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REGULARIZATION AND STABILITY OF SOLUTIONS OF SYSTEM OF LINEAR INTEGRAL FREDHOLM EQUATIONS OF THE FIRST KIND ON A SEMI-AXIS

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Dedicated to the 85th anniversary of the birth of Academician Vladimir Gavrilovich Romanov

Asanov A., Pirmatov A.Z., Bekeshova D.A.,Kadenova Z.A.,

Abstract: This work is devoted to the study of a system of linear Fredholm integral equations of the first kind. A theorem on the uniqueness of solutions on the semi-axis is proven. Lavrentiev regularizations are constructed and stability estimates for solutions of this system of equations on the semi-axis are obtained.

Keywords: linear, sustem, inteqral equations, Fredholm, first kind, solution, regularization, stability, semi-axis.

1. Introduction

Inverse and ill-posed problems are of great interest both in theoretical and practical research. The founders A.N. Tikhonov [1], V.K. Ivanov [2] and M.M. Lavrentiev [3] opened new inexhaustible fields of theory and applications of ill-posed and inverse problems.

One of the leaders of the Siberian school of inverse problems V.G. Romanov and his students made a great contribution to the development of ill-posed and inverse problems, such as new problems of electrodynamics [4], [5], acoustics [6], stability in inverse problems for hyperbolic methods [7], inverse problems and artificial intelligence [8], models of electromagnetic probe [9], medicine [10].

In the works of M.M. Lavrentiev [11],[12] regularizing operators for Fredholm integral equations of the first kind were constructed. Using this method, scientific results were obtained for integral equations of the first kind in the works [13]-[16].

Let's consider

$$Pu \equiv \int_{-\infty}^{a} P(t,s)u(s)ds, t \in (-\infty, a], \tag{1}$$

$$Pu = f(t), t \in (-\infty, a], \tag{2}$$

where

$$P(t,s) = \begin{cases} L(t,s), & -\infty < s \le t \le a, \\ M(t,s), & -\infty < t \le s \le a, \end{cases} \tag{3}$$

$$L(t,s) = (l_{ij}(t,s)) = \begin{pmatrix} l_{11}(t,s) & l_{12}(t,s) & \dots & l_{1n}(t,s) \\ l_{21}(t,s) & l_{22}(t,s) & \dots & l_{2n}(t,s) \\ \dots & \dots & \dots & \dots \\ l_{n1}(t,s) & l_{n2}(t,s) & \dots & l_{nn}(t,s) \end{pmatrix},$$

$$M(t,s) = (m_{ij}(t,s)) = \begin{pmatrix} m_{11}(t,s) & m_{12}(t,s) & \dots & m_{1n}(t,s) \\ m_{21}(t,s) & m_{22}(t,s) & \dots & m_{2n}(t,s) \\ \dots & \dots & \dots & \dots \\ m_{n1}(t,s) & m_{n2}(t,s) & \dots & m_{nn}(t,s) \end{pmatrix},$$

$$f(t) = (f_i(t)) = (f_1(t), ..., f_n(t))^T, u(t) = (u_i(t)) = (u_1(t), ..., u_n(t))^T.$$

$L(t,s)$ and $M(t,s)$ are the known matrix functions, $f(t)$ is known vector function, $u(t)$ is unknown vector function.

We introduce possible notation:

1. For vectors $u = (u_1, u_2, ..., u_n), v = (\vartheta_1, \vartheta_2, ..., \vartheta_n) \in R^n$ we define the scalar product by the equality

$$\langle u, \vartheta \rangle = \sum_{i=1}^{n} u_i v_i.$$

2. For $A = (a_{ij}) - n \times n$ - square matrix and $u = (u_1, u_2, ..., u_n) \in R^n$ we define the norm

$$\|A\| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

$$\|u\| = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{\frac{1}{2}}.$$

3. Through $L_{2,n}(-\infty, a]$ denote the space of all $n-$ dimensional vector functions $u(t) = (u_1(t), ..., u_n(t))$ satisfying the condition $u_i(t) \in L_2(-\infty, a]$ for all $i = 1, 2, ..., n$.

For $u(t) = (u_1(t), ..., u_n(t)) \in L_{2,n}(-\infty, a]$ define the norm

$$\|u(t)\|_{2,n} = \left(\int_{-\infty}^{a} \|u(t)\|^2 \, dt \right)^{\frac{1}{2}}.$$

We will assume that

$$\|K(t,s)\| \in L_2((-\infty, a] \times (-\infty, a]), \|f(t)\| \in L_2(-\infty, a].$$

2. The system of linear integral Fredholm equations of the first kind on the semi-axis

From relation (2), we write the system of linear integral Fredholm equations of the first kind (hereinafter referred to as the system of equations) (1) in the following form

$$\int_{-\infty}^{t} L(t,s)u(s)ds + \int_{t}^{a} M(t,s)u(s)ds = f(t). \tag{4}$$

Both parts of (3) are scalarly multiplied by the $u(t)-$ vector function and, integrating the results on $-\infty < t \le a$, we obtain

$$\int_{-\infty}^{a} \int_{-\infty}^{t} \langle L(t,s)u(s), u(t) \rangle \, dsdt + \int_{-\infty}^{a} \int_{t}^{a} \langle M(t,s)u(s), u(t) \rangle \, dsdt = \int_{-\infty}^{a} \langle f(t), u(t) \rangle \, dt,$$

$$\int_{-\infty}^{a} \int_{-\infty}^{t} \langle L(t,s)u(s), u(t) \rangle \, dsdt + \int_{-\infty}^{a} \int_{s}^{a} \langle M^*(s,t)u(t), u(s) \rangle \, dsdt =$$

$$= \int_{-\infty}^{a} \langle f(t), u(t) \rangle \, dt, \qquad (5)$$

here M*(s, t) is the matrix transposed to the matrix M(s, t). For the second integral in formula (5) we use the Dirichlet formula

$$\int_{-\infty}^{a} \int_{s}^{a} \langle M^*(s,t)u(s), u(t) \rangle \, dtds = \int_{-\infty}^{a} \int_{-\infty}^{t} \langle M^*(s,t)u(s), u(t) \rangle \, dsdt.$$

Then from (5) we obtain

$$\int_{-\infty}^{a} \int_{-\infty}^{t} \langle [L(t,s) + M^*(s,t)]u(s), u(t) \rangle \, dsdt = \int_{-\infty}^{a} \langle f(t), u(t) \rangle \, dt \qquad (6)$$

Denote

$$K(t,s) = \frac{1}{2}(L(t,s) + M^*(s,t)), \qquad (t,s) \in G = \{-\infty < s \leq t \leq a\}.$$

Then from the latter we get

$$2 \int_{-\infty}^{a} \int_{-\infty}^{t} \langle K(t,s)u(s), u(t) \rangle \, dsdt = \int_{-\infty}^{a} \langle f(t), u(t) \rangle \, dt, \qquad (7)$$

here

$$\int_{-\infty}^{a} \int_{-\infty}^{t} K^2(t,s) dsdt < +\infty.$$

Now we get a new matrix function as follows

$$Q(t,s) = \begin{cases} K(t,s), & -\infty < s \leq t \leq a, \\ K^*(s,t), & -\infty < t \leq s \leq a. \end{cases} \qquad (8)$$

From this it is clear

$$Q(t,s) = Q^*(s,t), (t,s) \in (-\infty, a] \times (-\infty, a].$$

Hence the following inequalities are valid:

$$\int_{-\infty}^{a} \int_{-\infty}^{a} \|Q(t,s)\|^2 \, dsdt < +\infty.$$

From here

$$Q(t,s) = \sum_{\nu=1}^{m} \lambda_\nu \begin{pmatrix} \varphi_1^{(\nu)}(t) \\ ... \\ \varphi_n^{(\nu)}(t) \end{pmatrix} (\varphi_1^{(\nu)}(s), ..., \varphi_n^{(\nu)}(s)), m \leqslant \infty, \qquad (9)$$

where, λ_ν- are the eigenvalues of the matrix kernel $Q(s,t)$ arranged in descending order of their absolute values, and are vector functions; $\varphi^\nu(t) = (\varphi_1^\nu(t), ..., \varphi_n^\nu(t))-$ are the eigen orthonormal vector functions corresponding to the eigen values λ_ν. Here, $Q(s,t)$ is the full core and $0 < \lambda_1 \leqslant \lambda_2 \leqslant ...$. Hence, the solution of system (2) will be unique in $L_{2,n}(-\infty, a]$. Next, we assume that all eigenvalues of the kernel are positive. Due to the complete continuity and self conjugacy of the operator Q generated by the matrix kernel Q , the orthonormal sequence of eigenvector functions is complete in $L_{2,n}(-\infty,a]$.

For $u(t)=(u_i(t))\in L_{2,n}(-\infty,a]$ we obtain

$$\|u(t)\|_{2,n}=\left(\sum_{i=1}^{n}\int_{-\infty}^{a}|u_i(t)|^2dt\right)^{\frac{1}{2}}=\left(\int_{-\infty}^{a}\|u(t)\|^2\,dt\right)^{\frac{1}{2}}=\left(\sum_{\nu=1}^{\infty}\left|u^{(\nu)}\right|^2\right)^{\frac{1}{2}},$$

where

$$u^{(\nu)}=\int_{-\infty}^{a}\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt=\int_{-\infty}^{a}\left(\sum_{i=1}^{n}u_i(t)\varphi_i^{(\nu)}(t)\right)dt,(\nu=1,2,...).$$

Let the correctness sets depending on the parameter α be the following

$$N_\alpha=\left\{u(t)\in L_{2,n}(-\infty,a]:\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\le c\right\},where\ c>0,\ 0<\alpha<\infty,$$

$$u^{(\nu)}=\int_{-\infty}^{a}\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt,\ (\nu=1,2,...). \tag{10}$$

Let $u(t)=(u_i(t))\in N_\alpha$. Then

$$2\int_{-\infty}^{a}\int_{a}^{t}\sum_{\nu=1}^{\infty}\lambda_\nu\left\langle\begin{pmatrix}\varphi_1^{(\nu)}(t)\\ \dots\\ \varphi_n^{(\nu)}(t)\end{pmatrix}(\varphi_1^{(\nu)}(s)...\varphi_n^{(\nu)}(s))\begin{pmatrix}u_1(s)\\ \dots\\ u_n(s)\end{pmatrix},u(t)\right\rangle dsdt=\int_{-\infty}^{a}\left[\sum_{i=1}^{n}f_i(t)u_i(t)\right]dt,$$

$$\|u(t)\|_{2,n}^2=\sum_{i=1}^{\infty}\left|u^{(\nu)}\right|^2=\sum_{i=1}^{\infty}\lambda_\nu^{\alpha}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\leqslant\lambda_1^{\alpha}\left(\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\right)\le c\lambda_1^{\alpha},$$

$$\|u(t)\|_{2,n}^2\le c\lambda_1^{\alpha}. \tag{11}$$

We will assume that $f(t)\in K(N_\alpha)$. Then the system (2) has a solution $u(t)=(u_i(t))\in N_\alpha$ and by virtue of (7), (8)and (9) we have:

$$2\int_{-\infty}^{a}\int_{a}^{t}\sum_{\nu=1}^{\infty}\lambda_\nu\left\langle\begin{pmatrix}\varphi_1^{(\nu)}(t)\\ \dots\\ \varphi_n^{(\nu)}(t)\end{pmatrix}(\varphi_1^{(\nu)}(s),...,\varphi_n^{(\nu)}(s))\begin{pmatrix}u_1(s)\\ \dots\\ u_n(s)\end{pmatrix},u(t)\right\rangle dsdt=\int_{-\infty}^{a}\left[\sum_{i=1}^{n}f_i(t)u_i(t)\right]dt$$

$$\sum_{\nu=1}^{\infty}2\int_{-\infty}^{a}\lambda_\nu\left[\int_{a}^{t}\left\langle u(s),\varphi^{(\nu)}(s)\right\rangle ds\right]\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt,$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|\int_{-\infty}^{a}\left\langle u(t),\varphi^{(\nu)}(t)\right\rangle dt\right|^2=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt,$$

$$\sum_{\nu=1}^{\infty}\lambda_\nu\left|u^{(\nu)}\right|^2=\int_{-\infty}^{a}\langle f(t),u(t)\rangle\,dt.$$

Here we use Holder's inequalities

$$\sum_{\nu=1}^{\infty} \lambda_\nu \left|u^{(\nu)}\right|^2 \leq \|f(t)\|_{2,n} \|u(t)\|_{2,n} . \tag{12}$$

Besides

$$\|u(t)\|_{2,n}^2 = \sum_{\nu=1}^{\infty} \frac{\left|u^{(\nu)}\right|^{\frac{2\alpha}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{1+\alpha}}} \lambda_\nu^{\frac{\alpha}{1+\alpha}} \left|u^{(\nu)}\right|^{\frac{2}{1+\alpha}} \leq \left(\sum_{\nu=1}^{\infty} \lambda_\nu |u_\nu|^2\right)^{\frac{\alpha}{1+\alpha}} \left(\sum_{\nu=1}^{\infty} \lambda_\nu^{-\alpha} \left|u^{(\nu)}\right|^2\right)^{\frac{1}{1+\alpha}} .$$

We apply Holder's inequality when $p = 1+\alpha, q = \frac{(1+\alpha)}{\alpha}$. From formula (11) and $u(t) \in N_\alpha$, from the last inequality we obtain the following

$$\|u(t)\|_{2,n}^2 \leq c^{\frac{1}{1+\alpha}} \left(\|f(t)\|_{2,n} \|u(t)\|_{2,n}\right)^{\frac{\alpha}{1+\alpha}} .$$

From here we obtain the stability estimate:

$$\|u(t)\|_{2,n} \leq c^{\frac{1}{2+\alpha}} \|f(t)\|_{2,n}^{\frac{\alpha}{2+\alpha}} , 0 < \alpha < \infty. \tag{13}$$

and the following theorem is proven.

Theorem 1. Let the operator Q, defined by formula (8) generated by the matrix kernel, be positive. Then estimate (13) is valid on the set, i.e. on the set $P(N_\alpha)$ ($P(N_\alpha)$ image when N_α displayed by the operator P) the operator P^{-1}, the inverse of P, is uniformly continuous with the Holder exponent $\frac{\alpha}{2+\alpha}$.

Here we consider the following systems equation

$$\epsilon u(t,\epsilon) + \int_{-\infty}^{a} P(t,s) u(s,\epsilon) ds = f(t),\ t \in (-\infty, \alpha]),\ \epsilon > 0. \tag{14}$$

We will show that on the set N_α. (14) - the system of equations will be a regularization for equation (2).

In the system of equations (14) we do the following

$$u(t,\epsilon) = u(t) + \xi(t,\epsilon), \tag{15}$$

where $u(t) \in N_\alpha -$ solution of (2), then

$$\epsilon \xi(t,\epsilon) + \int_{-\infty}^{a} P(t,s) \xi(s,\epsilon) ds = -\epsilon u(t). \tag{16}$$

We scalar multiply both sides (16) by $\xi(t,\epsilon)$ and integrating from $-\infty$ to a.

Taking into account (3) and (9) we obtain:

$$\epsilon \|\xi(t,\epsilon)\|_{2,n}^2 + \sum_{\nu=1}^{\infty} \lambda_\nu |\xi_\nu(\epsilon)|^2 \leq \epsilon \sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| |\xi_\nu(\epsilon)| , \tag{17}$$

where $\xi_i(\epsilon)$ - are the Fourier coefficients for the function $\xi(t,\epsilon)$, according to the orthonormal system $\varphi^{(\nu)}(t) = \left\{\varphi_i^{(\nu)}(t)\right\}$ that is

$$\xi_\nu(\epsilon) = \int_{-\infty}^{a} \left\langle \xi(t,\epsilon), \varphi^{(\nu)}(t) \right\rangle dt, (\nu = 1, 2, ...).$$

Here we apply Holder's inequality for $p = q = \frac{1}{2}$, from (17) we have the following:

$$\|\xi(t,\epsilon)\|_{2,n} \leq \|u(t)\|_{2,n}\,, \tag{18}$$

$$\sum_{\nu=1}^{\infty} \lambda_\nu \left|\xi_\nu(\epsilon)\right|^2 \leq \epsilon \left\|u(t)\right\|_{2,n}^2 \leq \epsilon c \lambda_1^\alpha, \epsilon > 0. \tag{19}$$

Besides

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| = \sum_{\nu=1}^{\infty} \frac{\left|\xi_\nu(\epsilon)\right|^{\frac{\alpha}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{2(1+\alpha)}}} \lambda_\nu^{\frac{\alpha}{2(1+\alpha)}} \left|u^{(\nu)}\right|^{\frac{1}{1+\alpha}} \left|\xi_\nu(t,\epsilon)\right|^{\frac{1}{1+\alpha}} \left|u^{(\nu)}\right|^{\frac{\alpha}{1+\alpha}}.$$

Here we apply the generalized Holder inequality to the right-hand side for

$$p = \frac{2(1+\alpha)}{\alpha}, q = 2(1+\alpha), m = 2(1+\alpha), n = \frac{2(1+\alpha)}{\alpha},$$

we get the following:

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| \leq \left(\sum_{\nu=1}^{\infty} \lambda_\nu \left|\xi_\nu(\epsilon)\right|^2\right)^{\frac{1}{p}} \left(\sum_{\nu=1}^{\infty} \frac{\left|u^{(\nu)}\right|^2}{\lambda_\nu^\alpha}\right)^{\frac{1}{q}} \|\xi(t,\epsilon)\|^{\frac{2}{q}} \|u(t)\|^{\frac{2}{p}},$$

Subsequently given $u(t) \in N_\alpha$, (18) and (19) we get

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| \leq (\epsilon c \lambda_1^\alpha)^{\frac{1}{p}} c^{\frac{1}{q}} (c\lambda_1^\alpha)^{\frac{p+q}{pq}}.$$

This $p = \frac{2(1+\alpha)}{\alpha}, q = 2(1+\alpha)$ we get

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| \leq c^{\frac{1}{2(1+\alpha)}} (c\lambda_1^\alpha)^{\frac{1}{2}} (\epsilon c \lambda_1^\alpha)^{\frac{\alpha}{2(1+\alpha)}},$$

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| \leq c^{\frac{1}{2(1+\alpha)}} c^{\frac{1}{2}} c^{\frac{\alpha}{2(1+\alpha)}} \lambda_1^{\frac{\alpha}{2}} \lambda_1^{\frac{\alpha^2}{2(1+\alpha)}} \epsilon^{\frac{\alpha}{2(1+\alpha)}},$$

$$\sum_{\nu=1}^{\infty} \left|u^{(\nu)}\right| \left|\xi_\nu(\epsilon)\right| \leq c \lambda_1^{\frac{\alpha(2\alpha+1)}{2(1+\alpha)}} \epsilon^{\frac{\alpha}{2(1+\alpha)}}. \tag{20}$$

Taking into account (17),(18) from (15) we have

$$\|u(t,\epsilon) - u(t)\|_{2,n} \leq c^{\frac{1}{2}} \lambda_1^{\frac{\alpha(2\alpha+1)}{4(1+\alpha)}} \epsilon^{\frac{\alpha}{4(1+\alpha)}}, 0 < \alpha < \infty. \tag{21}$$

The following theorems were proven.

Theorem 2. Let the operator Q, defined by formula (8) generated by the matrix kernel, be positive and $f(t) \in P(N_\alpha)$. Then estimate (21) is valid, where $u(t,\epsilon)-$ is the solution of the system (14), $u(t)$ is the solution of the system (2), $Q(t,s)$ is determined by formula (9).

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Information about the authors

1). Asanov Avyt, Kyrgyz Turkish University Manas, Bishkek,Chyngyz Aitmatov Avenue, 56
2). Kadenova Zuurakan Azhimamatovna, Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic, Chui Avenue, 265a
3). Bekeshova Damira. Institute of Mathematics of the National Academy of Sciences of the Kyrgyz Republic, Chui Avenue, 265a
4). Pirmatov Abdymanap Ziiaidinovich, Osh State University,Lenin Avenue 331