

**REPORT ON: RINGS WITH 2- ΔU PROPERTY
BY**

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For a ring R with associative and with nonzero identity element, the Jacobson radical, the set of idempotent elements and the set of unit elements of R are denoted by $J(R)$, $E(R)$ and $U(R)$, respectively.

Regard the following popular inclusions:

$$1 + J(R) \subseteq U(R)$$

$$1 + N(R) \subseteq U(R)$$

$$1 + N(R) + J(R) \subseteq U(R)$$

and

$$1 + \Delta(R) \subseteq U(R)$$

where

$$\begin{aligned} J(R) \subseteq \Delta(R) &= \{x \in R : x + u \in U(R) \text{ for all } u \in U(R)\} \\ &= \{x \in R : 1 - xu \text{ is invertible for all } u \in U(R)\} \\ &= \{x \in R : 1 - ux \text{ is invertible for all } u \in U(R)\} \end{aligned}$$

was handled by Lam [4, Exercise 4.24] and, recently, studied by Leroy-Matczuk [5].

As pointed out by the authors in [5, Theorems 3 and 6], $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by all units (quasi-invertible elements) of R . Moreover, $\Delta(R) = J(T)$, where T is the subring of R generated by units of R , and the equality $\Delta(R) = J(R)$ holds if and only if $\Delta(R)$ is an ideal of R . Here we note that $\Delta(R)$ is not necessarily an ideal, in general. We also remark that $1 + \Delta(R)$ is a normal subgroup of $U(R)$ containing in $1 + J(R)$.

Definition 1.: A ring R is said to be UU if $U(R) = 1 + N(R)$ ([1]).

Definition 2.: A ring R is said to be UJ if $U(R) = 1 + J(R)$ ([3], [6]).

Definition 3.: A ring R is said to be UNJ if $U(R) = 1 + N(R) + J(R)$ ([7]).

We also note that, if R is a UJ -ring, then $\Delta(R) = J(R)$. Because of these facts, it was natural introducing the notion of ΔU -ring.

Definition 4 .: A ring R is said to be ΔU if $U(R) = 1 + \Delta(R)$ ([9, 10])

In section 2 of the paper under review, the authors obtained some new characterizations of ΔU -rings. Remarkable observations are Propositions 2.2, 2.3 and 2.4 and Theorem 2.5.

Definition 5 .: Let $n \in \mathbb{N}$. Consider the following forms of the units of a ring R which belong to $J(R)$:

(i) $u - u^n \in J(R)$ for each $u \in U(R)$,

(ii) For each $u \in U(R)$, there exists $n \in \mathbb{N}$ such that $u - u^n \in J(R)$.

If a ring R satisfies the condition (i) (respectively, (ii)), then R is said to be an n - UJ ring (respectively, an ∞ - UJ ring) ([8]).

Notice that all UJ rings are n - UJ and every n - UJ ring is ∞ - UJ .

The notions of n - UJ and ∞ - UJ generalize 2- UJ rings [2].

In section 3 of the paper under review, the authors introduced the notion of 2- ΔU -rings that is a natural generalizations of 2- UJ rings and ΔU -rings. Remarkable charazrerizations are obtained in Propositions 3.4, 3.8, 3.12, 3.13, 3.14 and Theorems 3.10.

Definition 6 .: An element $a \in R$ is (strongly) Δ -clean provided that there exist an idempotent $e \in R$ and an element $d \in \Delta(R)$ such that $a = e + d$ (that commute, i.e. $ed = de$.) A ring R is strongly Δ -clean in case every element in R is strongly Δ -clean.

Notice that (strongly) Δ -clean rings are (strongly)clean, and R is a ΔU -ring iff all clean elements of R are Δ -clean [9].

In [9, Theorem 4.2], it is shown that R is a Δ -clean ring iff, for every $x \in R$, $x - x^2 \in \Delta(R)$ and $x - e \in \Delta(R)$ for some idempotent $e \in R$.

In section 3 of the paper under review, the authors also generalized previous result and some other results in [9]: Theorems 3.19,3.20,3.22 and Proposition 3.21.

Section 4 is devoted to some ring extensions (for instance, polynomial extensions, matrix extensions, trivial extensions and Morita contexts) of 2- ΔU -rings.

To summarize,

the results of this manuscript are very good, innovative and of interest to a wide range of mathematicians. The results obtained in this paper extends some known ones in the literature. The paper is clearly written. I strongly suggest to acceptance of the paper (WITH THE SUBMITTED VERSION).

A small comment to authors: I would like to offer the paper [8] add to the references list for the motivations of 2- UU -rings, 2- UJ -rings and 2- ΔU -rings. This may also help to solve in some questions in section 5.

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