

On $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups of finite groups*

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Abstract: In this paper, we investigate the influence of $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups on the structure of finite groups and some new results on the p -nilpotency and p -supersolvability of finite groups are obtained.

Keywords: s -permutable; $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups; p -supersolvable; p -nilpotent.

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1 Introduction

All groups mentioned in this paper are considered to be finite. Most of terminologies and notations are standard. The reader is referred to [9] and [10]. G always denotes a finite group and $|G|$ is the order of G . A group G is said to be p -supersolvable if all chief factors of G having order divisible by p are exactly of order p . It is known that the class of all p -supersolvable groups is a saturated formation.

Let H be a subgroup of G . H is said to be s -permutable or s -quasinormal in G , if H permutes with all Sylow subgroups of G (see [11]); H is said to be C -normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G (see [14]); H is said to be an \mathcal{H} -subgroup of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$ (see [5]); H is called a weakly \mathcal{H} -subgroup of G if it has a normal subgroup T such that $G = HT$ and $H \cap T$ is an \mathcal{H} -subgroup of G (see [1]); H is said to be weakly \mathcal{H} -embedded in G if G has a normal subgroup T such that $H^G = HT$ and $H \cap T$ is an \mathcal{H} -subgroup of G , where H^G is the normal closure of H in G (see [3]); H is called an $\mathcal{H}C$ -subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$ (see [15]); H is said to be weakly $\mathcal{H}C$ -embedded in G if G has a normal subgroup T such that $H^G = HT$ and $H^g \cap N_T(H) \leq H$ for

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all $g \in G$ (see [2]). Using these concepts, many interesting results on the structure of finite groups have been obtained in [1, 2, 3, 5, 14, 15, 16].

More recently, T. M. Al-Gafri and S. K. Nauman [7] introduced a new subgroup embedding property that extends all the above mentioned concepts as follows:

Definition 1.1. *A subgroup H of G is said to be an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G if there exists an s -permutable subgroup T of G such that $H^{sG} = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$, where H^{sG} is the intersection of all s -permutable subgroups of G containing H .*

In [7], the authors studied the structure of finite groups under the assumption that certain subgroups of prime power orders are $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups. In this paper, we continue the work and present some sufficient conditions for a group to be p -supersolvable and p -nilpotent.

2 Preliminaries

Lemma 2.1 ([7, Lemma 2.2]). *Suppose that H is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G .*

- (1) *If $H \leq K \leq G$, then H is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in K .*
- (2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G/N .*
- (3) *If H is a p -subgroup and N is a normal p' -subgroup of G , then HN and HN/N are $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups in G and G/N , respectively.*

Lemma 2.2 ([5, Theorem 6 (2)]). *Let H be an \mathcal{H} -subgroup of G . If H is subnormal in G , then H is normal in G .*

Lemma 2.3 ([6] and [11]). *Suppose that H be a subgroup of G and H is s -permutable in G . Then*

- (1) *H is subnormal in G .*
- (2) *If $K \leq G$ and K is s -permutable in G , then $H \cap K$ is s -permutable in G .*
- (3) *H/H_G is nilpotent.*

Lemma 2.4 ([13, Theorem A]). *If P is an s -permutable p -subgroup of G for some prime p , then $N_G(P) \geq O^p(G)$.*

Lemma 2.5 ([7, Theorem 3.1]). *Let P be a Sylow p -subgroup of a group G , for some prime p . Then G is p -nilpotent if and only if $N_G(P)$ is p -nilpotent and every maximal subgroup of P is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G .*

Lemma 2.6. *Let H be a p -subgroup of a group G for some prime p . If N is normal in G and $(|N|, p) = 1$, then $N_{G/N}(HN/N) = N_G(H)N/N$.*

Proof. By [7, Lemma 2.3], we have $N_G(HN) = N_G(H)N$. Consequently, $N_{G/N}(HN/N) = N_G(HN)/N = N_G(H)N/N$. \square

Lemma 2.7 ([4, Theorem 2.1.6]). *If G is p -supersoluble and $O_{p'}(G) = 1$, then the Sylow p -subgroup of G is normal in G .*

Lemma 2.8 ([12, Theorem 1.3]). *Let P be a Sylow p -subgroup of a group G , where p is a prime divisor of $|G|$. If every maximal subgroup of P has a p -nilpotent supplement in G , then G is p -nilpotent.*

Lemma 2.9 ([8, Theorem 8.3.1]). *Let P be a Sylow p -subgroup of G , where p is an odd prime divisor of $|G|$. Then G is p -nilpotent if and only if $N_G(Z(J(P)))$ is p -nilpotent, where $J(P)$ is the Thompson subgroup of P .*

3 Main results

Theorem 3.1. *Let P be a Sylow p -subgroup of a p -solvable G for some prime p . If every maximal subgroup of P is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G , then G is p -supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Note that $P/O_{p'}(G)$ is a Sylow p -subgroups of $G/O_{p'}(G)$, where $G/O_{p'}(G)$ is p -solvable. Let $M/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = (M \cap P)O_{p'}(G)$, where $M \cap P$ is a maximal subgroup of P . Since $M \cap P$ is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G and $O_{p'}(G)$ is a normal p' -subgroup of G , then $M/O_{p'}(G) = (M \cap P)O_{p'}(G)/O_{p'}(G)$ is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in $G/O_{p'}(G)$ by Lemma 2.1(3). Therefore $G/O_{p'}(G)$ is p -supersolvable by the minimal choice of G . Consequently, G is p -supersolvable, a contradiction.

(2) G has a unique minimal normal subgroup N such that N is an elementary abelian p -group and G/N is p -supersolvable.

Since G is p -solvable, we may assume that G has a minimal normal subgroup N . In view of step (1), N is an elementary abelian p -group and so $N \leq P$. If $N = P$, then G/N is a p' -group and hence G/N is p -supersolvable. Now assume that $N < P$. Clearly, P/N is a Sylow p -subgroup of G/N . Let M/N be a maximal subgroup of P/N . Then M is a maximal subgroup of P . By hypothesis, M is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G . Lemma 2.1(2)

implies that M/N is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G/N . Therefore the maximal subgroups of P/N are $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroups in G/N . Thus G/N is p -supersolvable by the minimal choice of G . If N_1 and N_2 are two distinct minimal normal subgroups of G such that $N_1, N_2 \leq P$, then G/N_1 and G/N_2 are p -supersolvable by the above proof. Since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$, it follows that G is p -supersolvable. This contradiction shows that N is a unique minimal normal subgroup of G .

(3) $|N| > p$, $\Phi(G) = 1$ and $N = F(G) = O_p(G)$.

If $\Phi(G) > 1$, then $N \leq \Phi(G)$ and $G/\Phi(G) \cong (G/N)/(\Phi(G)/N)$ is p -supersolvable by step (2). Since the class of all p -supersolvable groups is saturated, it follows that G is p -supersolvable. This contradiction shows that $\Phi(G) = 1$. Since the **fitting** subgroup of a group with unit Frattini subgroup coincides with the product of all abelian minimal normal subgroups, we have $N = F(G) = O_p(G)$. If $|N| = p$, then G/N is p -supersolvable by step (2) and so G is p -supersolvable, a contradiction.

(4) There exists a maximal subgroup R of P such that $N \not\leq R$.

If $N \leq \Phi(P)$, then $N \leq \Phi(G) = 1$, a contradiction. Hence there exists a maximal subgroup R of P such that $N \not\leq R$.

(5) $R \cap N \neq 1$.

Assume that $R \cap N = 1$. Then $|N| = |N|/|R \cap N| = |NR/R| = |P/R| = p$ by step (4), a contradiction.

(6) $R \cap N$ is not normal in G .

If not, we have $R \cap N = 1$ or $R \cap N = N$ by the minimal normality of N , which contradicts (4) and (5).

(7) There exists an s -permutable subgroup T of G such that RT is s -permutable in G and $R^g \cap N_T(R) \leq R$ for all $g \in G$, where $T > 1$.

By the hypothesis of the theorem, R is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G . Therefore G has an s -permutable subgroup T such that $R^{sG} = RT$ and $R^g \cap N_T(R) \leq R$ for all $g \in G$. By Lemma 2.3(2), RT is s -permutable in G . If $T = 1$, then R is s -permutable in G . By Lemma 2.4, $O^p(G) \leq N_G(R)$. It follows that R is normal in $PO^p(G) = G$. Consequently, $R \cap N$ is a normal subgroup of G , which contradicts (6).

(8) $N \not\leq T$.

Assume that $N \leq T$. Noticing that N is abelian, we have $(R \cap N)^g \cap N_G(R \cap N) = R^g \cap N \cap N_G(R \cap N) = R^g \cap N = R^g \cap N \cap T \cap P \leq R^g \cap N \cap T \cap N_G(R) = R^g \cap N \cap N_T(R)$. By step (7), we have $(R \cap N)^g \cap N_G(R \cap N) \leq R \cap N$. This shows that $R \cap N$ is an \mathcal{H} -subgroup of G . Obviously, $R \cap N$ is subnormal in G . In view of Lemma 2.2, $R \cap N$ is normal in G , which contradicts (6).

(9) T is a p -group.

Since T is s -permutable in G , it follows from Lemma 2.3(3) that T/T_G is nilpotent. If $T_G \neq 1$, then $N \leq T_G$ by step (2). Consequently, $N \leq T$, which contradicts (8). Hence $T_G \neq 1$ and T is nilpotent. In view of Lemma 2.3(1), $T \triangleleft \triangleleft G$. Let $T_{p'}$ be the normal Hall p' -subgroup of T . Obviously, $T_{p'} \triangleleft \triangleleft G$. Hence $T_{p'} \leq O_{p'}(G) = 1$. This implies that T is a p -group.

(10) The final contradiction.

By the maximality of R in P , $RT = R$ or P . Since RT is s -permutable in G , we have $RT \trianglelefteq PO^p(G) = G$ by Lemma 2.4. If $RT = R$, then $N \leq R$ by step (2), which contradicts (4). If $RT = P$, then $N = P = O_p(G)$ is elementary abelian by step (2) and so $T \trianglelefteq P$. Furthermore, $T \trianglelefteq PO^p(G) = G$ by Lemma 2.4. By step (2), $N = T$, contrary to step (8). \square

Theorem 3.2. *Let P be a Sylow p -subgroup of a group G , for some odd prime p . Then G is p -nilpotent if and only if every maximal subgroup P_1 of P not having a p -nilpotent supplement in G is an $\mathcal{SS}\mathcal{H}$ -subgroup in G and $N_G(P_1)$ is p -nilpotent.*

Proof. If G is p -nilpotent, then every maximal subgroup P_1 of P not having a p -nilpotent supplement in G is an $\mathcal{SS}\mathcal{H}$ -subgroup in G by [7, Lemma 2.5] and $N_G(P_1)$ is p -nilpotent. For the converse, we suppose that the result is false and let G be a counterexample of minimal order.

(1) If $P \leq K < G$, then K is p -nilpotent.

Let M be a maximal subgroup of P not having a p -nilpotent supplement in K . If M has a p -nilpotent supplement L in G , then M has a p -nilpotent supplement $L \cap K$ in K , a contradiction. Thus M is a maximal subgroup of P not having a p -nilpotent supplement in G . By hypothesis, M is an $\mathcal{SS}\mathcal{H}$ -subgroup in G . By Lemma 2.1(1), M is an $\mathcal{SS}\mathcal{H}$ -subgroup in K . Since $N_K(M) = K \cap N_G(M)$ and $N_G(M)$ is p -nilpotent, it follows that $N_K(M)$ is p -nilpotent. Therefore, K satisfies the hypothesis of the theorem, and so K is p -nilpotent by the minimal choice of G .

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. Let $M/O_{p'}(G)$ be a maximal subgroup of the Sylow p -subgroup $PO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$ not having a p -nilpotent supplement in $G/O_{p'}(G)$. Then $M = (M \cap P)O_{p'}(G)$, where $M \cap P$ is a maximal subgroup of P . If $M \cap P$ has a p -nilpotent supplement L in G , then $M/O_{p'}(G)$ has a p -nilpotent supplement $LO_{p'}(G)/O_{p'}(G)$ in $G/O_{p'}(G)$, a contradiction. Thus $M \cap P$ is a maximal subgroup of P not having a p -nilpotent supplement in G . By hypothesis, $M \cap P$ is an $\mathcal{SS}\mathcal{H}$ -subgroup in G . By Lemma 2.1(3) $M/O_{p'}(G) = (M \cap P)O_{p'}(G)/O_{p'}(G)$ is an

$\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in $G/O_{p'}(G)$. In view of Lemma 2.6,

$$N_{G/O_{p'}(G)}(M/O_{p'}(G)) = N_{G/O_{p'}(G)}((M \cap P)O_{p'}(G)/O_{p'}(G)) = N_G(M \cap P)O_{p'}(G)/O_{p'}(G).$$

Since $N_G(M \cap P)O_{p'}(G)/O_{p'}(G) \cong N_G(M \cap P)/(N_G(M \cap P) \cap O_{p'}(G))$ and $N_G(M \cap P)$ is p -nilpotent, it follows that $N_{G/O_{p'}(G)}(M/O_{p'}(G))$ is p -nilpotent. Therefore $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The minimal choice of G yields that $G/O_{p'}(G)$ is p -nilpotent, and so G is p -nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

Since G is not p -nilpotent, it follows from Lemma 2.9 that $N_G(Z(J(P)))$ is not p -nilpotent, where $J(P)$ is the Thompson subgroup of P . Noticing that $Z(J(P))$ is a characteristic subgroup of P , $P \leq N_G(Z(J(P)))$. In view of step (1), we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$.

(4) Let L be a normal p -subgroup of G such that $1 < L < P$, then G/L is p -nilpotent.

It is easy to see that the hypothesis of the theorem holds for G/L by Lemma 2.1(2). Hence G/L is p -nilpotent.

(5) G is p -solvable.

If $O_p(G) = P$, then G is p -solvable obviously. If $O_p(G) < P$, then $G/O_p(G)$ is p -nilpotent by step (4). Consequently, G is p -solvable.

(6) G has a unique minimal normal subgroup N such that G/N is p -nilpotent.

Let N be a minimal normal subgroup of G . Then by steps (2) and (5), $N \leq O_p(G)$. If $N = P$, then G/N is a p' -group and hence G/N is p -nilpotent. Now assume that $N < P$. In view of step (4), G/N is p -nilpotent. If N_1 and N_2 are two distinct minimal normal subgroups of G such that $N_1, N_2 \leq P$, then G/N_1 and G/N_2 are p -supersolvable by the above proof. Since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$, it follows that G is p -supersolvable. This contradiction shows that N is a unique minimal normal subgroup of G .

(7) $\Phi(G) = 1$ and $N = F(G) = O_p(G)$.

If $\Phi(G) > 1$, then $N \leq \Phi(G)$ and $G/\Phi(G) \cong (G/N)/(\Phi(G)/N)$ is p -nilpotent by step (6). Since the class of all p -nilpotent groups is saturated, it follows that G is p -nilpotent. This contradiction shows that $\Phi(G) = 1$. Since the fitting subgroup of a group with unit Frattini subgroup coincides with the product of all abelian minimal normal subgroups, we have $N = F(G) = O_p(G)$.

(8) $|N| > p$.

If $|N| = p$, then G/N is p -nilpotent by step (6) and so G is p -supersolvable. In view of Lemma 2.7, P is normal in G and so $P = O_p(G)$. By step (7), $|P| = p$. Then the maximal subgroup of P is 1. By the hypothesis of the theorem, 1 has a p -nilpotent

supplement G or $N_G(1) = G$ is a p -nilpotent, a contradiction.

(9) N has a p -nilpotent supplement M in G .

Since $\Phi(G) = 1$ by step (7), it follows that G has a maximal subgroup M such that $N \not\leq M$ and so $G = NM$. It is easy to see that $N \cap M$ is normal in G . Consequently, $N \cap M = 1$ and $M \cong G/N$ is p -nilpotent by step (6).

(10) There exists a maximal subgroup R of P such that R has no p -nilpotent supplement in G . Then R is an $\mathcal{S}\mathcal{S}\mathcal{H}$ -subgroup in G by hypothesis.

This follows from Lemma 2.8.

(11) $N \not\leq R$.

If $N \leq R$, then R has a p -nilpotent supplement M in G by step (9), contrary to step (10).

(12) The final contradiction.

See the similar arguments to those used in the proof of Theorem 3.1. \square

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