

RINGS WITH THE 2- ΔU PROPERTY

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ABSTRACT. Rings in which the square of each unit lies in $1 + \Delta(R)$ are said to be 2- ΔU rings, where $J(R) \subseteq \Delta(R) =: \{r \in R \mid r + U(R) \subseteq U(R)\}$. The set $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by units of R and is detailed studied in [21]. The class of 2- ΔU rings consists several rings including UJ -rings, 2- UJ rings and ΔU -rings, respectively, and we observe that ΔU -rings are UUC in terms of [2]. Furthermore, the structure of 2- ΔU rings is examined under various algebraic conditions. Moreover, the 2- ΔU property is explored under some extended constructions.

1. INTRODUCTION AND BASIC CONCEPTS

In the current paper, let R denote an associative not necessarily commutative ring with identity element. Typically, for such a ring R , the sets $U(R)$, $Nil(R)$, $C(R)$, $Id(R)$ and $J(R)$ represent the set of invertible elements in R , the set of nilpotent elements in R , the set of central elements in R , the set of idempotent elements in R and the Jacobson radical of R , respectively. Additionally, the ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are, respectively, denoted by $M_n(R)$ and $T_n(R)$. Traditionally, a ring is termed *abelian* if each idempotent element is central, meaning that $Id(R) \subseteq C(R)$.

The key instrument of the present study is the set $\Delta(R)$ which was handled by Lam in [20, Exercise 4.24] and recently investigated by Leroy-Matczuk in [21]. As pointed out by the authors in [21, Theorem 3 and 6], the subring $\Delta(R)$ is the largest Jacobson radical of R that is closed with respect to multiplication by all units (quasi-invertible elements) of R . Also, $J(R) \subseteq \Delta(R)$, and $\Delta(R) = J(T)$, where T is the subring of R generated by units of R , and the equality $\Delta(R) = J(R)$ holds if, and only if, $\Delta(R)$ is an ideal of R .

It is well known that $1 + J(R) \subseteq U(R)$. A ring R is said to be an UJ -ring if the reverse inclusion holds, i.e., $U(R) = 1 + J(R)$ (see [7] and [14]). Imitating [6], a ring R is said to be 2- UJ if, for each $u \in U(R)$, $u^2 = 1 + j$, where $j \in J(R)$. These rings are a common generalization of UJ rings. The authors showed there that for 2- UJ rings the notions of being semi-regular, exchange and clean rings are all equivalent.

In the other vein, recall that a ring R is called an UU -ring if $U(R) = 1 + Nil(R)$ (see, e.g., [9]). As a natural expansion of UU rings, Sheibani and Chen introduced in [25] the so-called 2- UU rings – a ring R is called 2- UU if the square of every unit is the sum of 1_R and a nilpotent. They showed that R is strongly 2-nil-clean if, and only if, R is an exchange 2- UU ring.

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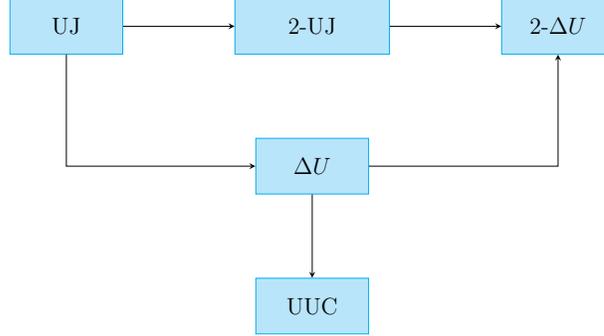
Let us also recollect certain classical concepts, needed for our successful presentation: a ring R is known to be *Boolean* if every element of R is idempotent. Also, as a more general setting, a ring R is said to be *regular* (resp., *unit-regular*) in the sense of von Neumann if, for every $a \in R$, there is an $x \in R$ (resp., $x \in U(R)$) such that $axa = a$ and, in addition, R is said to be *strongly regular* if, for each $a \in R$, $a \in a^2R$. Recall that a ring R is *exchange* if, for each $a \in R$, there exists $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$, and a ring R is *clean* if every element of R is a sum of an idempotent and an unit (cf. [23]). Notice that every clean ring is exchange, but the converse is manifestly *not* true in general; however, it is true in the abelian case (see, for more details, [23, Proposition 1.8]). Likewise, a ring R is called *semi-regular*, provided $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Note that semi-regular rings are exchange, but the opposite is generally *not* valid (see [23]).

Further, according to Chen ([3]), an element of a ring is called *J -clean*, provided that it can be written as the sum of an idempotent and an element from its Jacobson radical. Accordingly, a ring is termed *J -clean* in the case when each of its elements is J -clean or, equivalently, $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ (which is also called *semi-boolean* in the language of [24]). It was shown in [14] that a ring R is J -clean if, and only if, R is a clean UJ ring. Later on, in 2019, Karabacak et al. introduced new rings that are a non-trivial generalization of UJ rings; in fact, they named these rings ΔU (see [11]) and R is said to be a ΔU ring if $1 + \Delta(R) = U(R)$. Besides, due to Karabacak et al. ([11]), a ring R is called *Δ -clean*, provided every element of R is a sum of an idempotent and an element from the $\Delta(R)$. Thus, Δ -clean rings are clean, but the reciprocity is *not* fulfilled in all generality. They also showed in [11] that a ring R is Δ -clean if, and only if, R is a clean ΔU ring. Some other interesting results close to this material can be found in [15] as well.

As a proper expansion of some of the above concepts, we introduce the new class of *2- ΔU rings* as follows: a ring R is called *2- ΔU* if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (or, in an equivalent form, for each $u \in U(R)$, $u^2 = 1 + r$, where $r \in \Delta(R)$). Clearly, all ΔU rings, and hence the unit uniquely clean rings from [2] as well as the rings with only two units are *2- ΔU* . Also, *2- UJ* rings and hence *UJ* rings are *2- ΔU* , but the converse does *not* hold in general. Our motivating tool is to give a satisfactory description of these *2- ΔU* rings by comparing their crucial properties with these of *2- UU* and *2- UJ* rings, respectively, as well as to find some new exotic properties of *2- ΔU* rings that are not too characteristically or frequently seen in the existing literature.

We are now planning to give a brief program of our main material established in the sequel: In Section 2, we achieve to exhibit some major properties and characterizations of ΔU rings in various different aspects (see, for instance, Propositions 2.2, 2.3 and 2.4 and Theorem 2.5). In Section 3, we establish some fundamental characterizing properties of *2- ΔU* rings that are mainly stated and proved in Theorems 3.10, 3.19, 3.20 and 3.22 and the other statements associated with them. In Section 4, we give some extensions of *2- ΔU* rings; for instance, polynomial extensions, matrix extensions, trivial extensions and Morita contexts. We close our work in the final Section 5 with challenging questions, namely Problems 5.1, 5.2, 5.3 and 5.4.

Now, we have the following diagram which violates the relationships between the defined above sorts of rings:



2. ΔU RINGS

In this section, we reinvestigate some major properties of ΔU rings which were *not* found in [11], as well as we give a few close relations between ΔU rings and some related type of rings.

Definition 2.1 ([11]). A ring R is called ΔU if $1 + \Delta(R) = U(R)$.

Mimicking Calugareanu and Zhou (see [2]), a ring R is called UUC if every unit is uniquely clean.

We start with a series of preliminaries.

Proposition 2.2. *Let R be a ΔU ring. Then, the following three points hold:*

- (i) $U(R) + U(R) \subseteq \Delta(R)$.
- (ii) R is a UUC ring.
- (iii) $(U(R) + U(R)) \cap Id(R) = \{0\}$.

Proof. (i) Choose $x \in U(R) + U(R)$. So, $x = u_1 + u_2$, where $u_1, u_2 \in U(R) = 1 + \Delta(R)$ yielding $x = 1 + r_1 + 1 + r_2 = 2 + (r_1 + r_2)$, where $r_1, r_2 \in \Delta(R)$. On the other hand, we know that $2 \in \Delta(R)$ by [11, Proposition 2.4]. But, $\Delta(R)$ is a subring of R and thus $x \in \Delta(R)$, as required.

(ii) Assume that $u = e + v$, where $u, v \in U(R)$ and $e \in Id(R)$. It suffices to show that $e = 0$. To this target, as R is ΔU , we may write $u = 1 + r$ and $v = 1 + r'$, where $r, r' \in \Delta(R)$. Hence, $e = 0$ in view [21, Proposition 15], as needed.

(iii) This is clear combining (i) and (ii). □

Proposition 2.3. *Let R be a ΔU ring and set $\bar{R} := R/J(R)$. The following two items hold:*

- (i) For any $u_1, u_2 \in U(R)$, $u_1 + u_2 \neq 1$.
- (ii) For any $\bar{u}_1, \bar{u}_2 \in U(\bar{R})$, $\bar{u}_1 + \bar{u}_2 \neq \bar{1}$.

Proof. (i) This is immediate utilizing Proposition 2.2 and [2, Example 2.2].

(ii) Since R is ΔU , one sees that \bar{R} is ΔU and hence it is UUC . Now, the result follows directly from [2, Example 2.2]. □

Remember for completeness of the exposition that a ring R is called *semi-potent* if every one-sided ideal *not* contained in $J(R)$ contains a non-zero idempotent. Moreover, a semi-potent ring R is called *potent*, provided all idempotents lift modulo $J(R)$.

Proposition 2.4. *Let R be a potent ΔU ring and set $\bar{R} := R/J(R)$. Then, we have:*

- (i) *For any $\bar{e} = \bar{e}^2 \in \bar{R}$ and any $\bar{u}_1, \bar{u}_2 \in U(\bar{e}\bar{R}\bar{e})$, $\bar{u}_1 + \bar{u}_2 \neq \bar{e}$.*
- (ii) *There does not exist $\bar{e} = \bar{e}^2 \in \bar{R}$ such that $\bar{e}\bar{R}\bar{e} \cong M_2(S)$ for some ring S .*

Proof. (i) Given $\bar{e}, \bar{u}_1, \bar{u}_2$ as in (i), we can assume that $e^2 = e \in R$, because idempotents lift modulo $J(R)$. Thus, $\bar{e}\bar{R}\bar{e} \cong eRe/J(eRe)$. Finally, since eRe is ΔU in virtue of [11, Proposition 2.6], (i) follows automatically from Proposition 2.3(i).

- (ii) Note that, in a 2×2 matrix ring, it is always valid that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in U(M_2(S)) + U(M_2(S)).$$

Hence, there exist $\bar{u}_1, \bar{u}_2 \in U(\bar{e}\bar{R}\bar{e})$ such that $\bar{u}_1 + \bar{u}_2 = \bar{e}$. This, however, is a contradiction with (i), as expected. □

A ring R is called *reduced* if it contains no non-zero nilpotent elements, that is, $\text{Nil}(R) = (0)$.

We now come to the following criterion.

Theorem 2.5. *Let R be a semi-potent ring. Then, the following statements are equivalent:*

- (i) *R is a ΔU ring.*
- (ii) *$R/J(R)$ is Boolean.*
- (iii) *R is a UJ ring.*
- (iv) *$R/J(R)$ is a UU ring.*

Proof. (i) \Rightarrow (ii). Since R is semi-potent, $R/J(R)$ is too semi-potent (and, indeed, even potent). Also, $R/J(R)$ is ΔU . So, without loss of generality, it can be assumed that $J(R) = (0)$. Thus, using [11, Theorem 4.4], R is reduced and hence abelian.

Now, assume that there exists $x \in R$ such that $x - x^2 \neq 0$ in R . Since R is a semi-potent ring, there exists $e = e^2 \in R$ such that $e \in (x - x^2)R$. So, write $e = (x - x^2)y$ for some $y \in R$. Since e is central, it must be that

$$[er(1 - e)]^2 = 0 = [(1 - e)re]^2,$$

whence we have

$$er(1 - e) = 0 = e(1 - e)re.$$

We now can write:

$$e = ex.e(1 - e).ey,$$

so that both $ex, e(1 - x) \in U(eRe)$. But, we know that eRe is a ΔU ring. However, $ex + e(1 - x) = e$, which contradicts Proposition 2.3(i). Therefore, R is a Boolean ring, as desired.

(ii) \Rightarrow (iii). Let us assume $u \in U(R)$. Then, $\bar{u} \in U(\bar{R})$, where $\bar{R} = R/J(R)$. But, since \bar{R} is a Boolean ring, we obtain $\bar{u} = \bar{1}$ which implies $u - 1 \in J(R)$, as required.

(iii) \Rightarrow (i). This is evident, because we know that always $J(R) \subseteq \Delta(R)$.

(ii) \Rightarrow (iv). This is obvious, so we omit the necessary arguments.

(iv) \Rightarrow (ii). Knowing that $R/J(R)$ is semi-potent, one derives that it is also strongly nil-clean applying [12, Theorem 2.25]. Hence, $R/J(R)$ is an exchange UU ring consulting with [9, Theorem 4.3]. In conclusion, it is Boolean (see [9, Theorem 4.1]), as wanted. \square

As five consequences, we deduce:

Corollary 2.6. *A regular ring R is ΔU if, and only if, R is UJ if, and only if, R is UU if, and only if, R is Boolean.*

Proof. Since R is regular, $J(R) = (0)$ and R is semi-potent. So, the result follows from Theorem 2.5. \square

Corollary 2.7. *Let R be a potent ring. Then, the following are equivalent:*

- (i) R is a ΔU ring.
- (ii) $R/J(R)$ is a ΔU ring.
- (iii) $R/J(R)$ is a Boolean ring.
- (iv) R is a UJ ring.
- (v) $R/J(R)$ is a UJ ring.
- (vi) $R/J(R)$ is a UU ring.

Proof. We know that every potent ring is semi-potent. Then, issues (i), (iii), (iv) and (vi) are equivalent invoking Theorem 2.5. On the other side, issues (i) and (ii) are equivalent employing [11, Proposition 2.4]. Finally, issues (iv) and (v) are tantamount in conjunction with [14, Proposition 1.3]. \square

Corollary 2.8. *Let R be an Artinian ring. Then, the following are equivalent:*

- (i) R is a ΔU ring.
- (ii) R is a UJ ring.
- (iii) R is a UU ring.

Proof. We know that every Artinian ring is always clean. Also, since R is Artinian, we have $J(R) \subseteq Nil(R)$. \square

Corollary 2.9. *Let R be a finite ring. Then, the following conditions are equivalent:*

- (i) R is a ΔU ring.
- (ii) R is a UJ ring.
- (iii) R is a UU ring.

Proof. In fact, any finite ring is known to be Artinian. \square

Corollary 2.10. *For a ring R , the following two conditions are equivalent:*

- (i) R is a potent ΔU ring.
- (ii) R is a J -clean ring.

Proof. (ii) \Rightarrow (i). This is apparent by a combination of [14, Theorem 3.2] and Corollary 2.7.

(i) \Rightarrow (ii). Thanks to Corollary 2.7, we infer that $R/J(R)$ is Boolean. Therefore, for each $a \in R$, we receive $a - a^2 \in J(R)$. Besides, since R is a potent ring, there exists an idempotent $e \in R$ such that $a - e \in J(R)$. Thus, R is a J -clean ring, as promised. \square

Let $Nil_*(R)$ denote the *prime* radical (or, in other terms, the *lower* nil-radical) of a ring R , i.e., the intersection of all prime ideals of R . We know that $Nil_*(R)$ is a nil-ideal of R . It is also long known that a ring R is called *2-primal* if its lower nil-radical $Nil_*(R)$ consists precisely of all the nilpotent elements of R . For instance, it is well known that reduced rings and commutative rings are both 2-primal.

For an endomorphism α of a ring R , R is called α -compatible if, for any $a, b \in R$,

$$ab = 0 \iff a\alpha(b) = 0,$$

and in this case α is clearly injective.

Let R be a ring and $\alpha : R \rightarrow R$ a ring endomorphism; then, $R[x; \alpha]$ denotes the *skew polynomial ring* over R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[x] = R[x; 1_R]$ is the ordinary *polynomial ring* over R .

Our next result of interest is the following.

Proposition 2.11. *Let R be a 2-primal ring and let α be an endomorphism of R . If R is α -compatible, then*

$$\Delta(R[x; \alpha]) = \Delta(R) + J(R[x; \alpha]).$$

Proof. Suppose first that R is a reduced ring. As R is α -compatible, [4, Corollary 2.12] applies to get that $U(R[x; \alpha]) = U(R)$. Also, it is easy to see that $\Delta(R) \subseteq \Delta(R[x; \alpha])$. We claim that this is exactly an equality. In fact, let $r + r_0 \in \Delta(R[x; \alpha])$, where $r \in R[x; \alpha]x$ and $r_0 \in R$. Then, for any $u \in U(R)$, $r + r_0 + u \in U(R)$. This shows that $r = 0$ and $r_0 + u \in U(R)$. Thus, we conclude that $\Delta(R[x; \alpha]) \subseteq \Delta(R)$ and hence $\Delta(R[x; \alpha]) = \Delta(R)$, as claimed.

Now assume that R is 2-primal. Obviously,

$$Nil_*(R[x; \alpha]) = Nil_*(R)[x; \alpha] \subseteq J(R[x; \alpha])$$

consulting with [4, Lemma 2.2]. As R is 2-primal, $R/Nil_*(R)$ is reduced, and so we arrive at

$$J(R[x; \alpha]) = Nil_*(R[x; \alpha]) = Nil_*(R)[x; \alpha].$$

By the first part of the proof applied to $R/Nil_*(R)$ and referring to [21, Proposition 6(3)], we deduce that

$$\Delta(R) + Nil_*(R)[x; \alpha] = \Delta\left(\frac{R}{Nil_*(R)}[x; \alpha]\right) = \Delta\left(\frac{R[x; \alpha]}{J(R[x; \alpha])}\right) = \frac{\Delta(R[x; \alpha])}{J(R[x; \alpha])}.$$

Now, summarizing all the above, we conclude the pursued equality. \square

It is principally known that, for any two elements $a, b \in R$, $1 - ab$ is a unit if, and only if, $1 - ba$ is a unit. This result is attributed to as *Jacobson's lemma* for units. There are several analogous results in the literature as well.

We now have the validity of the following.

Corollary 2.12. *Let R be a ΔU ring and let $a, b \in R$. Then, $1 - ab \in \Delta(R)$ if, and only if, $1 - ba \in \Delta(R)$.*

Proof. Assuming that $1-ab \in \Delta(R)$, we can write $ab \in U(R) = 1+\Delta(R)$. Therefore, [11, Proposition 2.4] is applicable to get that $a \in U(R)$. Thus,

$$1 - ba = a^{-1}(1 - ab)a \in \Delta(R),$$

because $\Delta(R)$ is closed with respect to multiplication by all units (see [21, Theorem 3]). The converse implication is similar, concluding the proof. \square

3. 2- Δ U RINGS

In this section, we introduce the concept of 2- Δ U rings and investigate their elementary properties.

We now give our main definition.

Definition 3.1. A ring R is called 2- Δ U if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (equivalently, for each $u \in U(R)$, $u^2 = 1 + r$, where $r \in \Delta(R)$).

two more constructions clarify the given definition a bit more.

Example 3.2. Unambiguously, 2- UJ rings are 2- Δ U. But, the converse is definitely *not* true in general. For example, consider the ring $R = \mathbb{F}_2\langle x, y \rangle / \langle x^2 \rangle$. Thus, one calculates that $J(R) = (0)$, $\Delta(R) = \mathbb{F}_2x + xRx$ and $U(R) = 1 + \mathbb{F}_2x + xRx$. Therefore, R is Δ U by [11, Example 2.2], and hence it is 2- Δ U. But, it is readily seen that R is not 2- UJ , as asserted.

Example 3.3. The ring \mathbb{Z}_3 is 2- Δ U, but is *not* Δ U.

We are now in a position to explore some critical properties of the newly defined notion.

Proposition 3.4. A direct product $\prod_{i \in I} R_i$ of rings is 2- Δ U if, and only if, each direct component R_i is 2- Δ U.

Proof. As $\Delta(\prod_{i \in I} R_i) = \prod_{i \in I} \Delta(R_i)$ and $U(\prod_{i \in I} R_i) = \prod_{i \in I} U(R_i)$, the result follows without any difficult. \square

Proposition 3.5. Let R be a 2- Δ U ring. If T is a factor ring of R such that all units of T lift to units of R , then T is 2- Δ U.

Proof. Suppose that $f : R \rightarrow T$ is a ring epimorphism. Choosing $v \in U(T)$, there exists $u \in U(R)$ such that $v = f(u)$ and $u^2 = 1 + r \in 1 + \Delta(R)$. Thus,

$$v^2 = (f(u))^2 = f(u^2) = f(1 + r) = f(1) + f(r) = 1 + f(r) \in 1 + \Delta(T),$$

as required. \square

Example 3.6. A division ring R is 2- Δ U if, and only if, either $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.

Proof. Since R is a division ring, one has that $\Delta(R) = 0$, and the result follows from [6, Example 2.1]. \square

Remark 3.7. The condition "all units of T lift to units of R " in Proposition 3.5 is necessary and cannot be ignored. Indeed, the ring \mathbb{Z}_7 is a factor ring of the 2- Δ U ring \mathbb{Z} . But, \mathbb{Z}_7 is not 2- Δ U by Example 3.6. Note that not all of units of \mathbb{Z}_7 can lift to units of \mathbb{Z} .

Proposition 3.8. *Let R be a 2- ΔU ring. For a unital subring S of R , if $S \cap \Delta(R) \subseteq \Delta(S)$, then S is a 2- ΔU ring. In particular, the center of R is a 2- ΔU ring.*

Proof. Let $v \in U(S) \subseteq U(R)$. Since R is 2- ΔU , we have $v^2 - 1 \in \Delta(R) \cap S \subseteq \Delta(S)$. So, S is a 2- ΔU ring. Now, the rest follows from [21, Corollary 8]. \square

Proposition 3.9. *Let R be a 2- ΔU ring and $2 \in \Delta(R)$. Then, the following two relations are fulfilled:*

- (i) $(U(R))^2 + (U(R))^2 \subseteq \Delta(R)$.
- (ii) $[(U(R))^2 + (U(R))^2] \cap Id(R) = \{0\}$.

Proof. (i) Let $t \in (U(R))^2 + (U(R))^2$, so $t = u^2 + v^2$, where $u, v \in U(R)$. Since R is 2- ΔU , one may write that $t = 1 + r + 1 + s$, where $r, s \in \Delta(R)$. So, we have $t = 2 + (r + s)$. But, $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R , so that $t \in \Delta(R)$ follows.

- (ii) It follows immediately from (i) and [21, Proposition 15]. \square

Proposition 3.10. *Let $I \subseteq J(R)$ be an ideal of a ring R . Then, R is 2- ΔU if, and only if, so is R/I .*

Proof. Let R be a 2- ΔU ring and $u + I \in U(R/I)$. Thus, $u \in U(R)$ and hence $u^2 = 1 + r$, where $r \in \Delta(R)$. Therefore,

$$(u + I)^2 = u^2 + I = (1 + I) + (r + I),$$

where $r + I \in \Delta(R)/I = \Delta(R/I)$ in view of [21, Proposition 6].

Conversely, let R/I be a 2- ΔU ring and $u \in U(R)$. Thus, $u + I \in U(R/I)$ whence $(u + I)^2 = (1 + I) + (r + I)$, where $r + I \in \Delta(R/I)$. This means that $u^2 + I = (1 + r) + I$. So,

$$u^2 - (1 + r) \in I \subseteq J(R) \subseteq \Delta(R).$$

Consequently, $u^2 = 1 + r'$, where $r' \in \Delta(R)$. Hence, R is a 2- ΔU ring. \square

Corollary 3.11. *A ring R is 2- ΔU if, and only if, $R/J(R)$ is 2- ΔU .*

Proposition 3.12. *Let R be a 2- ΔU ring and e an idempotent of R . Then, eRe is too a 2- ΔU ring.*

Proof. Letting $u \in U(eRe)$, we have $u + (1 - e) \in U(R)$. Under validity of the hypothesis,

$$(u + (1 - e))^2 = u^2 + (1 - e) = 1 + r \in 1 + \Delta(R).$$

Thus, $u^2 - e \in \Delta(R)$.

Now, we need to show that $u^2 - e \in \Delta(eRe)$. To that end, let v be an arbitrary unit of eRe . One inspects that $v + 1 - e \in U(R)$. Note also that $u^2 - e \in \Delta(R)$ gives that $u^2 - e + v + 1 - e \in U(R)$ under presence of the definition of $\Delta(R)$. Taking $u^2 - e + v + 1 - e = t \in U(R)$, one can check that $et = te = ete = u^2 - e + v$, and so $ete \in U(eRe)$. This shows that $u^2 - e + U(eRe) \subseteq U(eRe)$, so that $u^2 - e \in \Delta(eRe)$ and $u^2 \in e + \Delta(eRe)$ implying eRe is a 2- ΔU ring, as asked for. \square

Proposition 3.13. *For any ring $R \neq 0$ and any integer $n \geq 2$, the ring $M_n(R)$ is not a 2- ΔU ring.*

Proof. Since it is well known that $M_2(R)$ is isomorphic to a corner ring of $M_n(R)$ whenever $n \geq 2$, it suffices to show that $M_2(R)$ is not a 2-ΔU ring in conjunction with Proposition 3.12. To this purpose, consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in U(M_2(R)).$$

Then, one verifies that

$$A^2 - I_2 = A \notin J(M_2(R)) = \Delta(M_2(R)),$$

as required. \square

A set $\{e_{ij} : 1 \leq i, j \leq n\}$ of non-zero elements of R is said to be a system of n^2 matrix units if $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{jj} = 1$ and $\delta_{js} = 0$ for $j \neq s$. In this case, $e := \sum_{i=1}^n e_{ii}$ is an idempotent of R and $eRe \cong M_n(S)$, where

$$S = \{r \in eRe : re_{ij} = e_{ij}r, \text{ for all } i, j = 1, 2, \dots, n\}.$$

Recall that a ring R is said to be *Dedekind finite* if $ab = 1$ ensures $ba = 1$ for any $a, b \in R$. In other words, all one-sided inverses in such a ring are necessarily two-sided.

Proposition 3.14. *Every 2-ΔU ring is Dedekind finite.*

Proof. If we assume to the contrary that R is *not* a Dedekind finite ring, then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Assuming $e_{ij} = a^i(1 - ba)b^j$ and $e = \sum_{i=1}^n e_{ii}$, there exists a non-zero ring S such that $eRe \cong M_n(S)$. However, according to Proposition 3.12, eRe is a 2-ΔU ring, whence $M_n(S)$ must also be a 2-ΔU ring, thus contradicting Proposition 3.13. \square

Example 3.15. A local ring R is 2-ΔU if, and only if, either $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$.

Proof. Assume one of the possibilities $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$. We, however, know that $R/J(R)$ is a division ring, so $R/J(R)$ is 2-ΔU viewing Example 3.6. Thus, R is 2-ΔU in accordance with Corollary 3.11.

Conversely, letting R be 2-ΔU, we directly check that $R/J(R) \cong \mathbb{Z}_2$ or $R/J(R) \cong \mathbb{Z}_3$ with the aid of Example 3.6. \square

As an obvious consequence, we derive:

Corollary 3.16. (i) *A semi-simple ring R is 2-ΔU if, and only if, $R \cong \bigoplus_{i=1}^n R_i$, where $R_i \cong \mathbb{Z}_2$ or $R_i \cong \mathbb{Z}_3$ for every index i .*

(ii) *A semi-local ring R is 2-ΔU if, and only if, $R/J(R) \cong \bigoplus_{i=1}^m R_i$, where $R_i \cong \mathbb{Z}_2$ or $R_i \cong \mathbb{Z}_3$ for every index i .*

Example 3.17. The ring \mathbb{Z}_m is 2-ΔU if, and only if, $m = 2^k 3^l$ for some positive integers k and l .

Lemma 3.18. *Let R be a 2-ΔU ring. If $J(R) = (0)$ and every non-zero right ideal of R contains a non-zero idempotent, then R is reduced.*

Proof. Suppose that the contradiction R is not reduced holds. Then, there exists a non-zero element $a \in R$ such that $a^2 = 0$. With [22, Theorem 2.1] at hand, there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T . Now,

Proposition 3.12 tells us that eRe is a 2- Δ U ring, and hence $M_2(T)$ is a 2- Δ U ring. This, however, contradicts Proposition 3.13. \square

A ring R is called π -regular if, for each $a \in R$, $a^n \in a^n R a^n$ for some integer $n \geq 1$. Regular rings are always π -regular. Also, a ring R is said to be *strongly π -regular*, provided that, for any $a \in R$, there exists $n \geq 1$ such that $a^n \in a^{n+1} R$.

We are now ready to attack the following pivotal result.

Theorem 3.19. *Let R be a ring. Then, the following three assertions are equivalent:*

- (i) R is a regular 2- Δ U ring.
- (ii) R is a π -regular reduced 2- Δ U ring.
- (iii) R has the identity $x^3 = x$ (i.e., R is a tripotent ring).

Proof. (i) \Rightarrow (ii). Since R is regular, $J(R) = (0)$, and every non-zero right ideal contains a non-zero idempotent. In virtue of Lemma 3.18, R is reduced. Also, every regular ring is π -regular.

(ii) \Rightarrow (iii). Notice that reduced rings are abelian, so R is abelian regular by virtue of [1, Theorem 3], and hence it is strongly regular. Thus, R is unit-regular, so that $\Delta(R) = (0)$ in accordance with [21, Corollary 16]. Therefore, we have $Nil(R) = J(R) = \Delta(R) = (0)$.

On the other hand, one knows that R is strongly π -regular. Choose $x \in R$. The application of [10, Proposition 2.5] insures that there are an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$ and $ex = xe \in Nil(R) = (0)$. So, we deduce

$$x = x - xe = x(1 - e) = u(1 - e) = (1 - e)u.$$

Since R is a 2- Δ U ring, $u^2 = 1$. It now follows that $x^2 = (1 - e)$. Hence, $x = x(1 - e) = x.x^2 = x^3$.

(iii) \Rightarrow (i). It is not so hard to verify that R is regular. Choosing $u \in U(R)$, we infer $u^3 = u$, that is, $u^2 = 1$, and thus R is a 2- Δ U ring, as asserted. \square

A ring R is termed *strongly 2-nil-clean* if every element in R is a sum of two idempotents and a nilpotent that commute (for more account, we refer to [5]).

Our next chief result is as follows.

Theorem 3.20. *The following four statements are equivalent for a ring R :*

- (i) R is a regular 2- Δ U ring.
- (ii) R is a strongly regular 2- Δ U ring.
- (iii) R is a unit-regular 2- Δ U ring.
- (iv) R has the identity $x^3 = x$ (i.e., R is a tripotent ring).

Proof. (i) \Rightarrow (ii). Observe that Lemma 3.18 gives R is reduced and hence abelian. Then, R is strongly regular.

(ii) \Rightarrow (iii). This is pretty obvious, so we drop off the arguments.

(iii) \Rightarrow (iv). Choose $x \in R$ and write $x = ue$ for some $u \in U(R)$ and $e \in Id(R)$. We know that every unit-regular ring is regular, so that R is regular 2- Δ U whence R is abelian.

On the other side, [21, Corollary 16] informs us that $\Delta(R) = 0$. Therefore, for any $u \in U(R)$, we have $u^2 = 1$. Then, $x^2 = u^2 e^2 = e$. So, R is a 2-Boolean

ring. Thus, [5, Corollary 3.4] enables us that R is strongly 2-nil-clean, and hence [5, Theorem 3.3] guarantees that R is tripotent, as formulated.

(iv) \Rightarrow (i). It is quite elementary looking at Theorem 3.19. \square

Proposition 3.21. *A ring R is Δ U if, and only if,*

- (i) $2 \in \Delta(R)$,
- (ii) R is a 2- Δ U ring,
- (iii) If $x^2 \in \Delta(R)$, then $x \in \Delta(R)$ for every $x \in R$.

Proof. \Rightarrow . As R is a Δ U ring, one has that $-1 = 1 + r$ for some $r \in \Delta(R)$. This insures $-2 \in \Delta(R)$ and so $2 \in \Delta(R)$. But, every Δ U ring is 2- Δ U. The conclusion now follows from [11, Proposition 2.4].

\Leftarrow . Let $u \in U(R)$. Then,

$$(u - 1)^2 + 2(u - 1) = (u - 1)(u + 1) = u^2 - 1 \in \Delta(R),$$

because R is a 2- Δ U ring. It follows from the facts $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R that $(u - 1)^2 \in \Delta(R)$. So, by (iii) it must be that $u - 1 \in \Delta(R)$ whence R is a Δ U ring, as stated. \square

Following Kosan et al. ([18]), a ring R is called *semi-tripotent* if, for each $a \in R$, $a = e + j$, where $e^3 = e$ and $j \in J(R)$ (or, equivalently, $R/J(R)$ satisfies the identity $x^3 = x$ and all idempotents lift modulo $J(R)$).

We now have all the ingredients necessary to prove our next basic result.

Theorem 3.22. *Let R be a ring. Then, the following three conditions are equivalent:*

- (i) R is a semi-regular 2- Δ U ring.
- (ii) R is an exchange 2- Δ U ring.
- (iii) R is a semi-tripotent ring.

Proof. (i) \Rightarrow (ii). Observe that [23, Proposition 1.6] assures that semi-regular rings are exchange.

(ii) \Rightarrow (iii). Monitoring [23, Corollary 2.4], $R/J(R)$ is exchange and idempotents lift modulo $J(R)$. Moreover, Proposition 3.10 ensures that $R/J(R)$ is 2- Δ U. So, with no loss of generality, it can be assumed that $J(R) = (0)$. Since R is an exchange ring, every non-zero one sided ideal contains a non-zero idempotent. However, Lemma 3.18 is a guarantor that R is reduced and so abelian. Thus, R is abelian clean. Hence, [17, Proposition 14] employs to write that $R/J(R) \cong M_n(D)$, where $1 \leq n \leq 2$ and D is a division ring. Therefore, [21, Theorem 11] helps us to get that $\Delta(R) = J(R)$ whence $\Delta(R) = (0)$. But, as R is 2- Δ U, we have $v^2 = 1$ for every $v \in U(R)$. Consequently, the conclusion follows now from [6, Theorem 3.3].

(iii) \Rightarrow (i). Adapting [6, Theorem 3.3], R is semi-regular 2- UJ meaning that R is semi-regular 2- Δ U. \square

As four valuable corollaries, we yield:

Corollary 3.23. *Let R be a 2- Δ U ring. Then, the following are equivalent:*

- (i) R is a semi-regular ring.
- (ii) R is an exchange ring.
- (iii) R is a clean ring.

Proof. By Theorem 3.22, (i) \Leftrightarrow (ii).

(iii) \Rightarrow (ii). This is easy, so we leave the arguments to the interested reader.

(ii) \Rightarrow (iii). If R is exchange 2- ΔU , then R is reduced via Lemma 3.18, and hence it is abelian. Therefore, R is abelian exchange, so it is clean, ending the implication. \square

Corollary 3.24. *Let R be a ring. The following are equivalent:*

- (i) R is a semi-regular 2- ΔU ring and $J(R)$ is nil.
- (ii) R is an exchange 2- ΔU ring and $J(R)$ is nil.
- (iii) R is a strongly 2-nil-clean ring.

Proof. (i) \Rightarrow (ii). This can easily be deduced, so the detailed argumentation is leaved.

(ii) \Rightarrow (iii). Since R is exchange 2- ΔU , $\Delta(R) = J(R)$. Then, for any $u \in U(R)$, we have

$$u^2 - 1 \in \Delta(R) = J(R) \subseteq Nil(R).$$

So, R is 2- UU ring. Therefore, R is exchange 2- UU ring, whence it is strongly 2-nil-clean in regard to [25, Theorem 4.1].

(iii) \Rightarrow (i). This follows from [6, Corollary 3.5] and knowing that every 2- UJ ring is 2- ΔU . \square

Corollary 3.25. *Let R be a ring. Then, the following are equivalent:*

- (i) R is a regular ΔU ring.
- (ii) R is a π -regular reduced ΔU ring.
- (iii) R is a Boolean ring.

Proof. This is an automatic consequence of Theorem 3.19 and [11, Theorem 4.4]. \square

Corollary 3.26. *Let R be a ring. The following are equivalent:*

- (i) R is a semi-regular ΔU ring.
- (ii) R is an exchange ΔU ring.
- (iii) R is a clean ΔU ring.

Proof. The result follows from [11, Theorem 4.2 and Corollary 4.7]. \square

4. SOME EXTENSIONS OF 2- ΔU RINGS

We say that C is an unital subring of a ring D if $\emptyset \neq C \subseteq D$ and, for any $x, y \in C$, the relations $x - y, xy \in C$ and $1_D \in C$ hold. Let D be a ring and C an unital subring of D , and designate by $R[D, C]$ the set

$$\{(d_1, \dots, d_n, c, c, \dots) : d_i \in D, c \in C, 1 \leq i \leq n\}.$$

Then, $R[D, C]$ forms a ring under the usual component-wise addition and multiplication. The ring $R[D, C]$ is called the *tail ring extension*.

We are now attacking the following two preliminary claims giving us some useful necessary and sufficient conditions.

Proposition 4.1. *$R[D, C]$ is a 2- ΔU ring if, and only if, D and C are 2- ΔU rings.*

Proof. Let $R[D, C]$ be a 2-ΔU ring. Firstly, we prove that D is a 2-ΔU ring. Let $u \in U(D)$. Then $\bar{u} = (u, 1, 1, \dots) \in U(R[D, C])$. By existing hypothesis, we have $(u^2 - 1, 0, 0, \dots) \in \Delta(R[D, C])$, so

$$(u^2 - 1, 0, 0, \dots) + U(R[D, C]) \subseteq U(R[D, C]).$$

Thus, for all $v \in U(D)$,

$$(u^2 - 1 + v, 1, 1, \dots) = (u^2 - 1, 0, 0, \dots) + (v, 1, 1, \dots) \in U(R[D, C]).$$

Hence, $u^2 - 1 + v \in U(D)$ forcing that $u^2 - 1 \in \Delta(D)$.

Now, we show that C is a 2-ΔU ring. To this aim, let $v \in U(C)$. Then, $(1, \dots, 1, 1, v, v, \dots) \in U(R[D, C])$. By assumption,

$$(0, \dots, 0, v^2 - 1, v^2 - 1, \dots) \in \Delta(R[D, C]),$$

and so

$$(0, \dots, 0, v^2 - 1, v^2 - 1, \dots) + U(R[D, C]) \subseteq U(R[D, C]).$$

Thus, for all $u \in U(C)$,

$$(1, 1, \dots, v^2 - 1 + u, v^2 - 1 + u, \dots) \in U(R[D, C]).$$

But then, we have $v^2 - 1 + u \in U(C)$ and hence $v^2 - 1 \in \Delta(C)$, as needed.

For the converse, assume that D and C are 2-ΔU rings. Let $\bar{u} = (u_1, u_2, \dots, u_n, v, v, \dots) \in U(R[D, C])$, where $u_i \in U(D)$ and $v \in U(C) \subseteq U(D)$. We must show that $\bar{u}^2 - 1 + U(R[D, C]) \subseteq U(R[D, C])$. In fact, for all $\bar{a} = (a_1, \dots, a_m, b, b, \dots) \in U(R[D, C])$ with $a_i \in U(D)$ and $b \in U(C) \subseteq U(D)$, take $z = \max\{m, n\}$, and thus we obtain

$$\bar{u}^2 - 1 + \bar{a} = (u_1^2 - 1 + a_1, \dots, u_z^2 - 1 + a_z, v^2 - 1 + b, v^2 - 1 + b, \dots).$$

Note that $u_i^2 - 1 + a_i \in U(D)$ for all $1 \leq i \leq z$ and $v^2 - 1 + b \in U(C) \subseteq U(D)$. We, thereby, deduce that $\bar{u}^2 - 1 + \bar{a} \in U(R[D, C])$. Thus, $\bar{u}^2 - 1 \in \Delta(R[D, C])$ and $\bar{u}^2 \in 1 + \Delta(R[D, C])$. This shows that $R[D, C]$ is a 2-ΔU ring, as required. \square

Let R be a ring and $\alpha : R \rightarrow R$ a ring endomorphism. As usual, $R[[x; \alpha]]$ denotes the ring of *skew formal power series* over R ; that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x; 1_R]]$ is the ring of *formal power series* over R .

Proposition 4.2. *A ring R is 2-ΔU if, and only if, so is $R[[x; \alpha]]$.*

Proof. Consider $I = R[[x; \alpha]]x$. Then, I is an ideal of $R[[x; \alpha]]$. A simple check gives that $J(R[[x; \alpha]]) = J(R) + I$, so $I \subseteq J(R[[x; \alpha]])$. Since $R[[x; \alpha]]/I \cong R$, the result follows with the help of Proposition 3.10. \square

Corollary 4.3. *A ring R is 2-ΔU if, and only if, so is $R[[x]]$.*

Our further main achievement is the following one.

Theorem 4.4. *Let R be a 2-primal ring and let α be an endomorphism of R such that R is α -compatible. The following two statements are equivalent:*

- (i) $R[x; \alpha]$ is a 2-ΔU ring.
- (ii) R is a 2-ΔU ring.

Proof. (ii) \Rightarrow (i). Let

$$u(x) = a_0 + a_1x + \cdots + a_nx^n = \sum_{i=0}^n a_i x^i$$

be in $U(R[x; \alpha])$. So, the usage of [4, Corollary 2.14] allows us to infer that $a_0 \in U(R)$ and $a_i \in \text{Nil}(R)$ for each $i \geq 1$. Then, by assumption, $a_0^2 = 1 + r$, where $r \in \Delta(R)$.

In the other vein, we know that

$$J(R[x; \alpha]) = \text{Nil}_*(R[x; \alpha]) = \text{Nil}_*(R)[x; \alpha] = \text{Nil}(R)[x; \alpha].$$

Now, we conclude that

$$\begin{aligned} (u(x))^2 &= a_0^2 + a_0a_1x + \cdots + a_0a_nx^n + a_1xa_0 + \cdots = (1 + r) + a_0a_1x + \cdots \\ &= 1 + (r + a_0a_1x + \cdots) \in 1 + \Delta(R) + J(R[x; \alpha]). \end{aligned}$$

On the other side, it must be that $\Delta(R) + J(R[x; \alpha]) = \Delta(R[x; \alpha])$ by Proposition 2.11. Thus, this means that $R[x; \alpha]$ is a $2\text{-}\Delta U$ ring, as required.

(i) \Rightarrow (ii). Let $u \in U(R) \subseteq U(R[x; \alpha])$. Then,

$$u^2 \in 1 + \Delta(R[x; \alpha]) = 1 + \Delta(R) + J(R[x; \alpha]).$$

Thus, one detects that $u^2 \in 1 + \Delta(R)$, and hence R is a $2\text{-}\Delta U$ ring, as needed. \square

The following consequence is now immediate.

Corollary 4.5. *Let R be a 2-primal ring. Then, the following are equivalent:*

- (i) $R[x]$ is a $2\text{-}\Delta U$ ring.
- (ii) R is a $2\text{-}\Delta U$ ring.

Let R be a ring and M a bi-module over R . The trivial extension of R and M is defined as

$$T(R, M) = \{(r, m) : r \in R \text{ and } m \in M\},$$

with addition defined componentwise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

Note that the trivial extension $T(R, M)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M \right\}$$

of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$, and also $T(R, R) \cong R[x]/\langle x^2 \rangle$. We, likewise, notice that the set of units of the trivial extension $T(R, M)$ is

$$U(T(R, M)) = T(U(R), M).$$

Besides, thanks to [11], we can write

$$\Delta(T(R, M)) = T(\Delta(R), M).$$

We proceed by proving the following.

Proposition 4.6. *Suppose R is a ring and M is a bi-module over R . Then, the following hold:*

- (i) *The trivial extension $T(R, M)$ is a $2\text{-}\Delta U$ ring if, and only if, R is a $2\text{-}\Delta U$ ring.*

- (ii) For $n \geq 2$, the quotient-ring $\frac{R[x; \alpha]}{\langle x^n \rangle}$ is a 2- ΔU ring if, and only if, R is a 2- ΔU ring.
- (iii) For $n \geq 2$, the quotient-ring $\frac{R[[x; \alpha]]}{\langle x^n \rangle}$ is a 2- ΔU ring if, and only if, R is a 2- ΔU ring.
- (iv) The upper triangular matrix ring $T_n(R)$ is a 2- ΔU if, and only if, R is a 2- ΔU ring.

Proof. (i) Set $A := T(R, M)$ and consider $I := T(0, M)$. It is not so hard to see that $I \subseteq J(A)$ such that $\frac{A}{I} \cong R$. So, the result follows directly from Proposition 3.10.

- (ii) Put $A := \frac{R[x; \alpha]}{\langle x^n \rangle}$. Considering the ideal $I := \frac{\langle x \rangle}{\langle x^n \rangle}$ of A , we routinely obtain that $I \subseteq J(A)$ with $\frac{A}{I} \cong R$. So, the wanted result follows automatically from Proposition 3.10.
- (iii) Knowing that the isomorphism $\frac{R[x; \alpha]}{\langle x^n \rangle} \cong \frac{R[[x; \alpha]]}{\langle x^n \rangle}$ holds, point (iii) follows immediately from (ii).
- (iv) Setting $I := \{(a_{ij}) \in T_n(R) \mid a_{ii} = 0\}$, we then have $I \subseteq J(T_n(R))$ and $T_n(R)/I \cong R^n$. Therefore, the desired result follows from Propositions 3.10 and 3.4.

□

Corollary 4.7. *Let R be a ring. Then, the following are equivalent:*

- (i) R is a 2- ΔU ring.
- (ii) For $n \geq 2$, the quotient-ring $\frac{R[x]}{\langle x^n \rangle}$ is a 2- ΔU ring.
- (iii) For $n \geq 2$, the quotient-ring $\frac{R[[x]]}{\langle x^n \rangle}$ is a 2- ΔU ring.

Example 4.8. The upper triangular ring $T_n(\mathbb{Z}_3)$ for all $n \geq 1$ is 2- ΔU (see Proposition 4.6(4) and Example 3.3). But, it is not a ΔU ring as Example 3.3 and [11, Corollary 2.9] show.

Suppose R is a ring and M is a bi-module over R . Putting

$$DT(R, M) := \{(a, m, b, n) \mid a, b \in R, m, n \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1 a_2, a_1 m_2 + m_1 a_2, a_1 b_2 + b_1 a_2, a_1 n_2 + m_1 b_2 + b_1 m_2 + n_1 a_2),$$

we then see that $DT(R, M)$ is a ring that is isomorphic to $T(T(R, M), T(R, M))$.

Moreover, we have

$$DT(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b \in R, m, n \in M \right\}.$$

We now establish the following isomorphism as rings: the map $\frac{R[x, y]}{\langle x^2, y^2 \rangle} \rightarrow DT(R, R)$ is defined by

$$a + bx + cy + dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, arrive at the following.

Corollary 4.9. *Let R be a ring and M a bi-module over R . Then, the following statements are equivalent:*

- (i) R is a 2- ΔU ring.
- (ii) $DT(R, M)$ is a 2- ΔU ring.
- (iii) $DT(R, R)$ is a 2- ΔU ring.
- (iv) $\frac{R[x, y]}{\langle x^2, y^2 \rangle}$ is a 2- ΔU ring.

Let A, B be two rings and let M, N be an (A, B) -bi-module and a (B, A) -bi-module, respectively. Also, we consider the two bi-linear maps $\phi : M \otimes_B N \rightarrow A$ and $\psi : N \otimes_A M \rightarrow B$ that apply to the following properties.

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, we define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring with obvious matrix operations that is called a *Morita context ring*. Denote two-sided ideals $Im\phi$ and $Im\psi$ to MN and NM , respectively, that are called the *trace ideals* of the Morita context ring.

The following assertion holds.

Proposition 4.10. *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring such that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Then, R is a 2- ΔU ring if, and only if, both A and B are 2- ΔU .*

Proof. One observes that [28, Lemma 3.1] can be applied to argue that $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$, and hence $\frac{R}{J(R)} \cong \frac{A}{J(A)} \times \frac{B}{J(B)}$. Thus, the result follows from Corollary 3.11 and Proposition 3.4. \square

Now, let R, S be two rings and let M be an (R, S) -bi-module such that the operation $(rm)s = r(ms)$ is valid for all $r \in R, m \in M$ and $s \in S$. Given such a bi-module M , we can set

$$T(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it obviously forms a ring with the usual matrix operations. The so-stated formal matrix $T(R, S, M)$ is called a *formal triangular matrix ring*.

It is worthy of noticing that, if we set $N = 0$ in Proposition 4.10, then we will obtain the following statement.

Corollary 4.11. *Let R, S be rings and let M be an (R, S) -bi-module. Then, the formal triangular matrix ring $T(R, S, M)$ is a 2- Δ U ring if, and only if, R and S are both 2- Δ U.*

Given a ring R and a central element s of R , the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition component-wise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + s x_1 y_2 & a_1 x_2 + x_1 b_2 \\ y_1 a_2 + b_1 y_2 & s y_1 x_2 + b_1 b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$. A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring* over R . It was observed by Krylov in [19] that a ring S is generalized matrix over R if, and only if, $S = K_s(R)$ for some $s \in C(R)$. Here $MN = NM = sR$, so that $MN \subseteq J(A) \iff s \in J(R)$ and $NM \subseteq J(B) \iff s \in J(R)$.

We can now extract the following.

Corollary 4.12. *Let R be a ring and $s \in C(R) \cap J(R)$. Then, $K_s(R)$ is a 2- Δ U ring if, and only if, R is 2- Δ U.*

Following Tang and Zhou (cf. [27]), for $n \geq 2$ and for $s \in C(R)$, the $n \times n$ formal matrix ring over R defined by s , and designated by $M_n(R; s)$, is the set of all $n \times n$ matrices over R with usual addition of matrices and with multiplication defined below:

For (a_{ij}) and (b_{ij}) in $M_n(R; s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where } (c_{ij}) = \sum s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, where $\delta_{jk}, \delta_{ij}, \delta_{ik}$ are the Kronecker delta symbols.

We now manage to prove the following.

Corollary 4.13. *Let R be a ring and $s \in C(R) \cap J(R)$. Then, $M_n(R; s)$ is a 2- Δ U ring if, and only if, R is 2- Δ U.*

Proof. If $n = 1$, then $M_n(R; s) = R$. So, in this situation, there is nothing to establish. That is why, suppose $n = 2$. Using the definition of $M_n(R; s)$, we have $M_2(R; s) \cong K_{s^2}(R)$. Evidently, $s^2 \in J(R) \cap C(R)$, so the assertion is true for $n = 2$ taking into account Corollary 4.12.

To proceed by induction, assume now that $n > 2$ and that the claim holds for $M_{n-1}(R; s)$. Set $A := M_{n-1}(R; s)$. Then, one inspects that $M_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a Morita context, where

$$M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \quad \text{and} \quad N = (M_{n1} \dots M_{n,n-1})$$

with $M_{in} = M_{ni} = R$ for all $i = 1, \dots, n-1$, and

$$\begin{aligned}\psi : N \otimes M &\rightarrow N, & n \otimes m &\mapsto snm \\ \phi : M \otimes N &\rightarrow M, & m \otimes n &\mapsto smn.\end{aligned}$$

Moreover, for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = (y_{n1} \dots y_{n,n-1}) \in N$, we may write

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \dots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \dots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \dots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in sA$$

as well as

$$yx = s^2 y_{n1} x_{1n} + s^2 y_{n2} x_{2n} + \dots + s^2 y_{n,n-1} x_{n-1,n} \in s^2 R.$$

Since $s \in J(R)$, we observe that $MN \subseteq J(A)$ and $NM \subseteq J(A)$. So, we receive that

$$\frac{M_n(R; s)}{J(M_n(R; s))} \cong \frac{A}{J(A)} \times \frac{R}{J(R)}.$$

Finally, the induction hypothesis along with Proposition 4.10 yield the desired conclusion after all. \square

A Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called *trivial*, if the context products are trivial, i.e., $MN = 0$ and $NM = 0$. We now see that

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial Morita context bearing in mind [13].

We, thus, obtain the following.

Corollary 4.14. *The trivial Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a 2- ΔU ring if, and only if, A and B are both 2- ΔU .*

Proof. It is plainly seen that the isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}$$

hold. Then, the rest of the proof follows by a combination of Propositions 4.6(i) and 3.4. \square

We shall now deal with group rings of 2- ΔU rings as follows.

As usual, for an arbitrary ring R and an arbitrary group G , the symbol RG stands for the *group ring* of G over R . Standardly, $\varepsilon(RG)$ denotes the kernel of the classical *augmentation map* $\varepsilon : RG \rightarrow R$, defined by $\varepsilon\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} a_g$, and this ideal is called the *augmentation ideal* of RG .

Besides, a group G is called a p -group if every element of G has order which is a power of the prime number p . Moreover, a group G is said to be *locally finite* if every finitely generated subgroup is finite.

We begin our work with two preliminary technicalities.

Lemma 4.15. [28, Lemma 2]. *Let p be a prime with $p \in J(R)$. If G is a locally finite p -group, then $\varepsilon(RG) \subseteq J(RG)$.*

Proposition 4.16. (i) *If RG is a 2- Δ U ring, then R is also a 2- Δ U ring.*

(ii) *If R is a 2- Δ U ring and G is a locally finite p -group, where p is a prime number such that $p \in J(R)$, then RG is a 2- Δ U ring.*

Proof. (i) Assume $u \in U(R)$, then $u \in U(RG)$. Thus, $u^2 = 1 + r$, where $r \in \Delta(RG)$. Since $r = 1 - u^2 \in R$, it suffices to verify that $r \in \Delta(R)$, which is pretty obvious as, for any $v \in U(R) \subseteq U(RG)$, we see that $v - r \in U(RG) \cap R \subseteq U(R)$. Therefore, $r \in \Delta(R)$.

(ii) Obviously, Lemma 4.15 gives us that $\varepsilon(RG) \subseteq J(RG)$. However, since $RG/\varepsilon(RG) \cong R$, Theorem 3.10 applies to conclude that RG is a 2- Δ U ring. \square

The following reversed implication is somewhat slightly curious.

Proposition 4.17. *If RG is a 2- Δ U ring with $2 \in \Delta(RG)$, then G is a 2-group.*

Proof. We first consider two claims:

Claim 1: Every element $g \in G$ has a finite order.

Assume the contrary, namely that there exists $g \in G$ with infinite order. Since RG is a 2- Δ U ring, we have $1 - g^2 \in \Delta(RG)$. Given $2 \in \Delta(RG)$, we then can write $1 + g^2 \in \Delta(RG)$, ensuring $1 + g + g^2 \in U(RG)$. Therefore, there exist integers $n < m$ and elements a_i with $a_n \neq 0 \neq a_m$ such that

$$(1 + g + g^2) \sum_n^m a_i g^i = 1.$$

This, however, leads to a contradiction, and thus every element $g \in G$ must have finite order, as expected.

Claim 2: For any $g \in G$ and $k \in \mathbb{N}$, we have $\sum_{i=0}^{2k} g^i \in U(RG)$.

We will show this only for $k = 1, 2$, because the general claim follows in a way of similarity.

For $k = 1$ and any $g \in G$, we have $1 - g^2 \in \Delta(RG)$. Since $2 \in \Delta(RG)$, we then can write $1 + g^2 \in \Delta(RG)$ and hence $1 + g + g^2 \in U(RG)$.

For $k = 2$ and any $g \in G$, observing that $g, g^2 \in U(RG)$, we get $1 - g^2 \in \Delta(RG)$ and hence $1 + g^2 \in \Delta(RG)$. Thus, $g + g^3 \in \Delta(RG)$. But, $1 - g^4 \in \Delta(RG)$ and, therefore, $1 + g^4 \in \Delta(RG)$.

Furthermore, since $\Delta(RG)$ is closed under addition, it follows that

$$2 + g + g^2 + g^3 + g^4 \in \Delta(RG),$$

assuring that

$$g + g^2 + g^3 + g^4 \in \Delta(RG)$$

and so

$$1 + g + g^2 + g^3 + g^4 \in U(RG).$$

Continuing this process, we can show that $\sum_{i=0}^{2k} g^i \in U(RG)$, as claimed.

If now $g \in G$ has an order p that does not divide 2, then p has to be odd, whence $p - 1 = 2k$. Consequently, by what we have shown above $\sum_{i=0}^{2k} g^i \in U(RG)$, and since $(1 - g)(\sum_{i=0}^{2k} g^i) = 0$, we deduce $1 - g = 0$, which is a contradiction. Thus, G must be a 2-group, as stated. \square

5. OPEN QUESTIONS

We finish our work with the following four questions which allude us.

A ring R is called UQ if $U(R) = 1 + QN(R)$ (see [8]).

Problem 5.1. Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + q$ where $q \in QN(R)$ (i.e., 2- UQ rings).

Problem 5.2. Characterize regular (or semi-regular) 2- UQ rings.

A ring R is called UNJ if $U(R) = 1 + Nil(R) + J(R)$ (see [16]).

Problem 5.3. Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + n + j$, where $n \in Nil(R)$ and $j \in J(R)$ (i.e., 2- UNJ rings).

Problem 5.4. Characterize regular (or semi-regular) 2- UNJ rings.

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