,s \in [0,T],\\
&(98) & \|u_m(t) - u_m(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)} &\leq c\,|t-s|,\ \forall t,s \in [0,T].
\end{aligned}$$

Proof. It is clear that (92)-(94) are consequences of Lemma 4, (84)-(87) and (91). For the proof of (95)-(96), see [16, Lemma 4.7]. Now, for all $t \in (t_i, t_{i+1}]$, $i \in \{0,...,m-1\}$, we have

$$\begin{aligned}
u_0 + \int_0^t \dot{u}_m(r)dr &= u_0 + \sum_{j=1}^{i}\int_{t_{j-1}^m}^{t_j^m} \dot{u}_m(r)dr + \int_{t_i^m}^{t} \dot{u}_m(r)dr\\
&= u_0 + \sum_{j=1}^{i} h_m \delta u_m^j + \left(\int_{t_i^m}^{t} dr\right)\delta u_m^{i+1}\\
&= u_m^i + (t - t_i^m)\,\delta u_m^{i+1} = u_m(t).
\end{aligned}$$

Therefore, using (94), we get

$$\|u_m(t) - u_m(s)\|_V \leq \left| \int_s^t \|\dot{u}_m(r)\|_V \, dr \right| \leq c\,|t-s|\,,$$

for all $t,\, s \in [0,T]$. Finally, (98) is a direct consequence of (97) and (24). $\square$

In the next we need the following result.

Lemma 6. *There exists $c>0$, such that for all $m,\, n \in \mathbb{N}^*$ with $m>n$,*

$$\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}} \leqslant c \int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V \, ds + c h_n,\ \ \forall t \in [0,T]\,. \tag{99}$$

Proof. Let $m,\, n \in \mathbb{N}^*$ with $m>n$. It is obvious that (99) holds for $t=0$. Now, for each $t \in (0,T]$, there exist two integers $q \in \{0,...,m-1\}$ and $p \in \{0,...,n-1\}$, such that

$$t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right]. \tag{100}$$

We use (68), (70), (85) and (86) to obtain

$$\tilde{\sigma}_m(t) = \mathcal{A}\varepsilon\left(\tilde{u}_m(t)\right) + h_m \mathcal{B}\left(\sigma_0, \varepsilon\left(u_0\right)\right) + \sigma_0 - \mathcal{A}\varepsilon\left(u_0\right),\ \forall t \in \left(t_0^m, t_1^m\right]. \tag{101}$$

On the other hand, let $t \in \left(t_q^m, t_{q+1}^m\right]$, $q \in \{1,...,m-1\}$. It follows from (68), (85) and (86), that

$$\begin{aligned}\tilde{\sigma}_m(t) \ &= \ \mathcal{A}\varepsilon\left(\tilde{u}_m(t)\right) + \sum_{l=1}^{q} \int_{t_{l-1}^m}^{t_l^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon\left(\tilde{u}_m(s)\right)\right) ds + h_m \mathcal{B}\left(\sigma_0, \varepsilon\left(u_0\right)\right) \\ &\quad + \sigma_0 - \mathcal{A}\varepsilon\left(u_0\right),\end{aligned}$$

which, with (101), gives

$$\left\{\begin{aligned} &\tilde{\sigma}_m(t) = \mathcal{A}\varepsilon\left(\tilde{u}_m(t)\right) + \int_0^t \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon\left(\tilde{u}_m(s)\right)\right) ds \\ &+ \int_t^{t_q^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon\left(\tilde{u}_m(s)\right)\right) ds + h_m \mathcal{B}\left(\sigma_0, \varepsilon\left(u_0\right)\right) + \sigma_0 - \mathcal{A}\varepsilon\left(u_0\right),\end{aligned}\right. \tag{102}$$

for all $t \in \left(t_q^m, t_{q+1}^m\right]$, $q \in \{0,...,m-1\}$. Therefore, we deduce from (102), (25), (26), (100) and (92) that

$$\begin{aligned}\|\tilde{\sigma}_m(t) - \tilde{\sigma}_n(t)\|_{\mathcal{Q}} \ &\leqslant \ c\,\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V \, ds \\ &\quad + c\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_{\mathcal{Q}} \, ds + c h_m + c h_n,\ \forall t \in [0,T]\,.\end{aligned}$$

Using Lemma 1 in the last inequality, we obtain

$$\|\tilde{\sigma}_m(t) - \tilde{\sigma}_n(t)\|_{\mathcal{Q}} \leqslant c\,\|\tilde{u}_m(t) - \tilde{u}_n(t)\|_V + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V \, ds + c h_n, \tag{103}$$

for all $t \in [0,T]$. To continue, we use (85), (86) and (89) to obtain

$$\mathcal{B}_m(t) \ = \ \int_0^{t_q^m} \mathcal{B}\left(\tilde{\sigma}_m(s), \varepsilon\left(\tilde{u}_m(s)\right)\right) ds + h_m \mathcal{B}\left(\sigma_0, \varepsilon\left(u_0\right)\right), \tag{104}$$

$$\mathcal{B}_n(t) \ = \ \int_0^{t_p^n} \mathcal{B}\left(\tilde{\sigma}_n(s), \varepsilon\left(\tilde{u}_n(s)\right)\right) ds + h_n \mathcal{B}\left(\sigma_0, \varepsilon\left(u_0\right)\right), \tag{105}$$

for all $t \in \left(t_q^m, t_{q+1}^m\right] \cap \left(t_p^n, t_{p+1}^n\right]$, $q \in \{0, ..., m-1\}$ and $p \in \{0, ..., n-1\}$. It follows from (104), (105) and (26), that

$$\begin{aligned}\|\mathcal{B}_m(t) - \mathcal{B}_n(t)\|_{\mathcal{Q}} \leqslant\; & c\int_0^t \|\tilde{\sigma}_m(s) - \tilde{\sigma}_n(s)\|_{\mathcal{Q}}\, ds + c\int_0^t \|\tilde{u}_m(s) - \tilde{u}_n(s)\|_V\, ds \\ & + \int_{t_q^m}^t \|\mathcal{B}(\tilde{\sigma}_m(s), \varepsilon(\tilde{u}_m(s)))\|_{\mathcal{Q}}\, ds + \int_{t_p^n}^t \|\mathcal{B}(\tilde{\sigma}_n(s), \varepsilon(\tilde{u}_n(s)))\|_{\mathcal{Q}}\, ds \\ & + ch_m + ch_n,\end{aligned}$$

and employing (103), (100) and (92), we obtain (99). □

Lemma 7. *There exists an element* $u \in W^{1,2}(0,T;V)$ *and two subsequences of* $\{u_m\}$ *and* $\{\tilde{u}_m\}$ *again denoted by* $\{u_m\}$ *and* $\{\tilde{u}_m\}$*, respectively, such that*

(106) $\quad u_m \rightharpoonup u$ *weakly in* $L^2(0,T;V)$.

(107) $\quad \dot{u}_m \rightharpoonup \dot{u}$ *weakly in* $L^2(0,T;V)$.

(108) $\quad \varepsilon(\dot{u}_m) \rightharpoonup \varepsilon(\dot{u})$ *weakly in* $L^2(0,T;\mathcal{Q})$.

(109) $\quad u_m \to u$ *strongly in* $C\left([0,T]; L^2\left(\Gamma_3;\mathbb{R}^d\right)\right)$.

(110) $\quad u_m \to u$ *strongly in* $C([0,T];V)$.

(111) $\quad \tilde{u}_m \to u$ *strongly in* $L^2(0,T;V)$.

Proof. We notice that $L^2(0,T;V)$ and $L^2(0,T;\mathcal{Q})$ are Hilbert spaces equipped with the respective canonical inner products

$$(w,v)_{L^2(0,T;V)} = \int_0^T (w(s), v(s))_V\, ds,\ \ (\vartheta,\zeta)_{L^2(0,T;\mathcal{Q})} = \int_0^T (\vartheta(s),\zeta(s))_{\mathcal{Q}}\, ds,$$

for all w, $v \in L^2(0,T;V)$ and for all ϑ, $\zeta \in L^2(0,T;\mathcal{Q})$. Therefore, using standard compactness argument see [3], it follows from (93) and (94) that there are two functions u and z belong to $L^2(0,T;V)$ and a subsequence of $\{u_m\}$ again denoted by $\{u_m\}$, such that

$$u_m \rightharpoonup u \text{ weakly } in\ L^2(0,T;V) \text{ and } \dot{u}_m \rightharpoonup z \text{ weakly } in\ L^2(0,T;V).$$

Now, since $v \longmapsto \left(w, \int_0^T v(s)\dot{\xi}(s)ds\right)_V$ and $v \longmapsto \left(w, -\int_0^T v(s)\xi(s)ds\right)_V$ are linear continuous functionals on $L^2(0,T;V)$, for all $w\in V$ and for all $\xi\in D(0,T)$, we deduce that

$$\begin{aligned}\left(w, \int_0^T u(s)\dot{\xi}(s)ds\right)_V &= \lim_{m\to\infty}\left(w, \int_0^T u_m(s)\dot{\xi}(s)ds\right)_V = \lim_{m\to\infty}\left(w, -\int_0^T \dot{u}_m(s)\xi(s)ds\right)_V \\ &= \left(w, -\int_0^T z(s)\xi(s)ds\right)_V.\end{aligned}$$

Therefore, z is the weak derivative of u with respect to the time variable. To continue, for each $\zeta\in L^2(0,T;\mathcal{Q})$, consider the functional $\phi_\zeta : L^2(0,T;V) \to \mathbb{R}$ defined by

$$\phi_\zeta(v) = \int_0^T (\varepsilon(v(s)), \zeta(s))_{\mathcal{Q}}\, ds,\ \forall v \in L^2(0,T;V).$$

It is clear that ϕ_ζ is a linear continuous functional on $L^2(0,T;V)$, so it follows from (107) that

$$\lim_{m\to+\infty}\int_0^T \left(\varepsilon\left(\dot{u}_m(s)\right),\zeta(s)\right)_{\mathcal{Q}} ds = \lim_{m\to+\infty}\phi_\zeta\left(\dot{u}_m\right) = \phi_\zeta\left(\dot{u}\right) = \int_0^T \left(\varepsilon\left(\dot{u}(s)\right),\zeta(s)\right)_{\mathcal{Q}} ds,$$

for all $\zeta\in L^2(0,T;\mathcal{Q})$, more precisely, we get the convergence (108). Now, it follows from (98) that $E=\left\{u_m:[0,T]\to L^2\left(\Gamma_3;\mathbb{R}^d\right),\ m\in\mathbb{N}^*\right\}$, the set of the traces of $\{u_m\}$ on Γ_3, is equicontinuous. Furthermore, from the fact that the trace map is compact operator and using (93), we deduce that $E(t)=\{u_m(t),\ u_m\in E\}$ is relatively compact for all $t\in[0,T]$. Thus, by applying a version of the Arzela-Ascoli theorem see [12], and taking another subsequence if necessary, we obtain (109). To prove (110) we need to show that the subsequence $\{u_m\}$, obtained in (106)-(109), is a Cauchy sequence in the Banach space $C([0,T];V)$. To this end, using (72), (34), (35), (39), (85), (87), (89) and (90), we conclude that $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$, $\left\{\tilde{\beta}_m\right\}$ and $\{f_m\}$ satisfy the following

$$\begin{cases} \left(\mathcal{F}\tilde{u}_m(t), v-\tilde{u}_m(t)\right)_V + \left(\mathcal{B}_m(t),\varepsilon\left(v-\tilde{u}_m(t)\right)\right)_{\mathcal{Q}} + \\ +j_{ad}(\tilde{\beta}_m(t),\tilde{u}_m(t),v-\tilde{u}_m(t)) + \psi(\tilde{u}_m(t),v-\tilde{u}_m(t)) \\ \geq \left(f_m(t),v-\tilde{u}_m(t)\right)_V, \end{cases} \tag{112}$$

for all $v\in V$ and for all $t\in[0,T]$. Now, let $m,\,n\in\mathbb{N}^*$, such that $m>n>T$, by taking $(\mathcal{B}_m,\,\tilde{u}_m,\,\tilde{\beta}_m,\,f_m,\,v)=(\mathcal{B}_m,\,\tilde{u}_m,\,\tilde{\beta}_m,\,f_m,\,\tilde{u}_n)$, $(\mathcal{B}_m,\tilde{u}_m,\,\tilde{\beta}_m,\,f_m,\,v)=(\mathcal{B}_n,\,\tilde{u}_n,\,\tilde{\beta}_n,\,f_n,\,\tilde{u}_m)$ in (112) and adding the two inequalities, we get

$$\begin{cases} \left(\mathcal{F}\tilde{u}_m(t)-\mathcal{F}\tilde{u}_n(t),\tilde{u}_m(t)-\tilde{u}_n(t)\right)_V \leqslant \left(\mathcal{B}_n(t)-\mathcal{B}_m(t),\varepsilon\left(\tilde{u}_m(t)-\tilde{u}_n(t)\right)\right)_{\mathcal{Q}} \\ \psi(\tilde{u}_n(t),\tilde{u}_m(t)-\tilde{u}_n(t))+\psi(\tilde{u}_m(t),\tilde{u}_n(t)-\tilde{u}_m(t)) \\ +j_{ad}(\tilde{\beta}_n(t),\tilde{u}_n(t),\tilde{u}_m(t)-\tilde{u}_n(t))+j_{ad}(\tilde{\beta}_m(t),\tilde{u}_m(t),\tilde{u}_n(t)-\tilde{u}_m(t)) \\ +\left(f_m(t)-f_n(t),\tilde{u}_m(t)-\tilde{u}_n(t)\right)_V,\ \forall t\in[0,T], \end{cases}$$

which combined with (65), (41), (43), (92) and using the inequality

$$ab\leq\frac{a^2}{m_{\mathcal{A}}}+\frac{m_{\mathcal{A}}}{4}b^2,\ \forall a,\ b\in\mathbb{R},$$

leads us to

$$\begin{cases} \left\|\tilde{u}_m(t)-\tilde{u}_n(t)\right\|_V^2 \leqslant c\left\|\mathcal{B}_m(t)-\mathcal{B}_n(t)\right\|_{\mathcal{Q}}^2 + c\left\|\tilde{u}_m(t)-\tilde{u}_n(t)\right\|_{L^2(\Gamma_3;\mathbb{R}^d)} \\ \qquad +c\left\|f_m(t)-f(t)\right\|_V^2 + c\left\|f(t)-f_n(t)\right\|_V^2, \end{cases} \tag{113}$$

for all $t\in[0,T]$. Using (95), it is straightforward to show that

$$\left\|\tilde{u}_m(t)-\tilde{u}_n(t)\right\|_V \leqslant \left\|u_m(t)-u_n(t)\right\|_V + ch_m + ch_n. \tag{114}$$

Also, from (95) and (24), we deduce that

$$\left\|\tilde{u}_m(t)-\tilde{u}_n(t)\right\|_{L^2(\Gamma_3;\mathbb{R}^d)} \leqslant \left\|u_m(t)-u_n(t)\right\|_{L^2(\Gamma_3;\mathbb{R}^d)} + ch_m + ch_n. \tag{115}$$

Now, it follows from (113), (114), (115), (99) and (96), that

$$\begin{aligned}\|\tilde{u}_m(t)-\tilde{u}_n(t)\|_V^2 &\leqslant c\,\|u_m(t)-u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}+c\int_0^t \|u_n(s)-u_m(s)\|_V^2\,ds\\ &\quad +ch_m+ch_n,\end{aligned}$$

which, together with (95), and using the fact that
$\|u_m(t)-u_n(t)\|_V^2 \leqslant c\,\|u_m(t)-\tilde{u}_m(t)\|_V^2+c\,\|\tilde{u}_m(t)-\tilde{u}_n(t)\|_V^2+c\,\|\tilde{u}_n(t)-u_n(t)\|_V^2,$
yields

$$\begin{aligned}\|u_m(t)-u_n(t)\|_V^2 &\leqslant c\,\|u_m(t)-u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}+c\int_0^t \|u_n(s)-u_m(s)\|_V^2\,ds\\ &\quad +ch_m+ch_n,\ \forall t\in[0,T].\end{aligned}$$

Now, we use Lemma 1, in the last inequality, to obtain

$$\begin{aligned}\|u_m(t)-u_n(t)\|_V^2 &\leqslant c\,\|u_m(t)-u_n(t)\|_{L^2(\Gamma_3;\mathbb{R}^d)}+c\int_0^t \|u_m(s)-u_n(s)\|_{L^2(\Gamma_3;\mathbb{R}^d)}\,ds+\\ &\quad +ch_n,\ \forall t\in[0,T].\end{aligned}$$

Thus, we get

$$\|u_m-u_n\|_{C([0,T];V)}^2 \leqslant c\,\|u_m-u_n\|_{C([0,T];L^2(\Gamma_3;\mathbb{R}^d))}+ch_n,$$

which combined with (109), implies that $\{u_m\}$ is a Cauchy sequence in $C([0,T];V)$, and using the convergence (106), we obtain (110). Finally, the convergence (111) is a consequence of (95) and (110). □

In the rest of this paper u is the function obtained in Lemma 7 and $\{u_m\},\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$, $\left\{\tilde{\beta}_m\right\}$ and $\{f_m\}$ represent appropriate subsequences of $\{u_m\}$, $\{\mathcal{B}_m\}$, $\{\tilde{u}_m\}$, $\left\{\tilde{\beta}_m\right\}$ and $\{f_m\}$, respectively, such that the convergences (106)-(111) hold. Let $\Lambda : C([0,T];\mathcal{Q})\to C([0,T];\mathcal{Q})$ be the operator defined by

$$\Lambda\zeta(t)=\mathcal{A}\varepsilon(u(t))+\int_0^t \mathcal{B}(\zeta(s),\varepsilon(u(s)))\,ds+\sigma_0-\mathcal{A}\varepsilon(u_0),\ \forall\, t\in[0,T], \tag{116}$$

for all $\zeta\in C([0,T];\mathcal{Q})$. We have the following result.

Lemma 8. *The operator Λ has a unique fixed point $\sigma\in C([0,T];\mathcal{Q})$. Moreover, we have*

$$\left\|\mathcal{B}_m-\widetilde{\mathcal{B}}\right\|_{L^2(0,T;\mathcal{Q})} \leqslant c\,\|\tilde{u}_m-u\|_{L^2(0,T;V)}+ch_m, \tag{117}$$

where

$$\widetilde{\mathcal{B}}(t)=\int_0^t \mathcal{B}(\sigma(s),\varepsilon(u(s)))\,ds,\ \forall\, t\in[0,T]. \tag{118}$$

Proof. Let $\zeta_1,\zeta_2\in C([0,T];\mathcal{Q})$ and let $t\in[0,T]$. Using (26) and (116), one has

$$\|\Lambda\zeta_1(t)-\Lambda\zeta_2(t)\|_{\mathcal{Q}}^2 \leqslant c\int_0^t \|\zeta_1(s)-\zeta_2(s)\|_{\mathcal{Q}}^2\,ds,$$

Reiterating the last inequality n times, we infer that

$$\|\Lambda^n\zeta_1-\Lambda^n\zeta_2\|_{C([0,T];\mathcal{Q})}^2 \leqslant \frac{(cT)^n}{n!}\,\|\zeta_1-\zeta_2\|_{C([0,T];\mathcal{Q})}^2.$$

Thus, for n sufficiently large, a power Λ^n of Λ is a contraction in the Banach space $C([0,T];\mathcal{Q})$. Which implies that the operator Λ has a unique fixed point $\sigma \in C([0,T];\mathcal{Q})$. Now, let σ be the fixed point of Λ, it follows from (116), (102), (25), (26) and (92), that

$$\begin{aligned}\|\tilde{\sigma}_m(t)-\sigma(t)\|_{\mathcal{Q}} &\leqslant c\|\tilde{u}_m(t)-u(t)\|_V + c\int_0^t \|\tilde{\sigma}_m(s)-\sigma(s)\|_{\mathcal{Q}}\,ds \\ &\quad + c\int_0^t \|\tilde{u}_m(s)-u(s)\|_V\,ds + ch_m,\ \forall t\in[0,T].\end{aligned}$$

Using Lemma 1, in the last inequality, one has

$$\|\tilde{\sigma}_m(t)-\sigma(t)\|_{\mathcal{Q}} \leqslant c\|\tilde{u}_m(t)-u(t)\|_V + c\int_0^t \|\tilde{u}_m(s)-u(s)\|_V\,ds + ch_m,$$

which together with (104) and (118), gives

$$\begin{aligned}\left\|\mathcal{B}_m(t)-\widetilde{\mathcal{B}}(t)\right\|_{\mathcal{Q}} &\leqslant c\int_0^t \|\tilde{\sigma}_m(s)-\sigma(s)\|_{\mathcal{Q}}\,ds + c\int_0^t \|\tilde{u}_m(s)-u(s)\|_V\,ds + ch_m \\ &\leqslant c\int_0^t \|\tilde{u}_m(s)-u(s)\|_V\,ds + ch_m,\end{aligned}$$

and inequality (117) follows. □

Now, consider the following problem.

Problem 5. *Find a function $\beta:[0,T]\to L^\infty(\Gamma_3)$, such that*

$$\dot{\beta}(t) = H_{ad}\left(\beta(t),\theta_\beta(t),R_\nu(u_\nu(t))\right),\ a.e.\ t\in(0,T), \tag{119}$$

$$\beta(0) = \beta_0, \tag{120}$$

where

$$\theta_\beta(t) = \int_0^t \beta(s)\,ds,\ \forall t\in[0,T]. \tag{121}$$

Lemma 9. *Problem (119)-(121) has a unique solution which satisfies*

$$\begin{cases} (i)\ \ \beta\in W^{1,\infty}\left(0,T;L^\infty(\Gamma_3)\right), \\ (ii)\ \ 0\le\beta(t)\le 1, \forall\ t\in[0,T],\ a.e.\ x\in\Gamma_3.\end{cases} \tag{122}$$

Moreover, we have

$$\left\|\tilde{\beta}_m-\beta\right\|_{L^2(0,T;L^2(\Gamma_3))} \leqslant c\|\tilde{u}_m-u\|_{L^2(0,T;V)} + ch_m. \tag{123}$$

Proof. The proof of the existence and uniqueness of a solution to Problem (119)-(121) such that (122) holds, may be obtained by arguments similar to those used in the proof of [17, Lemma 11]. It follows from (87), (69), (70) that

$$\tilde{\beta}_m(t) = h_m H_{ad}\left(\beta_0,\, h_m\beta_0,\, R_\nu(u_{0\nu})\right) + \beta_0,\ \forall t\in(t_0^m,t_1^m]. \tag{124}$$

Using (87), (69), (85) and (88), we deduce that

$$\begin{aligned}\tilde{\beta}_m(t) &= \sum_{j=1}^{i}\int_{t_{j-1}^m}^{t_j^m} H_{ad}\left(\tilde{\beta}_m(s),\,\tilde{\theta}_m(s),\,R_\nu(\tilde{u}_{m\nu}(s))\right)ds \\ &\quad + h_m H_{ad}\left(\beta_0,\,h_m\beta_0,\,R_\nu(u_{0\nu})\right)+\beta_0,\ \forall t\in\left(t_i^m,t_{i+1}^m\right],\ i\in\{1,...,m-1\},\end{aligned}$$

which, with (124), leads us to

$$(125)\quad \begin{cases} \tilde{\beta}_m(t) = \displaystyle\int_0^t H_{ad}\left(\tilde{\beta}_m(s),\, \tilde{\theta}_m(s),\, R_\nu\left(\tilde{u}_{m\nu}(s)\right)\right) ds \\ \qquad + \displaystyle\int_t^{t_i^m} H_{ad}\left(\tilde{\beta}_m(s),\, \tilde{\theta}_m(s),\, R_\nu\left(\tilde{u}_{m\nu}(s)\right)\right) ds \\ \qquad + h_m H_{ad}\left(\beta_0,\, h_m\beta_0,\, R_\nu\left(u_{0\nu}\right)\right) + \beta_0, \end{cases}$$

for all $t \in \left(t_i^m, t_{i+1}^m\right]$, $i \in \{0, ..., m-1\}$. On the other hand, from (88), we get

$$(126)\quad \tilde{\theta}_m(t) = \sum_{k=1}^{i+1} \int_{t_{k-1}^m}^{t_k^m} \tilde{\beta}_m(s) ds + h_m\beta_0 = \int_0^t \tilde{\beta}_m(s) ds + \int_t^{t_{i+1}^m} \tilde{\beta}_m(s) ds + h_m\beta_0,$$

for all $t \in \left(t_i^m, t_{i+1}^m\right]$, $i \in \{0, ..., m-1\}$. We use (121) and (126) to obtain

$$\left\|\tilde{\theta}_m(t) - \theta_\beta(t)\right\|_{L^2(\Gamma_3)} \leqslant \int_0^t \left\|\tilde{\beta}_m(s) - \beta(s)\right\|_{L^2(\Gamma_3)} ds + \; ch_m,$$

which combined with (119)-(120), (28), (24), (125) and (92), implies that

$$\begin{aligned} \left\|\tilde{\beta}_m(t) - \beta(t)\right\|_{L^2(\Gamma_3)} &\leqslant c\int_0^t \left\|\tilde{\beta}_m(s) - \beta(s)\right\|_{L^2(\Gamma_3)} ds + c\int_0^t \left\|\tilde{\theta}_m(s) - \theta_\beta(s)\right\|_{L^2(\Gamma_3)} ds \\ &\quad + c\int_0^t \left\|\tilde{u}_m(s) - u(s)\right\|_V ds + ch_m \\ &\leqslant c\int_0^t \left\|\tilde{\beta}_m(s) - \beta(s)\right\|_{L^2(\Gamma_3)} ds + c\int_0^t \left\|\tilde{u}_m(s) - u(s)\right\|_V ds + ch_m, \end{aligned}$$

for all $t \in [0, T]$. Using Lemma 1, in the last inequality, one has

$$\left\|\tilde{\beta}_m(t) - \beta(t)\right\|_{L^2(\Gamma_3)} \leqslant c\int_0^t \left\|\tilde{u}_m(s) - u(s)\right\|_V ds + ch_m,\ \forall\, t \in [0, T],$$

which gives inequality (123). $\square$

We have the following convergence results.

Lemma 10. *For all $v \in L^2(0, T; V)$, we have*

$$(127)\quad \lim_{m\to+\infty} \int_0^T \left(\mathcal{F}\tilde{u}_m(s), v(s) - \dot{u}_m(s)\right)_V ds = \int_0^T \left(\mathcal{F}u(s), v(s) - \dot{u}(s)\right)_V ds,$$

$$(128)\quad \lim_{m\to+\infty} \int_0^T \left(f_m(s), v(s) - \dot{u}_m(s)\right)_V ds = \int_0^T \left(f(s), v(s) - \dot{u}(s)\right)_V ds,$$

(129)

$$\lim_{m\to+\infty} \int_0^T \left(\mathcal{B}_m(s), \varepsilon\left(v(s) - \dot{u}_m(s)\right)\right)_{\mathcal{Q}} ds = \int_0^T \left(\widetilde{\mathcal{B}}(s), \varepsilon\left(v(s) - \dot{u}(s)\right)\right)_{\mathcal{Q}} ds.$$

Proof. Obviously, (66) and (111) imply that

$$\mathcal{F}\tilde{u}_m \to \mathcal{F}u \text{ strongly } in\ L^2(0, T; V),$$

which together with (107) gives (127). Using (96), yields

$$f_m \to f \text{ strongly } in\ L^2(0, T; V),$$

and having in mind (107), we get (128). Now, from (111) and (117), we conclude that

$$\mathcal{B}_m \to \widetilde{\mathcal{B}} \text{ strongly } in\ L^2\left(0,T;\mathcal{Q}\right),$$

which combined with (108), leads to (129). □

Lemma 11. *For all $v \in L^2\left(0,T;V\right)$, we have*

$$\lim_{m\to+\infty}\int_0^T \varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),v(s))ds = \int_0^T \varphi(\beta(s),u(s),v(s))ds. \tag{130}$$

$$\lim_{m\to+\infty}\int_0^T \Big(\varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),\dot{u}_m(s)) - \varphi(\beta(s),u(s),\dot{u}_m(s))\Big)\,ds = 0. \tag{131}$$

$$\liminf_{m\to+\infty}\int_0^T \varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),\dot{u}_m(s))ds \geq \int_0^T \varphi(\beta(s),u(s),\dot{u}(s))ds. \tag{132}$$

Proof. Using the properties of the functional φ defined by (35), (24), (92) and (122), we deduce the following

$$\begin{cases} \left|\displaystyle\int_0^T \Big[\varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),v(s)) - \varphi(\beta(s),u(s),v(s))\Big]\,ds\right| \\ \leqslant c\left(\left\|\tilde{\beta}_m-\beta\right\|_{L^2(0,T;L^2(\Gamma_3))} + \|\tilde{u}_m-u\|_{L^2(0,T;V)}\right)\|v\|_{L^2(0,T;V)}. \end{cases} \tag{133}$$

For all $v \in L^2\left(0,T;V\right)$. Therefore, the convergences (130)-(131) follow from (111), (123), (133) and (94). To continue, let $\Phi : L^2\left(0,T;V\right) \to \mathbb{R}$ be the functional defined by

$$\Phi\left(v\right) = \int_0^T \varphi(\beta(s),u(s),v(s))ds,\ \forall v \in L^2\left(0,T;V\right). \tag{134}$$

where β is the unique solution of Problem (119)-(121). Using (35), (40), (45) and (134), we find that Φ is convex and continuous. Thus, we deduce that Φ is weakly lower semicontinuous function on $L^2\left(0,T;V\right)$, see [3], which with (107) gives

$$\liminf_{m\to+\infty}\Phi\left(\dot{u}_m\right) \geq \Phi\left(\dot{u}\right). \tag{135}$$

On the other hand, one has

$$\begin{cases} \displaystyle\int_0^T \varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),\dot{u}_m(s))ds = \Phi\left(\dot{u}_m\right) \\ +\displaystyle\int_0^T \Big(\varphi(\tilde{\beta}_m(s),\tilde{u}_m(s),\dot{u}_m(s)) - \varphi(\beta(s),u(s),\dot{u}_m(s))\Big)\,ds. \end{cases} \tag{136}$$

Therefore, taking into account (131) and (135) when passing to the lim inf as $m \to +\infty$ in (136), we obtain (132). □

Fourth step. We have now all the ingredients to prove Theorem 1.

Proof. Let $t \in (0,T)$, let $r > 0$, such that $t+r \in (0,T)$, for each $w \in V$, we define a function $v \in L^2\left(0,T;V\right)$ by

$$v(s) = \begin{cases} w & \text{for } s \in \left(t,t+r\right), \\ \dot{u}(s) & \text{elsewhere.} \end{cases}$$

We use (67), (85), (87), (89), (90), (91), to obtain the following system

$$\text{(137)}\quad \begin{cases} \displaystyle\int_0^T (\mathcal{F}\tilde{u}_m(s), v(s) - \dot{u}_m(s))_V\, ds + \int_0^T (\mathcal{B}_m(s), \varepsilon\,(v(s) - \dot{u}_m(s)))_{\mathcal{Q}}\, ds + \\ \displaystyle +\int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), v(s)) ds - \int_0^T \varphi(\tilde{\beta}_m(s), \tilde{u}_m(s), \dot{u}_m(s)) ds \\ \displaystyle \geq \int_0^T (f_m(s), v(s) - \dot{u}_m(s))_V\, ds. \end{cases}$$

Passing to the lim sup as $m \to +\infty$ in (137), by applying Lemma 10 and Lemma 11, we obtain

$$\text{(138)}\quad \begin{cases} \displaystyle \frac{1}{r}\int_t^{t+r} (\mathcal{F}u(s), w - \dot{u}(s))_V\, ds \ + \frac{1}{r}\int_t^{t+r} \left(\widetilde{\mathcal{B}}(s), \varepsilon\,(w - \dot{u}(s))\right)_{\mathcal{Q}} ds \\ \displaystyle +\frac{1}{r}\int_t^{t+r} (\varphi(\beta(s), u(s), w) - \varphi(\beta(s), u(s), \dot{u}(s)))\, ds \\ \displaystyle \geq \frac{1}{r}\int_t^{t+r} (f(s), w - \dot{u}(s))_V\, ds, \text{ for all } w \in V. \end{cases}$$

Since $u_m\,(t) \to u\,(t)$ strongly in V, $\forall t \in [0, T]$, it follows from (84) that $u\,(0) = u_0$. Let σ be the fixed point of Λ defined by (116), let β be the unique solution of Problem (119)-(121). Now, letting $r \to 0$ in (138) and keeping in mind (64) and (118), we conclude that $\{u,\, \sigma,\, \beta\}$ is a solution of Problem (55)-(58). On the other hand, from (97), we have

$$\begin{aligned} \|u(t) - u(s)\|_V &\leqslant \|u(t) - u_m(t)\|_V + \|u_m(t) - u_m(s)\|_V + \|u_m(s) - u(s)\|_V \\ &\leqslant \|u(t) - u_m(t)\|_V + c\,|t - s| + \|u_m(s) - u(s)\|_V\,, \forall t, s \in [0, T]\,. \end{aligned}$$

Passing to the limit as $m \to +\infty$, in the last inequality and using the convergence (110), we get

$$\|u(t) - u(s)\|_V \leqslant c\,|t - s|\,,\ \forall t, s \in [0, T]\,.$$

Therefore, u satisfies the regularity (61). Also, from (55), we have

$$\begin{aligned} \|\sigma(t) - \sigma(s)\|_{\mathcal{Q}} &\leqslant c\,\|u(t) - u(s)\|_V + c\left|\int_s^t \left(\|u(r)\|_V + \|\sigma(r)\|_{\mathcal{Q}} + 1\right) dr\right| \\ &\leqslant c\,|t - s|\,,\ \forall t, s \in [0, T]\,, \end{aligned}$$

which gives

$$\text{(139)}\qquad \sigma \in W^{1,\infty}\,(0, T; \mathcal{Q})\,.$$

Furtheremore, taking $w = \dot{u}\,(t) \pm z$ where $z \in [D\,(\Omega)]^d$ in (56), we deduce that

$$\text{(140)}\qquad Div\sigma\,(t) = -f_0\,(t)\ \text{ in } \Omega,$$

for all $t \in [0, T]$. Now, the regularity (62) is a consequence of (139), (140) and (30 (i)). Finally, (63) follows from Lemma 9, which concludes the proof. □

6. Conclusion

In this paper, we have studied an adhesive quasistatic contact problem with normal compliance condition associated to a version of Coulomb's law of dry friction, for viscoplastic materials. We have shown the existence of a weak solution, under a smallness assumption depending only on the normal compliance functions, the elasticity operator and on the geometry of the problem. The uniqueness of the solution remains, as far as we know, an open question.

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