Normal and differentiable periodicity of linear shift operators under partial sharing

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Abstract. In this paper, we investigate the periodicity of linear shift operators associated with meromorphic functions under the framework of partial sharing conditions. Furthermore, we introduce the novel concept of differentiable periodicity, which provides a robust framework for analyzing the behavior of these operators under partial CM or IM sharing. By establishing comprehensive criteria for differentiable periodicity, this work offers a new perspective in the theory of shift operators. The findings not only extend but also refine and improve prior results, such as those presented in the papers of W J. Chen, Z. G. Huang (2022) and D. C. Pramanik, A. Sarkar (2024).

1. Introduction and background

At the outset, we introduce the following key notations, which are essential for establishing a comprehensive framework to analyze meromorphic functions and their behavior concerning shared values or sets.

Let $f(z)$ be a non-constant meromorphic function and $a \in \mathbb{C}$. Define the function $\nu_f^a : \mathbb{C} \to \mathbb{N}$ by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity d,} \end{cases}$$

and set $\nu_f^\infty(z) = \nu_{\frac{1}{f}}^0(z)$. Next, define the reduced counting function $\overline{\nu}_f^a : \mathbb{C} \to \mathbb{N}$ as:

$$\overline{\nu}_f^a(z) = \min\{\nu_f^a(z), 1\}$$

and similarly $\overline{\nu}_f^\infty(z) = \overline{\nu}_{\frac{1}{f}}^0(z)$. Next for $m \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$, define the reduced multiplicity counting function $\nu_{(f,a)}^{\leq m} : \mathbb{C} \cup \{\infty\} \to \mathbb{N}$ by

$$\nu_{(f,a)}^{\leq m}(z) = \begin{cases} 0 & \text{if } \nu_f^a(z) > m \\ \nu_f^a & \text{if } \nu_f^a(z) \leq m \end{cases}$$

and $\overline{\nu}_{(f,a)}^{\leq m}(z) = \min\left\{\nu_{(f,a)}^{\leq m}(z), 1\right\}$. When $m = \infty$, we interpret $\nu_{(f,a)}^{\leq m}(z)$ as $\nu_f^a(z)$.

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For $S \subset \mathbb{C} \cup \{\infty\}$, define the set $E(S, f)$ as

$$E(S, f) = \bigcup_{a \in S} \{(z, \nu_f^a(z)) : z \in \mathbb{C}\}.$$

In an analogous way, replacing $\nu_f^a(z)$ with $\overline{\nu}_f^a(z)$ in $E(S, f)$, define $\overline{E}(S, f)$ as:

$$\overline{E}(S, f) = \bigcup_{a \in S} \{(z, \overline{\nu}_f^a(z)) : z \in \mathbb{C}\}.$$

Using these definitions, we describe various types of set sharing between meromorphic functions f and g:
• If $E(S, f) = E(S, g)$, we say that f and g share the set S CM.
• If $\overline{E}(S, f) = \overline{E}(S, g)$, we say that f and g share the set S IM.
• If $E(S, f) \subseteq E(S, g)$, we say that f and g partially share S CM.
• If $\overline{E}(S, f) \subseteq \overline{E}(S, g)$, we say that f and g partially share S IM.
When S is a singleton, the above definition corresponds to value sharing.

For further information or a detailed explanation of these concepts, readers are encouraged to consult the original sources cited in the manuscript: [1]. Additionally, we assume that readers are familiar with the standard notations of value distribution theory on the entire complex plane $\mathbb{C}$, including $N(r, f)$, $m(r, f)$, $T(r, f)$, $S(r, f)$ etc. as outlined in [7] and [14]. We use the notation $N_{(k}(r, a, f(z))$ to denote the counting function of a-points of $f(z)$ of multiplicity $\geq k$. For $a \in \mathbb{C} \cup \infty$, we also use the notation $\overline{N}_*(r, a; f, g)$ to denote the reduced counting function of common a-points of f and g of different multiplicities.

The order and hyper-order of f are defined respectively as follows:

$$\rho(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r, f)}{\log r}, \qquad \rho_2(f) = \limsup_{r \longrightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

Further throughout the paper, we will assume $c \in \mathbb{C} \backslash \{0\}$ and the shift operator of f will be denoted by $f(z + c)$.

We begin our discussion with a result established by Qi [13].

Theorem A. [13] *Let f be a entire function of finite order, $c \in \mathbb{C}$ and a_i, $i = 1, 2$ be three distinct periodic functions with period c. If $f(z)$ and $f(z + c)$ share a_1, a_2 IM, then $f(z) = f(z + c)$.*

For meromorphic function some analogous results are as follows:

Theorem B. [8] *Let f be a meromorphic function of finite order and $c \in \mathbb{C}$. If $f(z)$ and $f(z+c)$ share three distinct periodic functions a_i, $i = 1, 2, 3$ with period c CM, then $f(z) = f(z + c)$.*

Theorem C. [9] *Under the same condition of* Theorem B*, if $f(z)$ and $f(z + c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z + c)$.*

In this respect, further continuations of research can be found in [3] and [11]. Recently, Chen et al. introduced new types of structures that had not been considered earlier. In their work, Chen and Huang [4] presented results focusing on partial and normal sharing, as outlined below:

Theorem D. [4] *Let $f(z)$ be a meromorphic function of finite order, $a \in \mathbb{C} \setminus \{0\}$ and l, s be integers with $0 \leq l < s$. If $f^{(l)}(z)$ and $f^{(s)}(z+c)$ share a CM and satisfy $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$ and $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$, then $f^{(l)}(z) \equiv f^{(s)}(z+c)$.*

Theorem E. [4] *Let $f(z)$ be a transcendental meromorphic function of finite order, $0 \leq l < s$. Suppose that, $f^{(l)}(z)$ and $f^{(s)}(z+c)$ share a finite value $a(\neq 0)$ IM and satisfy $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$ and $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$. If $N\left(r, \frac{1}{f^{(l)}(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) = S(r,f)$, then $f^{(l)}(z) \equiv f^{(s)}(z+c)$.*

Very recently, in [12] the authors have been able to successfully remove the conditions over counting function in Theorem E. The result is as follows:

Theorem F. [12] *Let $f(z)$ be a non-constant meromorphic function of finite order, $0 \leq l < s$. If $f^{(l)}(z)$ and $f^{(s)}(z+c)$ share a finite value $a(\neq 0)$ IM and satisfy $E(0, f^{(l)}(z)) \subset E(0, f^{(s)}(z+c))$ and $E(\infty, f^{(s)}(z+c)) \subset E(\infty, f^{(l)}(z))$, then $f^{(l)}(z) \equiv f^{(s)}(z+c)$.*

2. Motivation, main results and relevant examples

The following example shows that Theorem A can not be extended for meromorphic function under partial sharing.

Example 2.1. *Let us assume two distinct complex numbers a, b, $c = \pi i$ and $f(z) = \frac{ae^z - b}{e^z - 1}$. Then, $f(z+c) = \frac{ae^z + b}{e^z + 1}$. It is easy to check, $E(a, f(z)) = E(a, f(z+c))$ and $E(b, f(z)) = E(b, f(z+c))$, but $f(z) \neq f(z+c)$.*

We see that in the above example the poles are not shared. In light of the above example, an intriguing question arises:

Can the periodicity of a meromorphic function be achieved under partial sharing on $\mathbb{C}$? If so, what would be the nature of the pole sharing?

The purpose of this paper is to address these questions and contribute to this line of inquiry. Specifically, we aim to establish our results within a more generalized framework, namely the linear shift operator, which is defined as follows:

For a non-constant meromorphic function $f(z)$, a non negative integer j, $a_i \in \mathbb{C}$, $a_j \neq 0$ and $c \in \mathbb{C}$, define $L_j f(z)$ as

$$L_j f(z) = \sum_{i=0}^{j} a_i f(z + ic). \tag{2.1}$$

The following theorem is one of our main results, where we investigate the periodicity of the linear shift operator based solely on partial IM sharing which was not addressed earlier.

Theorem 2.1. *Let $f(z)$ be a non-constant meromorphic functions with $\rho_2(f) < 1$ and let a, $b \in \mathbb{C}$ be two distinct numbers. Let $\overline{E}(a, L_j f(z)) \subseteq \overline{E}(a, L_j f(z+c))$, $\overline{E}(b, L_j f(z)) \subseteq \overline{E}(b, L_j f(z+c))$, $E(\infty, L_j f(z+c)) \subseteq E(\infty, L_j f(z))$ with $\overline{N}(r,f) = S(r,f)$, then $L_j f(z) = L_j f(z+c)$.*

The following example shows that, by no means, the linear shift operator $L_j(z+c)$ in Theorem 2.1, can not be replaced by an arbitrary linear shift operator.

Example 2.2. *Consider the function* $L_0f(z) = f(z) = \frac{e^{\frac{2\pi iz}{c}}-1}{e^{\frac{2\pi iz}{c}}+1}$, *where* $c \in \mathbb{C}\setminus\{0\}$. *Choose a linear shift operator* $\tilde{L}_3f(z)$ *by*

$$\begin{aligned}\tilde{L}_3f(z) &= 3f(z+3c) - f(z+2c) - 2f(z+c) - f(z)\\ &= -\frac{e^{\frac{2\pi iz}{c}}-1}{e^{2\frac{\pi iz}{c}}+1}.\end{aligned}$$

Clearly, $E(1, L_0f(z)) = E(1, \tilde{L}_3f(z))$, $E(0, L_0f(z)) = E(0, \tilde{L}_3f(z))$ *and* $E(\infty, L_0f(z)) = E(\infty, \tilde{L}_3f(z))$, *but* $f(z) \neq \tilde{L}_3f(z)$.

The next example shows that for entire functions, in Theorem 2.1, $f(z+c)$ can not be replaced by $\Delta_cf(z) = f(z+c) - f(z)$.

Example 2.3. *Take* $L_0f(z) = f(z) = \left(\frac{e^{\frac{\pi iz}{c}}-1}{2}\right)^2 + 1$. *Then* $\Delta_cf(z) = e^{\frac{\pi iz}{c}}$. *Here,* f *and* Δ_cf *share* 1, 0 *IM, but* $f \neq \Delta_cf$.

Next, we define differentiable periodic function.

Definition 2.1. *Let* $F(z)$ *be a non-constant meromorphic function and* l, s *two non-negative integers.* $F(z)$ *is said to be differentiably periodic if* $F^{(l)}(z) \equiv F^{(s)}(z+c)$, *where* $0 \leq l < s$.

In light of Theorem D, it becomes of significant interest to investigate the differentiable periodic property of $L_jf(z)$ as defined in (2.1) as this perspective provides a deeper understanding of the behavior of the linear shift operator L_j under certain sharing conditions. Moreover, by relaxing the condition from CM sharing of the non-zero value a to partial CM sharing of a, it can be shown that a similar result to Theorem D holds. The following theorem formalizes this extension:

Theorem 2.2. *Let* $f(z)$ *be a non-constant meromorphic function with* $\rho_2(f) < 1$, $a \in \mathbb{C}\setminus\{0\}$ *and* l, s *be two integers such that* $0 \leq l < s$. *If* $E(a, L_jf^{(l)}(z)) \subseteq E(a, L_jf^{(s)}(z+c))$, $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$, $E(\infty, L_jf^{(s)}(z+c)) \subseteq E(\infty, L_jf^{(l)}(z))$, *then* $L_jf^{(l)}(z) \equiv L_jf^{(s)}(z+c)$.

Note 2.1. *We can see that* Theorem 2.2 *is not only extending* Theorem D, *but also improving it by relaxing sharing condition.*

The following example shows that, sharing of the non-zero element in Theorem 2.2 can not be dropped.

Example 2.4. *Set* $L_0f(z) = \sin z$, *then* $L_0f'(z+\frac{\pi}{2}) = -\sin z$. *Clearly,* $E(0, L_0f(z)) = E(0, L_0f'(z+\frac{\pi}{2}))$ *and* $E(\infty, L_0f(z)) = E(\infty, L_0f'(z+\frac{\pi}{2}))$, *but* $E(1, L_0f(z)) \not\subset E(1, L_0f'(z+\frac{\pi}{2}))$ *and* $L_0f(z) \neq L_0f'(z+\frac{\pi}{2})$.

In view of Theorem 2.1, it is timely and significant to investigate the differentiable periodic property of $L_jf(z)$ under the framework of partial IM value sharing over $\mathbb{C}$.

Theorem 2.3. *Under the same situation of* Theorem 2.2, *if* $\overline{E}(a, L_jf^{(l)}(z)) \subseteq \overline{E}(a, L_jf^{(s)}(z+c))$, $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$ *and* $E(\infty, L_jf^{(s)}(z+c)) \subseteq E(\infty, L_jf^{(l)}(z))$ *and* $\overline{N}\left(r, \frac{1}{L_jf^{(l)}(z)}\right) = S(r, f)$, *then* $L_jf^{(l)}(z) \equiv L_jf^{(s)}(z+c)$.

The following example shows that sharing structure of ∞ in Theorem 2.3 can not be reversed.

Example 2.5. *Take* $L_0f(z) = \frac{2e^{\frac{2\pi iz}{c}}}{e^{\frac{2\pi iz}{c}}-1}$, *where* $c \in \mathbb{C} \setminus \{0\}$. *Then,* $L_0f'(z+c) = \frac{-4e^{\frac{2\pi iz}{c}}}{(e^{\frac{2\pi iz}{c}}-1)^2}$. *Clearly,* $\overline{E}(1, L_0f(z)) \subset \overline{E}(1, L_0f'(z+c))$, $E(0, L_0f(z)) = E(0, L_0f'(z+c))$ *and* $\overline{N}\left(r, \frac{1}{L_0f(z)}\right) = S(r, f)$, *but* $E(\infty, L_0f'(z+c)) \not\subset E(\infty, L_0f(z))$ *and* $L_0f(z) \neq L_0f'(z+c)$.

The next theorem extends and refines Theorems E and F to the case of differentiably periodic linear shift operators. In fact, to prove the theorem, we have adopted a different approach than the one used in [4] and [12].

Theorem 2.4. *Under the same situation of* Theorem 2.2, *if* $\overline{E}(a, L_jf^{(l)}(z)) = \overline{E}(a, L_jf^{(s)}(z+c))$, $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$ *and* $E(\infty, L_jf^{(s)}(z+c)) \subseteq E(\infty, L_jf^{(l)}(z))$, *then* $L_jf^{(l)}(z) \equiv L_jf^{(s)}(z+c)$.

The following two examples show that Theorem 2.1 cannot be directly extended to set sharing structure.

Example 2.6. *Let* $L_0f(z) = \frac{2e^{\frac{\pi iz}{c}}}{1+e^{\frac{2\pi iz}{c}}}$, $c \in \mathbb{C} \setminus \{0\}$. *Then we see that for a non-zero constant* α, $E(\{\alpha, -\alpha\}, L_0f(z)) = E(\{\alpha, -\alpha\}, L_0f(z+c))$, $E(\infty, L_0f(z)) = E(\infty, L_0f(z+c))$ *but* $L_0f(z) \neq L_0f(z+c)$.

Example 2.7. *Let* $L_0f(z) = \frac{1+e^{\frac{\pi iz}{c}}}{1-e^{\frac{2\pi iz}{c}}}$, $c \in \mathbb{C} \setminus \{0\}$. *Then we see that for a non-zero constant* α, $E\left(\{\alpha, \frac{1}{\alpha}\}, L_0f(z)\right) = E\left(\{\alpha, \frac{1}{\alpha}\}, L_0f(z+c)\right)$, $E(\infty, L_0f(z)) = E(\infty, L_0f(z+c))$ *but* $L_0f(z) \neq L_0f(z+c)$.

In view of Example 2.6 and 2.7, the following question is inevitable:

Question 2.1. *Can we can replace the value sharing condition with set sharing condition in the* Theorems 2.1-2.4*?*

Below we are presenting the following chart, which provides an in-depth comparison of the related theorems in terms of sufficient conditions in a compact form. Throughout the chart, we assume that $a(\neq 0)$, b are two distinct complex numbers.

Theorems	Scaling of sharing			Additional Supposition
D	$E(a, f^{(l)}(z)) = E(a, f^{(s)}(z+c))$	$E\left(0, f^{(l)}(z)\right) \subset E(0, f^{(s)}(z+c))$	$E\left(\infty, f^{(s)}(z+c)\right) \subset E(\infty, f^{(l)}(z))$	None.
E	$\overline{E}(a, f^{(l)}(z)) = \overline{E}(a, f^{(s)}(z+c))$	$E\left(0, f^{(l)}(z)\right) \subset E(0, f^{(s)}(z+c))$	$E\left(\infty, f^{(s)}(z+c)\right) \subset E(\infty, f^{(l)}(z))$	$N\left(r, \frac{1}{f^{(l)}(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) = S(r, f).$
F	$\overline{E}(a, f^{(l)}(z)) = \overline{E}(a, f^{(s)}(z+c))$	$E\left(0, f^{(l)}(z)\right) \subset E(0, f^{(s)}(z+c))$	$E\left(\infty, f^{(s)}(z+c)\right) \subset E(\infty, f^{(l)}(z))$	None.
2.1	$\overline{E}\left(a, L_j f(z)\right) = \overline{E}(a, L_j f(z+c))$	$\overline{E}\left(b, L_j f(z)\right) = \overline{E}(b, L_j f(z+c))$	$E\left(\infty, f^{(s)}(z+c)\right) \subset E(\infty, f^{(l)}(z))$	$\overline{N}(r, f(z)) = S(r, f).$
2.2	$E\left(a, L_j f^{(l)}(z)\right) \subseteq E(a, L_j f^{(s)}(z+c))$	$E\left(0, L_j f^{(l)}(z)\right) \subseteq E(0, L_j f^{(s)}(z+c))$	$E\left(\infty, L_j f^{(s)}(z+c)\right) \subseteq E\left(\infty, L_j f^{(l)} f(z)\right)$	None.
2.3	$\overline{E}\left(a, L_j f^{(l)}(z)\right) \subseteq E(a, L_j f^{(s)}(z+c))$	$E\left(0, L_j f^{(l)}(z)\right) \subseteq E(0, L_j f^{(s)} f(z+c))$	$E\left(\infty, L_j f^{(s)}(z+c)\right) \subseteq E\left(\infty, L_j f^{(l)} f(z)\right)$	$\overline{N}\left(r, \frac{1}{L_j f^{(l)}(z))}\right) = S(r, f).$
2.4	$\overline{E}\left(a, L_j f^{(l)}(z)\right) = E(a, L_j f^{(s)}(z+c))$	$E\left(0, L_j f^{(l)}(z)\right) \subseteq E(0, L_j f^{(s)} f(z+c))$	$E\left(\infty, L_j f^{(s)}(z+c)\right) \subseteq E\left(\infty, L_j f^{(l)} f(z)\right)$	None.

3. Lemmas

In this section, we are going to present some lemmas, which will be needed to proceed further.

Lemma 3.1. [6] *Let $f(z)$ be a meromorphic function with $\rho_2(f) < 1$ and $c \in \mathbb{C}$. Then we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r \frac{f(z)}{f(z+c)}\right) = S(r, f),$$

where $S(r, f) = o(T(r, f))$, for all r, outside of a possible exceptional set S with finite linear measure.

Lemma 3.2. [5] *Let $f(z)$ be a non-constant meromorphic function of $\rho_2(f) < 1$ and $c \in \mathbb{C}$. Then*

$$N(r, f(z+c)) = N(r, f) + S(r, f),\ N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f)$$

and

$$\overline{N}(r, f(z+c)) = \overline{N}(r, f) + S(r, f),\ \overline{N}\left(r, \frac{1}{f(z+c)}\right) = \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r, f).$$

Lemma 3.3. [15] *Let $f(z)$ be a non-constant meromorphic function and let $d_j, (j = 1, 2, \ldots, q)$ be q distinct complex numbers. Then we have*

$$m\left(r, \sum_{i=1}^{q} \frac{1}{f - d_i}\right) = \sum_{i=1}^{q} m\left(r, \frac{1}{f - d_i}\right) + O(1).$$

Lemma 3.4. [2] *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$, then we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f).$$

Lemma 3.5. *Let $f(z)$ be a meromorphic function of $\rho_2(f) < 1$, then $S(r, L_j f(z))$ and $S(r, L_j f(z+c))$ can be replaced by $S(r, f)$.*

Proof. In view of *Lemmas 3.1, 3.4*, we have

$$\begin{aligned} T(r, L_j f(z)) &= m(r, L_j f(z)) + N(r, L_j f(z)) \\ &\leq m\left(r, \frac{L_j f(z)}{f}\right) + m(r, f(z)) + N(r, L_j f(z)) + S(r, f) \\ &\leq (j+2) T(r, f(z)) + S(r, f) \end{aligned}$$

and so the lemma follows. Similar result holds for $S(r, L_j f(z+c))$. □

Lemma 3.6. *Let $f(z)$ be a non-constant meromorphic function with $\rho_2(f) < 1$, $c \in \mathbb{C} \setminus \{0\}$ and l, $s \geq 0$. Then, $S(r, L_j f^{(l)}(z))$ and $S(r, L_j f^{(s)}(z+c))$ can be replaced by $S(r, f)$.*

Proof. In view of *Lemmas 3.1. 3.4* and logarithmic derivative theorem, we obtain

$$\begin{aligned} T(r, L_j f^{(l)}(z)) &= m(r, L_j f^{(l)}(z)) + N(r, L_j f^{(l)}(z)) \\ &\leq m\left(r, \frac{L_j f^{(l)}(z)}{L_j f(z)}\right) + m\left(r, \frac{L_j f(z)}{f(z)}\right) + N(r, L_j f^{(l)}(z)) + S(r, f) \\ &\leq (l + j + 1)T(r, f(z)) + S(r, f). \end{aligned}$$

Hence, the lemma follows. Similar results holds for $S(r, L_j f^{(s)}(z+c))$. □

Lemma 3.7. *Let $f(z)$ be a non-constant meromorphic function with $\rho_2(f) < 1$. For any complex number b we have,*

$$m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(l)}(z) - b}\right) \leq S(r, f), \quad \text{where } 0 \leq l < s.$$

Proof. Using logarithmic derivative theorem and *Lemmas 3.1, 3.6*, we obtain

$$\begin{aligned} m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(l)}(z) - b}\right) &\leq m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(s)}(z)}\right) + m\left(r, \frac{L_j f^{(s)}(z)}{L_j f^{(l)}(z) - b}\right) \\ &= S(r, f). \end{aligned}$$

□

Lemma 3.8. *Under the same situation of* Lemma 3.7, *if $a \in \mathbb{C} \setminus \{0\}$ and $0 \leq l < s$; then*

$$\begin{aligned} & m\left(r, \frac{1}{L_j f^{(l)}(z)}\right) + m\left(r, \frac{1}{L_j f^{(l)}(z) - a}\right) \\ & \leq m(r, L_j f^{(l)}(z)) + N(r, L_j f^{(s)}(z+c)) - N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r, f). \end{aligned}$$

Proof. Set

$$F(z) = \frac{1}{L_j f^{(l)}(z)} + \frac{1}{L_j f^{(l)}(z) - a}.$$

Further, using *Lemmas 3.6, 3.7* we get

$$\begin{aligned} m(r, L_j f^{(s)}(z+c)F(z)) &\leq m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(l)}(z)}\right) + m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(l)}(z) - a}\right) \\ &\leq S(r, f). \end{aligned}$$

Thus,

$$\begin{aligned} m(r, F(z)) &\leq m(r, L_j f^{(s)}(z+c)F(z)) + m\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) \\ &\leq m\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r, f). \end{aligned}$$

Next, using *Lemma 3.3*, the First Fundamental Theorem and the above inequality we obtain,

$$\begin{aligned} T(r, L_j f^{(s)}(z+c)) &= T\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + O(1) \\ &= N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + m\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f) \\ &\geq m(r, F(z)) + N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f) \\ &= m\left(r, \frac{1}{L_j f^{(l)}(z)}\right) + m\left(r, \frac{1}{L_j f^{(l)}(z) - a}\right) \\ &\quad + N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f). \end{aligned}$$

Thus, from the above inequality using *Lemmas 3.1, 3.6, 3.7* we have

$$\begin{aligned} & m\left(r, \frac{1}{L_j f^{(l)}(z)}\right) + m\left(r, \frac{1}{L_j f^{(l)}(z) - a}\right) \\ \leq\ & T(r, L_j f^{(s)}(z+c)) - N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f) \\ =\ & m(r, L_j f^{(s)}(z+c)) + N(r, L_j f^{(s)}(z+c)) - N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f) \\ \leq\ & m\left(r, \frac{L_j f^{(s)}(z+c)}{L_j f^{(l)}(z)}\right) + m(r, L_j f^{(l)}(z)) + N(r, L_j f^{(s)}(z+c)) \\ & -N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f) \\ \leq\ & m(r, L_j f^{(l)}(z)) + N(r, L_j f^{(s)}(z+c)) - N\left(r, \frac{1}{L_j f^{(s)}(z+c)}\right) + S(r,f). \end{aligned}$$

□

Lemma 3.9. *Let $f(z)$ a meromorphic function of $\rho_2(f) < 1$ and $a \in \mathbb{C} \setminus \{0\}$. If $E(\infty, L_j f^{(s)}(z+c)) \subseteq E(\infty, L_j f^{(l)}(z))$ for $0 \leq l < s$, then $\overline{N}(r, L_j f^{(l)}(z)) = \overline{N}(r, L_j f^{(s)}(z+c)) = S(r,f)$.*

Proof. Since $E(\infty, L_j f^{(s)}(z+c)) \subseteq E(\infty, L_j f^{(l)}(z))$, we deduce

$$N(r, L_j f^{(s)}(z+c)) \leq N(r, L_j f^{(l)}(z)) + S(r,f).$$

Using *Lemma 3.6* we get,

$$\begin{aligned} N(r, L_j f^{(s)}(z+c)) &= N(r, L_j f^{(s)}(z)) + S(r,f) \\ &= N(r, L_j f^{(l)}(z)) + (s-l)\overline{N}(r, L_j f^{(l)}(z)) + S(r,f) \\ &\leq N(r, L_j f^{(l)}(z)) + S(r,f), \end{aligned}$$

i.e., $\overline{N}(r, L_j f^{(l)}(z)) = \overline{N}(r, L_j f^{(s)}(z+c)) = S(r,f)$. □

4. Proofs of the Theorems

Proof of Theorem 2.1. The proof of the theorem is inspired by the idea of Lin-Ishizaki [10].
On the contrary, we assume that, $L_jf(z) \not\equiv L_jf(z+c)$. Take a function

$$\Phi(z) = \frac{L_jf'(z)(L_jf(z) - L_jf(z+c))}{(L_jf(z) - a)(L_jf(z) - b)}. \tag{4.1}$$

• Let z_0 be an a- point of $L_jf(z)$ of order p and same of $L_jf(z+c)$ of order q, then z_0 is a zero of $\Phi(z)$ of order $(p-1) + min\{p, q\} - p$, i.e., z_0 is not a pole of $\Phi(z)$.
• Similarly, b-points of $L_jf(z)$ are not poles of Φ.
• Let z_1 be a pole of $L_jf(z)$ of order r. Then, a simple calculation yields that z_1 is a simple pole of $\Phi(z)$, regardless of whether it is a pole of $L_jf(z+c)$ or not. Hence, by *Lemma 3.2*,

$$N(r, \Phi) \leq \overline{N}(r, L_jf(z)) \leq (j+1)\overline{N}(r, f) = S(r, f).$$

Further, using *Lemmas 3.1, 3.5* and logarithmic derivative theorem we obtain

$$\begin{aligned} m(r, \Phi) &\leq m\left(r, \frac{L_jf'(z)}{L_jf(z) - a}\right) + m\left(r, \frac{L_jf(z) - L_jf(z+c)}{L_jf(z) - b}\right) \\ &\leq m\left(r, 1 - \frac{L_jf(z+c) - b}{L_jf(z) - b}\right) = S(r, f). \end{aligned}$$

Hence.

$$T(r, \Phi) = S(r, f). \tag{4.2}$$

Now, we denote by $N_0\left(r, \frac{1}{L_jf'(z)}\right)$ counting function of those zeros of $L_jf'(z)$, which are not zeros of $(L_jf(z)-a)(L_jf(z)-b)$ and similar notation for $L_jf(z+c)$. Now, from (4.1) and (4.2) we have

$$N_0\left(r, \frac{1}{L_jf'(z)}\right) \leq N\left(r, \frac{1}{\Phi}\right) = S(r, f). \tag{4.3}$$

Here,

$$\Phi(z+c) = \frac{L_jf'(z+c)(L_jf(z+c) - L_jf(z+c+c))}{(L_jf(z+c) - a)(L_jf(z+c) - b)}.$$

From the above using *Lemma 3.4*, we get

$$N_0\left(r, \frac{1}{L_jf'(z+c)}\right) \leq N\left(r, \frac{1}{\Phi(z+c)}\right) \leq T(r, \Phi(z+c)) = S(r, f). \tag{4.4}$$

Let us consider a function

$$H(z) = \frac{L_jf'(z)(L_jf(z+c) - a)(L_jf(z+c) - b)}{L_jf'(z+c)(L_jf(z) - a)(L_jf(z) - b)}. \tag{4.5}$$

Let z_0 be an a-point (b-point) of $L_jf(z)$ of multiplicity m and that of $L_jf(z+c)$ of multiplicity n. Then z_0 is a pole of $H(z)$ of order $(m+n-1-m-n+1) = 0$, i.e., z_0 is not a pole of $H(z)$. Note that, when $m > 1$, then zeros of $L_jf(z)$ are zeros of $L_jf'(z)$ of order $(m-1)$.

Let z_1 be a zero of $L_jf'(z+c)$. Then, the following cases arise:
Case 1: Suppose, z_1 is a common zero of $(L_jf(z)-a)$ $((L_jf(z)-b))$ and $(L_jf(z+c)-a)((L_jf(z+c)-b))$. By the similar arguments that were used before, we can show that z_1 is not a pole of $H(z)$.
Case 2: z_1 is a zero of $(L_jf(z+c)-a)((L_jf(z+c)-b))$ but not of $(L_jf(z)-a)((L_jf(z)-b))$. In that case also, z_1 is not a pole of $H(z)$.
Case 3: If z_1 is not zero of $(L_jf(z+c)-a)((L_jf(z+c)-b))$. Then by (4.4), we can say that z_1 is not a pole of $H(z)$. Therefore, combining the above cases we obtain

$$N(r,H)=S(r,f).$$

Now, using *Lemma 3.1* and logarithmic derivative theorem we have

$$\begin{aligned} m(r,H(z)) &\leq m\left(r,\frac{L_jf'(z)}{L_jf'(z+c)}\right)+m\left(r,\frac{L_jf(z+c)-a}{L_jf(z)-a}\right)\\ &\quad +m\left(r,\frac{L_jf(z+c)-b}{L_jf(z)-b}\right)\\ &\leq S(r,f). \end{aligned}$$

Hence, using the above facts we have

$$T(r,H)=S(r,f). \tag{4.6}$$

We denote by $\overline{N}_{\times}\left(r,\frac{1}{(L_jf(z+c)-a)(L_jf(z+c)-b)}\right)$, the reduced counting function of zeros of $(L_jf(z+c)-a)$ (resp. $(L_jf(z+c)-b)$) which are not the zeros of $(L_jf(z)-a)$ (resp. $(L_jf(z)-b)$). By a simple calculation we have,

$$\overline{N}_{\times}\left(r,\frac{1}{(L_jf(z+c)-a)(L_jf(z+c)-b)}\right)\leq \overline{N}\left(r,\frac{1}{H(z)}\right)\leq T(r,H)=S(r,f).$$

Next (4.1) can be rewritten as

$$\frac{H(z)L_jf'(z+c)}{(L_jf(z+c)-a)(L_jf(z+c)-b)}=\frac{L_jf'(z)}{(L_jf(z)-a)(L_jf(z)-b)}. \tag{4.7}$$

Henceforth by $\overline{N}_{(m,n)}(r,p;L_jf(z),L_jf(z+c))=\overline{N}_{(m,n)}(r,p)$, $p\in\{a,b\}$; we mean the reduced counting function of those common zeros of $(L_jf(z)-p)$ with multiplicity m and of $(L_jf(z+c)-p)$ with multiplicity n.
Claim: $\overline{N}_{(m,n)}(r,p)=S(r,f)$, $p\in\{a,b\}$.

Without loss of generality take $p=a$. Suppose on the contrary, there exist m,n such that $\overline{N}_{(m,n)}(r,p)\neq S(r,f)$. Let $z_2\in\mathbb{C}$ such that $L_jf(z_2)=L_jf(z_2+c)=a$. Then (4.7) gives us,

$$H(z_2)=\frac{m}{n}.$$

Case 1: $m=n$. Then, $H(z_2)=1$. If $H(z)\not\equiv 1$, then using (4.6) we have

$$\overline{N}_{(m,n)}(r,a)\leq N\left(r,\frac{1}{H(z)-1}\right)=S(r,f),$$

which is a contradiction. Hence, $H(z) \equiv 1$. Therefore, (4.7) can be written as

$$\frac{L_jf'(z+c)}{(L_jf(z+c)-a)(L_jf(z+c)-b)} = \frac{L_jf'(z)}{(L_jf(z)-a)(L_jf(z)-b)},$$

which implies that $L_jf(z)$ and $L_jf(z+c)$ share a, b and ∞ CM and using Theorem B we have $L_jf(z) \equiv L_jf(z+c)$, a contradiction.

Case 2: Suppose that $m \neq n$, a contradiction. So, there exist $z_3 \in \mathbb{C}$ such that $H(z_3) = \frac{m}{n}$. Using similar arguments as in *Case 1*, we obtain $H(z) \equiv \frac{m}{n}$. Then (4.7) can be written as

$$\frac{mL_jf'(z+c)}{(L_jf(z+c)-a)(L_jf(z+c)-b)} = \frac{nL_jf'(z)}{(L_jf(z)-a)(L_jf(z)-b)}.$$

Integrating both sides of the above equation we get

$$\left(\frac{L_jf(z+c)-a}{L_jf(z+c)-b}\right)^m = A\left(\frac{L_jf(z)-a}{L_jf(z)-b}\right)^n,$$

where $A(\neq 0) \in \mathbb{C}$. Therefore, by Valiron-Mohon'ko theorem and *Lemma 3.4*, $m = n$.

Hence, the claim is true and $\overline{N}_{(m,n)}(r,p) = S(r,f)$, for $p \in \{a,b\}$. Now,

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{L_jf(z)-a}\right) &= \sum_{m,n}\overline{N}_{(m,n)}(r,a)\\
&= \sum_{m\leq 2,n}\overline{N}_{(m,n)}(r,a) + \sum_{m\geq 3,n}\overline{N}_{(m,n)}(r,a)\\
&\leq \frac{1}{3}\sum_{m\geq 3,n}\frac{1}{m}N_{(m}\left(r,a;L_jf(z)\right)\\
&\leq \frac{1}{3}T(r,L_jf(z)) + S(r,f).
\end{aligned}$$

Similarly,

$$\overline{N}\left(r,\frac{1}{L_jf(z)-b}\right) \leq \frac{1}{3}T(r,L_jf(z)) + S(r,f).$$

Now, by the Second Fundamental Theorem we get

$$\begin{aligned}
T(r,L_jf(z)) &\leq \overline{N}(r,L_jf(z)) + \overline{N}\left(r,\frac{1}{L_jf(z)-a}\right) + \overline{N}\left(r,\frac{1}{L_jf(z)-b}\right)\\
&\quad +S(r,f)\\
&\leq \frac{2}{3}T(r,L_jf(z)) + (j+1)\overline{N}(r,f) + S(r,f),
\end{aligned}$$

which is not possible. Hence, $L_jf(z) \equiv L_jf(z+c)$. □

Proof of Theorem 2.2. Set

$$\Psi(z) \equiv \frac{L_jf^{(s)}(z+c)}{L_jf^{(l)}(z)}. \tag{4.8}$$

Since, $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$ and $E(\infty, L_jf^{(s)}(z+c)) \subseteq E(\infty, L_jf^{(l)}(z))$, $\Psi(z)$ is an entire function. Hence,

$$N(r, \Psi(z)) = 0.$$

Further, by *Lemma 3.1, 3.6* and logarithmic derivative theorem, we know that

$$m(r, \Psi(z)) \leq m\left(r, \frac{L_jf^{(s)}(z+c)}{L_jf^{(s)}(z)}\right) + m\left(r, \frac{L_jf^{(s)}(z)}{L_jf^{(l)}(z)}\right) = S(r, f).$$

Therefore,

$$T(r, \Psi) = S(r, f).$$

If possible, let $L_jf^{(l)}(z) \not\equiv L_jf^{(s)}(z+c)$. Hence, $\Psi(z) \neq 1$.
Case 1: Let $\Psi(z)$ is constant and z_0 be an a-point of $L_jf^{(l)}(z)$. Since, $E(a, L_jf^{(l)}(z)) \subseteq E(a, L_jf^{(s)}(z+c))$, we can deduce that $\Psi(z_0) = 1$, which implies that $\Psi(z) = 1$, which is not possible. Therefore, a is an e.v.p. of $L_jf^{(l)}(z)$.
Case 2: $\Psi(z)$ is not constant. If z_0 is a-point of $L_jf^{(l)}(z)$ and $L_jf^{(s)}(z+c)$, by the same argument used as in *Case 1*, $\Psi(z_0) = 1$. Then by the First Fundamental Theorem, we obtain,

$$\overline{N}\left(r, \frac{1}{L_jf^{(l)}(z) - a}\right) \leq N\left(r, \frac{1}{\Psi(z) - 1}\right) \leq T(r, \Psi) + O(1) = S(r, f).$$

From (4.8), we have

$$\frac{\Psi'(z)}{\Psi(z)} = \frac{L_jf^{(s+1)}(z+c)}{L_jf^{(s)}(z+c)} - \frac{L_jf^{(l+1)}(z)}{L_jf^{(l)}(z)}. \tag{4.9}$$

Let z_0 be an a-point of $L_jf^{(l)}(z)$ of multiplicity $p \geq 2$. Since, $E(a, L_jf^{(l)}(z)) \subseteq E(a, L_jf^{(s)}(z+c))$, z_0 is also an a-point of $L_jf^{(s)}(z+c)$ with multiplicity $p \geq 2$. Then, from (4.9), it follows that z_0 is a zero of $\frac{\Psi'(z)}{\Psi(z)}$ of multiplicity $p-1$. Hence,

$$\begin{aligned} N_{(2}\left(r, \frac{1}{L_jf^{(l)}(z) - a}\right) &\leq 2N\left(r, \frac{1}{\frac{\Psi'(z)}{\Psi(z)}}\right) \\ &\leq 2T\left(r, \frac{\Psi'(z)}{\Psi(z)}\right) + S(r, f) \\ &\leq 2N\left(r, \frac{\Psi'(z)}{\Psi(z)}\right) + S(r, f) = S(r, f). \end{aligned}$$

Combining the above cases we can say,

$$N\left(r, \frac{1}{L_jf^{(l)}(z) - a}\right) = S(r, f). \tag{4.10}$$

Since, $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$ and $E(\infty, L_jf^{(s)}(z+c)) \subseteq E(\infty, L_jf^{(l)}(z))$, we deduce that

$$N\left(r, \frac{1}{L_jf^{(l)}(z)}\right) - N\left(r, \frac{1}{L_jf^{(s)}(z+c)}\right) \leq 0$$

and

$$N(r, L_jf^{(s)}(z+c)) - N(r, L_jf^{(l)}(z)) \leq 0.$$

Adding $N\left(r, \frac{1}{L_jf^{(l)}(z)}\right) + N\left(r, \frac{1}{L_jf^{(l)}(z)-a}\right)$ with the both sides of the inequality of *Lemma 3.8* and applying (4.10) and the First Fundamental Theorem we obtain,

$$\begin{aligned} 2T(r, L_jf^{(l)}(z)) &\leq N\left(r, \frac{1}{L_jf^{(l)}(z)}\right) + N\left(r, \frac{1}{L_jf^{(l)}(z)-a}\right) \\ &+m(r, L_jf^{(l)}(z)) + N(r, L_jf^{(s)}(z+c)) \\ &-N\left(r, \frac{1}{L_jf^{(s)}(z+c)}\right) + S(r,f). \end{aligned}$$

i.e.,

$$\begin{aligned} T(r, L_jf^{(l)}(z)) &\leq N\left(r, \frac{1}{L_jf^{(l)}(z)-a}\right) + \left(N\left(r, \frac{1}{L_jf^{(l)}(z)}\right) - N\left(r, \frac{1}{L_jf^{(s)}(z+c)}\right)\right) \\ &+(N(r, L_jf^{(s)}(z+c))) - N(r, L_jf^{(l)}(z)) + S(r,f) \\ &= S(r,f), \end{aligned}$$

which is not possible. Therefore, $L_jf^{(l)}(z) \equiv L_jf^{(s)}(z+c)$. □

Proof of Theorem 2.3. Let us consider the following auxiliary function

$$U(z) = \frac{L_jf^{(l+1)}(z)(L_jf^{(l)}(z) - L_jf^{(s)}(z+c))}{L_jf^{(l)}(z)(L_jf^{(l)}(z)-a)}. \tag{4.11}$$

First assume that $U \not\equiv 0$.

Let z_0 be a-point of $L_jf^{(l)}(z)$ of multiplicity p and that of $L_jf^{(s)}(z+c)$ of multiplicity q. Therefore, z_0 is a zero of U of multiplicity $(p-1)+\min\{p,q\}-p \geq 0$. Hence, z_0 is not a pole of U. Further, as $E(0, L_jf^{(l)}(z)) \subseteq E(0, L_jf^{(s)}(z+c))$, if z_1 is a zero of $L_jf^{(l)}(z)$ of multiplicity m, it is a zero of $L_jf^{(s)}(z+c)$ of multiplicity same. Therefore, z_1 is a zero of U of multiplicity $(m-1)+m-m = m-1 \geq 0$. Hence, z_1 is not a pole of U. Next, let z_2 be a pole of $L_jf^{(l)}(z)$ of multiplicity n. By a simple calculation we can say that z_2 is a simple pole of U. Using *Lemma 3.9* we have,

$$N(r, U) \leq \overline{N}(r, L_jf^{(l)}(z)) = S(r,f).$$

Hence,

$$N(r, U) = S(r,f).$$

Now, using *Lemmas 3.6, 3.7* and logarithmic derivative theorem we obtain

$$\begin{aligned} m(r, U) &\leq m\left(r, \frac{L_jf^{(l+1)}(z)}{L_jf^{(l)}(z)-a}\right) + m\left(r, 1 - \frac{L_jf^{(s)}(z+c)}{L_jf^{(l)}(z)}\right) \\ &= S(r,f). \end{aligned}$$

Therefore,

$$T(r, U) = S(r,f). \tag{4.12}$$

Further, using *Lemmas 3.6, 3.7* and logarithmic derivative theorem we have

$$\begin{aligned} m\left(r,\frac{1}{L_jf^{(l)}(z)}\right) &\leq m\left(r,\frac{U}{L_jf^{(l)}(z)}\right)+m\left(r,\frac{1}{U}\right)+S(r,f)\\ \leq\ & m\left(r,\frac{L_jf^{(l+1)}(z)}{L_jf^{(l)}(z)(L_jf^{(l)}(z)-a)}\right)+m\left(r,1-\frac{L_jf^{(s)}(z+c)}{L_jf^{(l)}(z)}\right)+S(r,f)\\ =\ & S(r,f). \end{aligned}$$

Clearly from (4.11) and (4.12) using the First Fundamental Theorem we get

$$N\left(r,\frac{1}{L_jf^{(l)}(z)}\right)-\overline{N}\left(r,\frac{1}{L_jf^{(l)}(z)}\right)\leq N\left(r,\frac{1}{U}\right)\leq T(r,U)=S(r,f).$$

By the hypothesis of Theorem 2.3 we get

$$N\left(r,\frac{1}{L_jf^{(l)}(z)}\right)=S(r,f).$$

Hence,

$$T\left(r,\frac{1}{L_jf^{(l)}(z)}\right)=S(r,f),$$

which is a contradiction. Therefore, $U\equiv 0$, i.e., $L_jf^{(l)}(z)\equiv L_jf^{(s)}(z+c)$. □

Proof of Theorem 2.4. Clearly, $\rho_2(L_jf(z))\leq\rho_2(f)<1$. Proceeding as the same way of *Lemma 4 of* [12] we can prove that $L_jf(z)$ is transcendental. Suppose that, $L_jf^{(l)}(z)\not\equiv L_jf^{(s)}(z+c)$, i.e., $\Psi(z)\not\equiv 1$, since otherwise we are done. Next considering the auxiliary function (4.8) and using the same arguments as used in the proof of Theorem 2.2, we have

$$T(r,\Psi)=S(r,f).$$

Let z_0 be a common a-point of $L_jf^{(l)}(z)$ and $L_jf^{(s)}(z+c)$. Then, $\Psi(z_0)=1$.

Therefore, using the First Fundamental Theorem we have

$$\begin{aligned} \overline{N}\left(r,\frac{1}{L_jf^{(l)}(z)-a}\right) &= \overline{N}\left(r,\frac{1}{L_jf^{(s)}(z+c)-a}\right)\leq N\left(r,\frac{1}{\Psi(z)-1}\right)\\ &\leq T(r,\Psi(z))+S(r,f)=S(r,f). \end{aligned}\tag{4.13}$$

From (4.8), we have

$$(L_jf^{(s)}(z+c)-a)=\Psi(z)\left(L_jf^{(l)}(z)-\frac{a}{\Psi(z)}\right).$$

Therefore, by (4.13) we have

$$\overline{N}\left(r,\frac{1}{L_jf^{(l)}(z)-\frac{a}{\Psi(z)}}\right)\leq\overline{N}\left(r,\frac{1}{L_jf^{(s)}(z+c)-a}\right)=S(r,f).$$

By *Lemma 3.4*, we know $L_jf(z)$ is transcendental and so we have $\Psi(z)$ is small function with respect to $L_jf(z)$, by the Second Fundamental Theorem for small function, (4.13) and the above inequality we have,

$$\begin{aligned} T(r, L_j f^{(l)}(z)) &\leq \overline{N}(r, L_j f^{(l)}(z)) + \overline{N}\left(r, \frac{1}{L_j f^{(l)}(z) - a}\right) \\ &+ \overline{N}\left(r, \frac{1}{L_j f^{(l)}(z) - \frac{a}{\Psi(z)}}\right) + S(r, f) \\ &= S(r, f), \end{aligned}$$

which is contradiction. Hence, $L_j f^{(l)}(z) \equiv L_j f^{(s)}(z+c)$. □

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