RINGS WITH 2-ΔU PROPERTY

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ABSTRACT. Rings in which the square of each unit lies in $1 + \Delta(R)$, are said to be 2-ΔU, where $J(R) \subseteq \Delta(R) =: \{r \in R | r + U(R) \subseteq U(R)\}$. The set $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by units of R and is studied in [19]. The class of 2-ΔU rings consists several rings including UJ-rings, 2-UJ rings and ΔU-rings, and we observe that ΔU-rings are UUC. The structure of 2-ΔU rings is studied under various conditions. Moreover, the 2-ΔU property is studied under some algebraic constructions.

1. INTRODUCTION AND BASIC CONCEPTS

In the current paper, let R denote an associative ring with identity element, not necessarily commutative. Typically, for such a ring R, the sets $U(R)$, $Nil(R)$, $C(R)$, $Id(R)$ and $J(R)$ represent the set of invertible elements, the set of nilpotent elements, the set of central elements, the set of idempotent elements in R and the Jacobson radical of R respectively. Additionally, the ring of $n \times n$ matrices over R and the ring of $n \times n$ upper triangular matrices over R are denoted by $M_n(R)$ and $T_n(R)$, respectively. Traditionally, a ring is termed *abelian* if each idempotent element is central, meaning that $Id(R) \subseteq C(R)$.

The main material of the present study is the set $\Delta(R)$ was handled by Lam [18, Exercise 4.24] and recently, studied by Leroy-Matczuk [19]. As pointed out by the authors in [19, Theorem 3 and 6], $\Delta(R)$ is the largest Jacobson radical subring of R which is closed with respect to multiplication by all units (quasi-invertible elements) of R. Also, $J(R) \subseteq \Delta(R)$, and $\Delta(R) = J(T)$, where T is the subring of R generated by units of R, and the equality $\Delta(R) = J(R)$ holds if, and only if, $\Delta(R)$ is an ideal of R.

It is well-known that $1 + J(R) \subseteq U(R)$. A ring R is said to be an UJ-ring if the reverse inclusion holds, i.e. $U(R) = 1 + J(R)$ [13]. By [6], a ring R is said to be 2-UJ if for each $u \in U(R)$, $u^2 = 1 + j$ where $j \in J(R)$. These rings are generalization of UJ rings. They showed that for 2-UJ rings the notions semi-regular, exchange and clean rings are equivalent. Recall that a ring R is called an UU-ring if $U(R) = 1 + Nil(R)$ (see [8]). As a generalization of UU rings Sheibani and Chen in [23] introduced 2-UU rings. A ring R is called 2-UU if the square of every unit is the sum of 1_R and a nilpotent. They showed that R is strongly 2-nil-clean if, and only if, R is an exchange 2-UU ring. A ring R is said to be regualr (unit-regular) in the sense of von Neumann if for every $a \in R$, there is an $x \in R$ (resp., $x \in U(R)$) such that $axa = a$, and R is said to be strongly regular if for each $a \in R$, $a \in a^2R$. Recall that a ring R is exchange if for each $a \in R$, there exist $e^2 = e \in aR$ such that $1 - e \in (1 - a)R$, and a ring R is clean if every element

of R is a sum of an idempotent and an unit [21]. Notice that every clean ring is exchange, but the converse is not true in general, whereas the converse is true in the abelian case (see [21, Proposition 1.8]). A ring R is called semi-regular if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. Semi-regular rings are exchange, but the converse is not true in general (see [21]). A ring R is Boolean if every element of R is idempotent.

According to Chen [3], an element of a ring is called J-clean provided that it can be written as the sum of an idempotent and an element from its Jacobson radical. A ring is J-clean in the case where each of its element is J-clean or equivalently $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ (which is also called semi-boolean in [22]). It was shown in [13] that a ring R is J-clean if, and only if, R is a clean UJ ring. In 2019, Fatih Karabacak et al. introduced new rings that are a generalization of UJ rings. They called these rings ΔU [10]. A ring R is said to be ΔU if $1 + \Delta(R) = U(R)$. Due to Fatih Karabacak et al. [10], a ring R is called Δ-clean if every element of R is a sum of an idempotent and an element from the $\Delta(R)$. Δ-clean rings are clean, but the converse is not true in general. They showed in [10] that a ring R is Δ-clean if, and only if, R is a clean ΔU ring.

As a generalization of the above concepts, we introduce 2-ΔU rings. A ring R is called 2-ΔU if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (equivalently, for each $u \in U(R)$, $u^2 = 1 + r$ where $r \in \Delta(R)$). Clearly all ΔU rings and hence unit uniquely clean rings and rings with only two units are 2-ΔU. Also, 2-UJ rings and hence UJ rings are 2-ΔU but, the converse is not true in general. Our motivating tool is to give a satisfactory description of 2-ΔU rings by comparing their crucial properties with these of 2-UU and 2-UJ rings, respectively, as well as to find some new exotic properties of 2-ΔU rings that are not too characteristically frequently seen in the existing literature.

We are now planning to give a brief program of our results established in the sequel: In section 2, we achieve to exhibit some major properties and characterizations of ΔU rings in various different aspects (see, for instance, Propositions 2.2, 2.3 and 2.4 and Theorem 2.5). In section 3, we establish some fundamental characterization properties of 2-ΔU rings that are mainly stated and proved in Theorems 3.10, 3.19, 3.20 and 3.22 and the other statments associated with them. In section 4, we give some extensions of 2-ΔU rings, for instance, polynomial extensions, matrix extensions, trivial extensions and Morita contexts. We close our work in the final section with challenging questions, namely Problems 5.1, 5.2, 5.3 and 5.4.

Now, we have the following diagram which violates the relationships between the defined above classes of rings:
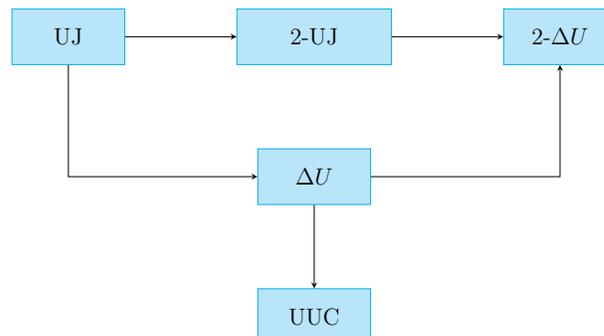
2. ΔU RINGS

In this section, we investigate some properties of ΔU rings. We give relations between ΔU rings and some related rings.

Definition 2.1 ([10]). A ring R is called ΔU if $1 + \Delta(R) = U(R)$.

By Calugareanu and Zhou, a ring R is called UUC if every unit is uniquely clean [2].

Proposition 2.2. *Let R be a ΔU ring. Then:*
  (i) $U(R) + U(R) \subseteq \Delta(R)$.
  (ii) R *is a UUC ring.*
  (iii) $(U(R) + U(R)) \cap Id(R) = 0$.

Proof.   (i) Let $x \in U(R) + U(R)$. Then $x = u_1 + u_2$ where, $u_1, u_2 \in U(R) = 1 + \Delta(R)$, so $x = 1 + r_1 + 1 + r_2 = 2 + (r_1 + r_2)$ where, $r_1, r_2 \in \Delta(R)$. On the other hand, we know that $2 \in \Delta(R)$ by [10, Proposition 2.4]. Also, $\Delta(R)$ is a subring of R. Then, $x \in \Delta(R)$.
  (ii) Assume that $u = e + v$ where, $u, v \in U(R)$ and $e \in Id(R)$. It suffices to show that $e = 0$. As R is ΔU, $u = 1 + r$ and $v = 1 + r'$ where, $r, r' \in \Delta(R)$. Hence, $e = 0$ by [19, Proposition 15].
  (iii) This is clear by (i) and (ii).
  $\square$

Proposition 2.3. *Let R be a ΔU ring and let $\bar{R} = R/J(R)$. The following hold:*
  (i) *For any $u_1, u_2 \in U(R)$, $u_1 + u_2 \neq 1$.*
  (ii) *For any $\bar{u}_1, \bar{u}_2 \in U(\bar{R})$, $\bar{u}_1 + \bar{u}_2 \neq \bar{1}$.*

Proof.   (i) This is clear by Proposition 2.2 and [2, Example 2.2].
  (ii) Since R is ΔU, $\bar{R}$ is ΔU and hence it is UUC, then the result follows from [2, Example 2.2].
  $\square$

A ring R is called semi-potent if every one-sided ideal not contained in $J(R)$ contains a non-zero idempotent. Moreover, a semi-potent ring R is called potent if idempotents lift modulo $J(R)$.

Proposition 2.4. *Let R be a potent ΔU ring, and $\bar{R} = R/J(R)$. Then, we have:*
  (i) *For any $\bar{e} = \bar{e}^2 \in \bar{R}$ and any $\bar{u}_1, \bar{u}_2 \in U(\bar{e}\bar{R}\bar{e})$, $\bar{u}_1 + \bar{u}_2 \neq \bar{e}$.*
  (ii) *There does not exist $\bar{e} = \bar{e}^2 \in \bar{R}$ such that $\bar{e}\bar{R}\bar{e} \cong M_2(S)$ for some ring S.*

Proof.   (i) Given $\bar{e}, \bar{u}_1, \bar{u}_2$ as in (i), we can assume that $e^2 = e \in R$, because idempotents lift modulo $J(R)$. Then, $\bar{e}\bar{R}\bar{e} \cong eRe/J(eRe)$. Finally, since eRe is ΔU by [10, Proposition 2.6], (i) follows directly by Proposition 2.3(i).
  (ii) Note that, in a 2×2 matrix ring, it is always true that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in U(M_2(S)) + U(M_2(S)).$$

Hence, there exist $\bar{u}_1, \bar{u}_2 \in U(\bar{e}\bar{R}\bar{e})$ such that $\bar{u}_1 + \bar{u}_2 = \bar{e}$. This is a contradiction with (i).
  $\square$

A ring R is called reduced if it contains no non-zero nilpotent elements.

Theorem 2.5. *Let R be a semi-potent ring. The following statements are equivalent:*

    (i) *R is a ΔU ring.*
    (ii) *$R/J(R)$ is Boolean.*
    (iii) *R is a UJ ring.*
    (iv) *$R/J(R)$ is a UU ring.*

Proof. $(i) \Rightarrow (ii)$ Since R is semi-potent, $R/J(R)$ is semi-potent (indeed, potent). Also, $R/J(R)$ is ΔU. So without loss of generality it can be assumed that $J(R) = 0$. Then by [10, Theorem 4.4], R is reduced and hence it is abelian. Now, assume that there exists $x \in R$ such that $x - x^2 \neq 0$ in R. Since R is a semi-potent ring, there exists $e = e^2 \in R$ such that $e \in (x - x^2)R$. So, write $e = (x - x^2)y$ for some $y \in R$. Since e is central, $[er(1-e)]^2 = 0 = [(1-e)re]^2$, we have $er(1-e) = 0 = e(1-e)re$. We can write:

$$e = ex.e(1-e).ey,$$

so that both $ex, e(1-x) \in U(eRe)$. But, we know that eRe is a ΔU ring. However, $ex + e(1-x) = e$, which contradicts Proposition 2.3(i). Therefore, R is a Boolean ring, as desired.

    $(ii) \Rightarrow (iii)$ Let us assume $u \in U(R)$. Then, $\bar{u} \in U(\bar{R} = R/J(R))$. Since $\bar{R}$ is a Boolean ring, we have $\bar{u} = \bar{1}$, which implies $u - 1 \in J(R)$.

    $(iii) \Rightarrow (i)$ This is clear, because always we have $J(R) \subseteq \Delta(R)$.

    $(ii) \Rightarrow (iv)$ This is obvious.

    $(iv) \Rightarrow (ii)$ We know that $R/J(R)$ is semi-potent. Then, it is strongly nil-clean by [11, Theorem 2.25]. Hence $R/J(R)$ is an exchange UU ring by [8, Theorem 4.3]. Thus, it is Boolean by [8, Theorem 4.1]. $\qquad\square$

Corollary 2.6. *A regular ring R is ΔU if, and only if, R is UJ if, and only if, R is UU if, and only if, R is Boolean.*

Proof. Since R is regular, $J(R) = 0$ and R is semi-potent. So the result follows from Theorem 2.5. $\qquad\square$

Corollary 2.7. *Let R be a potent ring. Then, the following are equivalent:*

    (i) *R is a ΔU ring.*
    (ii) *$R/J(R)$ is a ΔU ring.*
    (iii) *$R/J(R)$ is a Boolean ring.*
    (iv) *R is a UJ ring.*
    (v) *$R/J(R)$ is a UJ ring.*
    (vi) *$R/J(R)$ is a UU ring.*

Proof. We know that every potent ring is semi-potent. Then, (i), (iii), (iv) and (vi) are equivalent by Theorem 2.5. On the other hand (i) and (ii) are equivalent by [10, Proposition 2.4]. Also, (iv) and (v) are equivalent by [13, Proposition 1.3]. $\qquad\square$

Corollary 2.8. *Let R be an Artinian ring. Then, the following are equivalent:*

    (i) *R is a ΔU ring.*
    (ii) *R is a UJ ring.*
    (iii) *R is a UU ring.*

Proof. We know that every Artinian ring is always clean. Also, since R is Artinian, we have $J(R) \subseteq Nil(R)$. □

Corollary 2.9. *Let R be a finite ring. Then, the following conditions are equivalent:*

(i) *R is a ΔU ring.*
(ii) *R is a UJ ring.*
(iii) *R is a UU ring.*

Proof. In fact, any finite ring is Artinian. □

Corollary 2.10. *For a ring R, the following two conditions are equivalent:*

(i) *R is a potent ΔU ring.*
(ii) *R is a J-clean ring.*

Proof. $(ii) \Rightarrow (i)$ This is clear by [13, Theorem 3.2] and Corollary 2.7.

$(i) \Rightarrow (ii)$ Applying Corollary 2.7, we know that $R/J(R)$ is Boolean. Therefore, for each $a \in R$, we have $a - a^2 \in J(R)$. Since R is a potent ring, there exists an idempotent $e \in R$ such that $a - e \in J(R)$. Thus, R is a J-clean ring, as wanted. □

Let $Nil_*(R)$ denote the prime radical (or lower nil-radical) of a ring R, i.e. the intersection of all prime ideals of R. We know that $Nil_*(R)$ is a nil-ideal of R. It is long known that a ring R is called 2-*primal* if its lower nil-radical $Nil_*(R)$ consists precisely of all the nilpotent elements of R. For instance, it is well known that both reduced rings and commutative rings are both 2-primal. For an endomorphism α of a ring R, R is called α-*compatible* if, for any $a, b \in R$, $ab = 0 \iff a\alpha(b) = 0$, and in this case α is clearly injective.

Let R be a ring and $\alpha : R \to R$ is a ring endomorphism and $R[x; \alpha]$ denotes the ring of skew polynomial over R, with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[x] = R[x; 1_R]$ is the ring of polynomial over R.

Proposition 2.11. *Let R be a 2-primal ring and α be an endomorphism of R. If R is α-compatible, then*

$$\Delta(R[x; \alpha]) = \Delta(R) + J(R[x; \alpha]).$$

Proof. Suppose first that R is a reduced ring. As R is α-compatible, by [4, Corollary 2.12] we have $U(R[x; \alpha]) = U(R)$. Also, it is easy to see that, $\Delta(R) \subseteq \Delta(R[x; \alpha])$. Let $r + r_0 \in \Delta(R[x; \alpha])$, where $r \in R[x; \alpha]x$ and $r_0 \in R$. Then, for any $u \in U(R)$, $r + r_0 + u \in U(R)$. This shows that $r = 0$ and $r_0 + u \in U(R)$. Thus, we conclude that $\Delta(R[x; \alpha]) \subseteq \Delta(R)$ and hence $\Delta(R[x; \alpha]) = \Delta(R)$. Now assume that R is 2-primal. Clearly $Nil_*(R[x; \alpha]) = Nil_*(R)[x; \alpha] \subseteq J(R[x; \alpha])$ by [4, Lemma 2.2]. As R is 2-primal, $R/Nil_*(R)$ is reduced, so we have $J(R[x; \alpha]) = \text{Nil}_*(R[x; \alpha]) = \text{Nil}_*(R)[x; \alpha]$. By the first part of the proof applied to $R/Nil_*(R)$ and using [19, Proposition 6(3)], we have:

$$\Delta(R) + Nil_*(R)[x; \alpha] = \Delta(\frac{R}{Nil_*(R)}[x; \alpha]) = \Delta(\frac{R[x; \alpha]}{J(R[x; \alpha])}) = \frac{\Delta(R[x; \alpha])}{J(R[x; \alpha])}.$$

By the above we conclude the desired equality. □

For any two elements $a, b \in R$, $1 - ab$ is an unit if, and only if, $1 - ba$ is an unit. This result is known as Jacobson's lemma for units. There are several analogous results in the literature. We have the following.

Corollary 2.12. *Let R be a ΔU ring and let $a, b \in R$. Then, $1 - ab \in \Delta(R)$ if, and only if, $1 - ba \in \Delta(R)$.*

Proof. Assuming that $1 - ab \in \Delta(R)$, we have $ab \in U(R)$. Therefore, by [10, Proposition 2.4], $a \in U(R)$. Thus,

$$1 - ba = a^{-1}(1 - ab)a \in \Delta(R),$$

because $\Delta(R)$ is closed with respect to multiplication by all units (see [19, Theorem 3]). The converse is similar.   □

3. 2-ΔU RINGS

In this section, we introduce 2-ΔU rings and investigate its elementary properties. We now give our main definition.

Definition 3.1. A ring R is called 2-ΔU if the square of each unit is a sum of an idempotent and an element from the $\Delta(R)$ (equivalently, for each $u \in U(R)$, $u^2 = 1 + r$ where $r \in \Delta(R)$).

Example 3.2. Clearly, 2-UJ rings are 2-ΔU. But, the converse is not true in general. For example, consider the ring $R = F_2\langle x, y\rangle / \langle x^2 \rangle$. Then $J(R) = 0$, $\Delta(R) = F_2 x + xRx$ and $U(R) = 1 + F_2 x + xRx$. Then R is ΔU by [10, Example 2.2] and hence it is 2-ΔU. But clearly R is not 2-UJ.

Example 3.3. The ring $\mathbb{Z}_3$ is 2-ΔU, but is not ΔU.

Proposition 3.4. *A direct product $\prod_{i \in I} R_i$ of rings is 2-ΔU if, and only if, each R_i is 2-ΔU.*

Proof. As $\Delta(\prod_{i \in I} R_i) = \prod_{i \in I} \Delta(R_i)$ and $U(\prod_{i \in I} R_i) = \prod_{i \in I} U(R_i)$, the result follows.   □

Proposition 3.5. *Let R be a 2-ΔU ring. If T is a factor ring of R such that units of T lift to units of R, then T is 2-ΔU.*

Proof. Suppose that $f : R \to T$ is a ring epimorphism. Let $v \in U(T)$. Then there exists $u \in U(R)$ such that $v = f(u)$ and $u^2 = 1 + r \in 1 + \Delta(R)$. Thus, we have, $v^2 = (f(u))^2 = f(u^2) = f(1 + r) = f(1) + f(r) = 1 + f(r) \in 1 + \Delta(T)$ □

Example 3.6. A division ring R is 2-ΔU if, and only if, $R \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. Since R is a division ring, $\Delta(R) = 0$, the result follows from [6, Example 2.1].   □

Remark 3.7. The condition "units of T lift to units of R" in Proposition 3.5 is necessary. The ring $\mathbb{Z}_7$ is a factor ring of the 2-ΔU ring $\mathbb{Z}$. But, $\mathbb{Z}_7$ is not 2-ΔU by Example 3.6. Not that, not all of units of $\mathbb{Z}_7$ can lift to units of $\mathbb{Z}$.

Proposition 3.8. *Let R be a 2-ΔU ring. For a unital subring S of R, if $S \cap \Delta(R) \subseteq \Delta(S)$, then S is a 2-ΔU ring. In particular, the center of R is a 2-ΔU ring.*

Proof. Let $v \in U(S) \subseteq U(R)$. Since R is 2-ΔU, we have, $v^2 - 1 \in \Delta(R) \cap S \subseteq \Delta(S)$. So S is a 2-ΔU ring. The rest follows from [19, Corollary 8].   □

Proposition 3.9. *Let R be a 2-ΔU ring and $2 \in \Delta(R)$. Then:*

(i) $(U(R))^2 + (U(R))^2 \subseteq \Delta(R)$.

(ii) $[(U(R))^2 + (U(R))^2] \cap \mathrm{Id}(R) = 0$.

Proof. (i) Let $t \in (U(R))^2 + (U(R))^2$, so $t = u^2 + v^2$ where $u, v \in U(R)$. Since R is 2-ΔU, $t = 1 + r + 1 + s$ where $r, s \in \Delta(R)$. So, we have $t = 2 + (r + s)$. Since, $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R, $t \in \Delta(R)$.

(ii) It is clear by (i) and [19, Proposition 15]. □

Theorem 3.10. *Let $I \subseteq J(R)$ be an ideal of a ring R. Then R is 2-ΔU if, and only if, is so R/I.*

Proof. Let R be a 2-ΔU ring and $u + I \in U(R/I)$. Then $u \in U(R)$ and hence $u^2 = 1 + r$ where $r \in \Delta(R)$. Thus $(u + I)^2 = u^2 + I = (1 + I) + (r + I)$, where $r + I \in \Delta(R)/I = \Delta(R/I)$ by [19, Proposition 6]. Conversely, let R/I be a 2-ΔU ring and $u \in U(R)$. Then $u + I \in U(R/I)$ and hence $(u + I)^2 = (1 + I) + (r + I)$, where $r+I \in \Delta(R/I)$. Thus $u^2+I = (1+r)+I$. So $u^2 - (1+r) \in I \subseteq J(R) \subseteq \Delta(R)$. Therefore $u^2 = 1 + r'$, where $r' \in \Delta(R)$. Hence R is a 2-ΔU ring. □

Corollary 3.11. *A ring R is 2-ΔU if, and only if, $R/J(R)$ is 2-ΔU.*

Proposition 3.12. *Let R be a 2-ΔU ring and e be an idempotent of R. Then eRe is a 2-ΔU ring.*

Proof. Let $u \in U(eRe)$. Then $u+(1-e) \in U(R)$. By the hypothesis, $(u+(1-e))^2 = u^2 + (1 - e) = 1 + r \in 1 + \Delta(R)$. Then we have $u^2 - e \in \Delta(R)$. Now we show that $u^2 - e \in \Delta(eRe)$. Let v be an arbitrary unit of eRe. Clearly, $v + 1 - e \in U(R)$. Note that $u^2 - e \in \Delta(R)$ gives us that $u^2 - e + v + 1 - e \in U(R)$, by the definition of $\Delta(R)$. Take $u^2 - e + v + 1 - e = t \in U(R)$. One can check that $et = te = ete = u^2 - e + v$, and so $ete \in U(eRe)$. It shows that $u^2 - e + U(eRe) \subseteq U(eRe)$. Then we have $u^2 - e \in \Delta(eRe)$ and hence $u^2 \in e + \Delta(eRe)$, which implies eRe is a 2-ΔU ring. □

Proposition 3.13. *For any ring $R \neq 0$ and any integer $n \geq 2$, the ring $M_n(R)$ is not a 2-ΔU ring.*

Proof. Since it is well known that $M_2(R)$ is isomorphic to a corner ring of $M_n(R)$ (for $n \geq 2$), it suffices to show that $M_2(R)$ is not a 2-ΔU ring by virtue of Proposition 3.12. To this target, consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in U(M_2(R)).$$

Then, $A^2 - I_2 = A \notin J(M_2(R)) = \Delta(M_2(R))$, as required. □

A set $\{e_{ij} : 1 \leq i, j \leq n\}$ of nonzero elements of R is said to be a system of n^2 matrix units if $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{jj} = 1$ and $\delta_{js} = 0$ for $j \neq s$. In this case, $e := \sum_{i=1}^{n} e_{ii}$ is an idempotent of R and $eRe \cong M_n(S)$, where

$$S = \{r \in eRe : re_{ij} = e_{ij}r, \text{ for all } i, j = 1, 2, ..., n\}.$$

Recall that, a ring R is said to be a Dedekind-finite if $ab = 1$ implies $ba = 1$ for any $a, b \in R$. In other words, all one-sided inverses in the ring are two-sided.

Proposition 3.14. *Every 2-ΔU ring is Dedekind-finite.*

Proof. If we assume to the contrary that R is not a Dedekind-finite ring, then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Assuming $e_{ij} = a^i(1 - ba)b^j$ and $e = \sum_{i=1}^{n} e_{ii}$, there exists a nonzero ring S such that $eRe \cong M_n(S)$. However, owing to Proposition 3.12, eRe is a 2-ΔU ring, so $M_n(S)$ must also be a 2-ΔU ring, which contradicts Proposition 3.13. $\square$

Example 3.15. A local ring R is 2-ΔU if, and only if, $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. Assume that $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. We know that $R/J(R)$ is a division ring, so $R/J(R)$ is 2-ΔU by Example 3.6. Thus, R is 2-ΔU by Corollary 3.11. Conversely, let R be 2-ΔU and hence $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ by Example 3.6. $\square$

Corollary 3.16. *(i)A semi-simple ring R is 2-ΔU if, and only if, $R \cong \bigoplus_{i=1}^{n} R_i$ where $R_i \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ for every i.*

(ii)A semi-local ring R is 2-ΔU if, and only if, $R/J(R) \cong \bigoplus_{i=1}^{m} R_i$ where $R_i \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ for every i.

Example 3.17. The ring $\mathbb{Z}_m$ is 2-ΔU if, and only if, $m = 2^k 3^l$ for some positive integers k and l.

Lemma 3.18. *Let R be a 2-ΔU ring. If $J(R) = 0$ and every nonzero right ideal of R contains a nonzero idempotent, then R is reduced.*

Proof. Suppose that R is not reduced. Then there exists a nonzero element $a \in R$ such that $a^2 = 0$. By [20, Theorem 2.1], there is an idempotent $e \in RaR$ such that $eRe \cong M_2(T)$ for some non-trivial ring T. By Proposition 3.12, eRe is a 2-ΔU ring and hence $M_2(T)$ is a 2-ΔU ring. This contradicts to Proposition 3.13. $\square$

A ring R is called π-regular if for each $a \in R$, $a^n \in a^n R a^n$ for some integer $n \geq 1$. Regular rings are π-regular. Also, a ring R is said to be strongly π-regular provided that for any $a \in R$ there exists $n \geq 1$, such that $a^n \in a^{n+1}R$.

Theorem 3.19. *Let R be a ring. The following are equivalent:*

   (i) *R is a regular 2-ΔU ring.*
   (ii) *R is a π-regular reduced 2-ΔU ring.*
   (iii) *R has the identity $x^3 = x$ (i.e., R is a tripotent ring).*

Proof. $(i) \Rightarrow (ii)$ Since R is regular, $J(R) = 0$, and every nonzero right ideal contains a nonzero idempotent. In view of Lemma 3.18, R is reduced. Also, clearly every regular ring is π-regular.

$(ii) \Rightarrow (iii)$ Notice that reduced rings are abelian, so R is abelian regular by [1, Theorem 3] and hence it is strongly regular. Then R is unit-regular, so $\Delta(R) = 0$ by [19, Corollary 16]. Thus, we have $Nil(R) = J(R) = \Delta(R) = 0$. On the other hand, R is strongly π-regular. Let $x \in R$. In view of [9, Proposition 2.5], there is an idempotent $e \in R$ and a unit $u \in R$ such that $x = e + u$, $ex = xe \in Nil(R) = 0$. So we have $x = x - xe = x(1 - e) = u(1 - e) = (1 - e)u$. Since R is a 2-ΔU ring, $u^2 = 1$. It follows that $x^2 = (1 - e)$. Hence, $x = x(1 - e) = x.x^2 = x^3$.

$(iii) \Rightarrow (i)$ It is clear that R is regular. Let $u \in U(R)$. Then we have $u^3 = u$, which implies that $u^2 = 1$, and thus, R is a 2-ΔU ring. $\square$

A ring R is strongly 2-nil-clean if every element in R is the sum of two idempotents and a nilpotent that commute (for details see [5]).

Theorem 3.20. *The following are equivalent for a ring R:*

(i) *R is a regular 2-ΔU ring.*
(ii) *R is a strongly regular 2-ΔU ring.*
(iii) *R is a unit-regular 2-ΔU ring.*
(iv) *R has the identity $x^3 = x$.*

Proof. $(i) \Rightarrow (ii)$ In view of Lemma 3.18, R is reduced and hence it is abelian. Then R is strongly regular.

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (iv)$ Let $x \in R$. Then, $x = ue$ for some $u \in U(R)$ and $e \in Id(R)$. We know that every unit-regular ring is regular, so R is regular 2-ΔU and hence R is abelian. On the other hand, by [19, Corollary 16], we have $\Delta(R) = 0$. Therefore, for any $u \in U(R)$, we have $u^2 = 1$. Then, $x^2 = u^2 e^2 = e$. So, R is a 2-Boolean ring. Thus, by [5, Corollary 3.4], R is strongly 2-nil-clean and hence by [5, Theorem 3.3], R is tripotent.

$(iv) \Rightarrow (i)$ It is clear by Theorem 3.19.   □

Proposition 3.21. *A ring R is a ΔU ring if, and only if,*

(i) *$2 \in \Delta(R)$,*
(ii) *R is a 2-ΔU ring,*
(iii) *If $x^2 \in \Delta(R)$, then $x \in \Delta(R)$ for every $x \in R$.*

Proof. $\Rightarrow$ As R is a ΔU ring, then $-1 = 1 + r$ for some $r \in \Delta(R)$. This implies that $-2 \in \Delta(R)$ and so $2 \in \Delta(R)$. Clearly every ΔU ring is 2-ΔU. The result follows from [10, Proposition 2.4].

$\Leftarrow$ Let $u \in U(R)$. Then $(u-1)^2 + 2(u-1) = (u-1)(u+1) = u^2 - 1 \in \Delta(R)$ as R is a 2-ΔU ring. It follows from $2 \in \Delta(R)$ and $\Delta(R)$ is a subring of R that $(u-1)^2 \in \Delta(R)$. So by (iii), we have $u - 1 \in \Delta(R)$ and hence R is a ΔU ring.   □

Due to Kosan et al. [16], a ring R is called semi-tripotent if for each $a \in R$, $a = e + j$ where $e^3 = e$ and $j \in J(R)$ (equivalently, $R/J(R)$ has the identity $x^3 = x$ and idempotents lift modulo $J(R)$).

Theorem 3.22. *Let R be a ring. Then, the following are equivalent:*

(i) *R is a semi-regular 2-ΔU ring.*
(ii) *R is an exchange 2-ΔU ring.*
(iii) *R is a semi-tripotent ring.*

Proof. $(i) \Rightarrow (ii)$ In view of [21, Proposition 1.6], semi-regular rings are exchange.

$(ii) \Rightarrow (iii)$ By [21, Corollary 2.4], $R/J(R)$ is exchange and idempotents lift modulo $J(R)$. Also, by Theorem 3.10, $R/J(R)$ is 2-ΔU. So, without loss of generality, it can be assumed that $J(R) = 0$. Since R is an exchange ring, every nonzero one sided ideal contains a nonzero idempotent. By Lemma 3.18, R is reduced and hence it is abelian. Thus, R is abelian weakly clean. Hence by [15, Proposition 14], $R/J(R) \cong M_n(D)$ where $1 \le n \le 2$ and D is a division ring. Then by [19, Theorem 11], we have $\Delta(R) = J(R)$ and hence $\Delta(R) = 0$. As R is 2-ΔU, we have $v^2 = 1$ for every $v \in U(R)$. Then the result follows from [6, Theorem 3.3].

$(iii) \Rightarrow (i)$ By [6, Theorem 3.3], R is semi-regular 2-UJ and hence R is semi-regular 2-ΔU.   □

Corollary 3.23. *Let R be a 2-ΔU ring. Then the following are equivalent:*

   (i) *R is a semi-regular ring.*
   (ii) *R is an exchange ring.*
   (iii) *R is a clean ring.*

Proof. By Theorem 3.22, (i) $\Leftrightarrow$ (ii).

   $(iii) \Rightarrow (ii)$ This is clear.

   $(ii) \Rightarrow (iii)$ If R is exchange 2-ΔU, then R is reduced by Lemma 3.18 and hence it is abelian. Therefore, R is abelian exchange, so it is clean. $\qquad\square$

Corollary 3.24. *Let R be a ring. The following are equivalent:*

   (i) *R is a semi-regular 2-ΔU ring and $J(R)$ is nil.*
   (ii) *R is an exchange 2-ΔU ring and $J(R)$ is nil.*
   (iii) *R is a strongly 2-nil-clean ring.*

Proof. $(i) \Rightarrow (ii)$ This is clear.

   $(ii) \Rightarrow (iii)$ Since R is exchange 2-ΔU, $\Delta(R) = J(R)$. Then, for any $u \in U(R)$, we have $u^2 - 1 \in \Delta(R) = J(R) \subseteq Nil(R)$. So, R is 2-UU ring. Therefore, R is exchange 2-UU ring and hence it is strongly 2-nil-clean by [23, Theorem 4.1].

   $(iii) \Rightarrow (i)$ This is clear by [6, Corollary 3.5] and we know that every 2-UJ ring is 2-ΔU. $\qquad\square$

Corollary 3.25. *Let R be a ring. Then, the following are equivalent:*

   (i) *R is a regular ΔU ring.*
   (ii) *R is a π-regular reduced ΔU ring.*
   (iii) *R is a Boolean ring.*

Proof. This is obvious by Theorem 3.19 and [10, Theorem 4.4]. $\qquad\square$

Corollary 3.26. *Let R be a ring. The following are equivalent:*

   (i) *R is a semi-regular ΔU ring.*
   (ii) *R is an exchange ΔU ring.*
   (iii) *R is a Δ-clean ring.*

Proof. The result follows from [10, Theorem 4.2 and Corollary 4.7]. $\qquad\square$

4. Some extensions of 2-ΔU rings

   We say that C is an unital subring of a ring D if $\emptyset \neq C \subseteq D$ and, for any $x, y \in C$, the relations $x - y$, $xy \in C$ and $1_D \in C$ hold. Let D be a ring and C an unital subring of D, and denote by $R[D, C]$ the set $\{(d_1, \ldots, d_n, c, c, \ldots) : d_i \in D, c \in C, 1 \leq i \leq n\}$. Then, $R[D, C]$ forms a ring under the usual component-wise addition and multiplication. The ring $R[D, C]$ is called the tail ring extension.

Proposition 4.1. *$R[D, C]$ is a 2-ΔU ring if, and only if, D and C are 2-ΔU rings.*

Proof. Let $R[D, C]$ be a 2-ΔU ring. Firstly, we prove that D is a 2-ΔU ring. Let $u \in U(D)$. Then $\bar{u} = (u, 1, 1, \ldots) \in U(R[D, C])$. By the hypothesis, we have $(u^2 - 1, 0, 0, \ldots) \in \Delta(R[D, C])$, so $(u^2 - 1, 0, 0, \ldots) + U(R[D, C]) \subseteq U(R[D, C])$. Thus, for all $v \in U(D)$, $(u^2 - 1 + v, 1, 1, \ldots) = (u^2 - 1, 0, 0, \ldots) + (v, 1, 1, \ldots) \in U(R[D, C])$. Hence, $u^2 - 1 + v \in U(D)$, which implies that $u^2 - 1 \in \Delta(D)$. Now, we show that C is a 2-ΔU ring. Let $v \in U(C)$. Then $(1, \ldots, 1, 1, v, v, \ldots) \in U(R[D, C])$. By

hypothesis, $(0,\ldots,0,v^2-1,v^2-1,\ldots) \in \Delta(R[D,C])$, so $(0,\ldots,0,v^2-1,v^2-1,\ldots) + U(R[D,C]) \subseteq U(R[D,C])$. Thus, for all $u \in U(C)$, $(1,1,\ldots,v^2-1+u,v^2-1+u,\ldots) \in U(R[D,C])$. Then, we have $v^2-1+u \in U(C)$ and hence $v^2-1 \in \Delta(C)$. For the converse, assume that D and C are 2-ΔU rings. Let $\bar{u} = (u_1,u_2,\ldots,u_n,v,v,\ldots) \in U(R[D,C])$, where $u_i \in U(D)$ and $v \in U(C) \subseteq U(D)$. We must show that $\bar{u}^2 - 1 + U(R[D,C]) \subseteq U(R[D,C])$. In fact, for all $\bar{a} = (a_1,\ldots,a_m,b,b,\ldots) \in U(R[D,C])$ with $a_i \in U(D)$ and $b \in U(C) \subseteq U(D)$, take $z = \max\{m,n\}$, then we have, $\bar{u}^2 - 1 + \bar{a} = (u_1^2 - 1 + a_1,\ldots,u_z^2 - 1 + a_z, v^2 - 1 + b, v^2 - 1 + b,\ldots)$. Note that $u_i^2 - 1 + a_i \in U(D)$ for all $1 \le i \le z$ and $v^2 - 1 + b \in U(C) \subseteq U(D)$. We deduce that $\bar{u}^2 - 1 + \bar{a} \in U(R[D,C])$. Thus, $\bar{u}^2 - 1 \in \Delta(R[D,C])$ or $\bar{u}^2 \in 1 + \Delta(R[D,C])$. This shows that $R[D,C]$ is a 2-ΔU ring.   □

Let R be a ring and $\alpha : R \to R$ is a ring endomorphism and $R[[x;\alpha]]$ denotes the ring of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x;1_R]]$ is the ring of formal power series over R.

Proposition 4.2. *A ring R is 2-ΔU if, and only if, so is $R[[x;\alpha]]$.*

Proof. Consider $I = R[[x;\alpha]]x$. Then I is an ideal of $R[[x;\alpha]]$. Note that $J(R[[x;\alpha]]) = J(R) + I$, so $I \subseteq J(R[[x;\alpha]])$. Since $R[[x;\alpha]]/I \cong R$, the result follows by Theorem 3.10.   □

Corollary 4.3. *A ring R is 2-ΔU if, and only if, so is $R[[x]]$.*

Theorem 4.4. *Let R be a 2-primal ring and α be an endomorphism of R such that R is α-compatible. The following are equivalent:*
   (i) *$R[x;\alpha]$ is a 2-ΔU ring.*
   (ii) *R is a 2-ΔU ring.*

Proof. $(ii) \Rightarrow (i)$ Let $u(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{i=0}^n a_i x^i$ in $U(R[x;\alpha])$. So by [4, Corollary 2.14], $a_0 \in U(R)$ and $a_i \in \mathrm{Nil}(R)$ for each $i \ge 1$. Then by hypothesis $a_0^2 = 1 + r$ where $r \in \Delta(R)$. On the other hand, we know that $J(R[x;\alpha]) = \mathrm{Nil}_*(R[x;\alpha]) = \mathrm{Nil}_*(R)[x;\alpha] = \mathrm{Nil}(R)[x;\alpha]$. Now, we have

$$(u(x))^2 = a_0^2 + a_0 a_1 x + \cdots + a_0 a_n x^n + a_1 x a_0 + \cdots = (1+r) + a_0 a_1 x + \ldots$$

$$= 1 + (r + a_0 a_1 x + \ldots) \in 1 + \Delta(R) + J(R[x;\alpha]).$$

On the other hand, we have $\Delta(R) + J(R[x;\alpha]) = \Delta(R[x;\alpha])$ by Proposition 2.11. Then, this shows that $R[x;\alpha]$ is a 2-ΔU ring.
$(i) \Rightarrow (ii)$ Let $u \in U(R) \subseteq U(R[x;\alpha])$. Then $u^2 \in 1 + \Delta(R[x;\alpha]) = 1 + \Delta(R) + J(R[x;\alpha])$. Thus, we have $u^2 \in 1 + \Delta(R)$ and hence R is a 2-ΔU ring.   □

Corollary 4.5. *Let R be a 2-primal ring. Then, the following are equivalent:*
   (i) *$R[x]$ is a 2-ΔU ring.*
   (ii) *R is a 2-ΔU ring.*

Let R be a ring and M a bi-modulo over R. The trivial extension of R and M is defined as

$$T(R,M) = \{(r,m) : r \in R \text{ and } m \in M\},$$

with addition defined componentwise and multiplication defined by

$$(r, m)(s, n) = (rs, rn + ms).$$

The trivial extension $T(R, M)$ is isomorphic to the subring

$$\left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} : r \in R \text{ and } m \in M \right\}$$

of the formal 2×2 matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$, and also $T(R, R) \cong R[x]/\langle x^2 \rangle$. We also note that the set of units of the trivial extension $T(R, M)$ is

$$U(T(R, M)) = T(U(R), M).$$

Also by [10], we have

$$\Delta(T(R, M)) = T(\Delta(R), M).$$

Proposition 4.6. *Let R be a ring and M a bi-modulo over R. Then, the following hold:*

(i) *The trivial extension* $\mathrm{T}(R, M)$ *is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*

(ii) *For $n \geq 2$, the quotient-ring $\dfrac{R[x; \alpha]}{\langle x^n \rangle}$ is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*

(iii) *For $n \geq 2$, the quotient-ring $\dfrac{R[[x; \alpha]]}{\langle x^n \rangle}$ is a 2-ΔU ring if, and only if, R is a 2-ΔU ring.*

(iv) *The upper triangular matrix ring $T_n(R)$ is a 2-ΔU if, and only if, R is a 2-ΔU ring.*

Proof. (i) Set $A = \mathrm{T}(R, M)$ and consider $I := \mathrm{T}(0, M)$. It is not too hard to verify that $I \subseteq J(A)$ such that $\dfrac{A}{I} \cong R$. So, the result follows directly from Theorem 3.10.

(ii) Put $A = \dfrac{R[x; \alpha]}{\langle x^n \rangle}$. Considering $I := \dfrac{\langle x \rangle}{\langle x^n \rangle}$, we obtain that $I \subseteq J(A)$ such that $\dfrac{A}{I} \cong R$. So, the result follows automatically Theorem 3.10.

(iii) Knowing that the isomorphism $\dfrac{R[x; \alpha]}{\langle x^n \rangle} \cong \dfrac{R[[x; \alpha]]}{\langle x^n \rangle}$ is true, (iii) follows automatically from (ii).

(iv) Let $I = \{(a_{ij}) \in T_n(R) \,|\, a_{ii} = 0\}$. Then, we have $I \subseteq J(T_n(R))$ with $T_n(R)/I \cong R^n$. Therefore, the desired result follows from Theorem 3.10 and Proposition 3.4. $\qquad\square$

Corollary 4.7. *Let R be a ring. Then, the following are equivalent:*

(i) *R is a 2-ΔU ring.*

(ii) *For $n \geq 2$, the quotient-ring $\dfrac{R[x]}{\langle x^n \rangle}$ is a 2-ΔU ring.*

(iii) *For $n \geq 2$, the quotient-ring $\dfrac{R[[x]]}{\langle x^n \rangle}$ is a 2-ΔU ring.*

Example 4.8. The upper triangular ring $T_n(\mathbb{Z}_3)$ for all $n \geq 1$ is 2-ΔU by Proposition 4.6(4) and Example 3.3. But, it is not ΔU by Example 3.3 and [10, Corollary 2.9].

Let R be a ring and M a bi-modulo over R. Let

$$\mathrm{DT}(R, M) := \{(a, m, b, n)| a, b \in R, m, n \in M\}$$

with addition defined componentwise and multiplication defined by

$$(a_1, m_1, b_1, n_1)(a_2, m_2, b_2, n_2) = (a_1 a_2, a_1 m_2 + m_1 a_2, a_1 b_2 + b_1 a_2, a_1 n_2 + m_1 b_2 + b_1 m_2 + n_1 a_2).$$

Then, $\mathrm{DT}(R, M)$ is a ring which is isomorphic to $\mathrm{T}(\mathrm{T}(R, M), \mathrm{T}(R, M))$. Also, we have

$$\mathrm{DT}(R, M) = \left\{ \begin{pmatrix} a & m & b & n \\ 0 & a & 0 & b \\ 0 & 0 & a & m \\ 0 & 0 & 0 & a \end{pmatrix} | a, b \in R, m, n \in M \right\}.$$

We have the following isomorphism as rings: $\dfrac{R[x, y]}{\langle x^2, y^2 \rangle} \to \mathrm{DT}(R, R)$ defined by

$$a + bx + cy + dxy \mapsto \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix}.$$

We, thereby, detect the following.

Corollary 4.9. *Let R be a ring and M a bi-modulo over R. Then the following statements are equivalent:*

(i) *R is a 2-ΔU ring.*
(ii) *$\mathrm{DT}(R, M)$ is a 2-ΔU ring.*
(iii) *$\mathrm{DT}(R, R)$ is a 2-ΔU ring.*
(iv) *$\dfrac{R[x, y]}{\langle x^2, y^2 \rangle}$ is a 2-ΔU ring.*

Let A, B be two rings and M, N be (A, B)-bi-modulo and (B, A)-bi-modulo, respectively. Also, we consider the bi-linear maps $\phi : M \otimes_B N \to A$ and $\psi : N \otimes_A M \to B$ that apply to the following properties.

$$Id_M \otimes_B \psi = \phi \otimes_A Id_M, Id_N \otimes_A \phi = \psi \otimes_B Id_N.$$

For $m \in M$ and $n \in N$, define $mn := \phi(m \otimes n)$ and $nm := \psi(n \otimes m)$. Now the 4-tuple $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ becomes to an associative ring with obvious matrix operations that is called a *Morita context ring*. Denote two-side ideals $Im\phi$ and $Im\psi$ to MN and NM, respectively, that are called the *trace ideals* of the *Morita context ring*.

Proposition 4.10. *Let $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context ring such that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. Then, R is a 2-ΔU ring if, and only if, both A and B are 2-ΔU.*

Proof. In view of [26, Lemma 3.1], we argue that $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$ and hence $\dfrac{R}{J(R)} \cong \dfrac{A}{J(A)} \times \dfrac{B}{J(B)}$. Then, the result follows from Corollary 3.11 and Proposition 3.4. □

Now, let R, S be two rings, and let M be an (R, S)-bi-modulo such that the operation $(rm)s = r(ms)$ is valid for all $r \in R$, $m \in M$ and $s \in S$. Given such a bi-modulo M, we can set

$$\mathrm{T}(R, S, M) = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R, m \in M, s \in S \right\},$$

where it forms a ring with the usual matrix operations. The so-stated formal matrix $\mathrm{T}(R, S, M)$ is called a *formal triangular matrix ring*. In Proposition 4.10, if we set $N = 0$, then we will obtain the following corollary.

Corollary 4.11. *Let R, S be rings and let M be an (R, S)-bi-modulo. Then, the formal triangular matrix ring $\mathrm{T}(R, S, M)$ is a 2-ΔU ring if, and only if, both R and S are 2-ΔU.*

Given a ring R and a central element s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition component-wise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + s x_1 y_2 & a_1 x_2 + x_1 b_2 \\ y_1 a_2 + b_1 y_2 & s y_1 x_2 + b_1 b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$. A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ with $A = B = M = N = R$ is called a *generalized matrix ring* over R. It was observed by Krylov in [17] that a ring S is a generalized matrix ring over R if, and only if, $S = K_s(R)$ for some $s \in C(R)$. Here $MN = NM = sR$, so $MN \subseteq J(A) \iff s \in J(R)$, $NM \subseteq J(B) \iff s \in J(R)$.

Corollary 4.12. *Let R be a ring and $s \in C(R) \cap J(R)$. Then, $K_s(R)$ is a 2-ΔU ring if, and only if, R is 2-ΔU.*

Following Tang and Zhou (cf. [25]), for $n \geq 2$ and for $s \in C(R)$, the $n \times n$ formal matrix ring over R defined by s, and denoted by $\mathrm{M}_n(R; s)$, is the set of all $n \times n$ matrices over R with usual addition of matrices and with multiplication defined below:
For (a_{ij}) and (b_{ij}) in $\mathrm{M}_n(R; s)$,

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad \text{where } (c_{ij}) = \sum s^{\delta_{ikj}} a_{ik} b_{kj}.$$

Here, $\delta_{ijk} = 1 + \delta_{ik} - \delta_{ij} - \delta_{jk}$, where δ_{jk}, δ_{ij}, δ_{ik} are the Kroncker delta symbols.

Corollary 4.13. *Let R be a ring and $s \in C(R) \cap J(R)$. Then, $\mathrm{M}_n(R; s)$ is a 2-ΔU ring if, and only if, R is 2-ΔU.*

Proof. If $n = 1$, then $\mathrm{M}_n(R; s) = R$. So, in this case, there is nothing to prove. Let $n = 2$. By the definition of $\mathrm{M}_n(R; s)$, we have $\mathrm{M}_2(R; s) \cong K_{s^2}(R)$. Apparently, $s^2 \in J(R) \cap C(R)$, so the claim holds for $n = 2$ with the help of Corollary 4.12.

To proceed by induction, assume now that $n > 2$ and that the claim holds for $M_{n-1}(R; s)$. Set $A := M_{n-1}(R; s)$. Then, $M_n(R; s) = \begin{pmatrix} A & M \\ N & R \end{pmatrix}$ is a *Morita context*, where

$$M = \begin{pmatrix} M_{1n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \quad \text{and} \quad N = (M_{n1} \ldots M_{n,n-1})$$

with $M_{in} = M_{ni} = R$ for all $i = 1, \ldots, n-1$, and

$$\psi : N \otimes M \to N, \quad n \otimes m \mapsto snm$$
$$\phi : M \otimes N \to M, \quad m \otimes n \mapsto smn.$$

Besides, for $x = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{n-1,n} \end{pmatrix} \in M$ and $y = (y_{n1} \ldots y_{n,n-1}) \in N$, we write

$$xy = \begin{pmatrix} s^2 x_{1n} y_{n1} & s x_{1n} y_{n2} & \cdots & s x_{1n} y_{n,n-1} \\ s x_{2n} y_{n1} & s^2 x_{2n} y_{n2} & \cdots & s x_{2n} y_{n,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ s x_{n-1,n} y_{n1} & s x_{n-1,n} y_{n2} & \cdots & s^2 x_{n-1,n} y_{n,n-1} \end{pmatrix} \in sA$$

and

$$yx = s^2 y_{n1} x_{1n} + s^2 y_{n2} x_{2n} + \cdots + s^2 y_{n,n-1} x_{n-1,n} \in s^2 R.$$

Since $s \in J(R)$, we see that $MN \subseteq J(A)$ and $NM \subseteq J(A)$. Thus, we obtain that

$$\frac{M_n(R; s)}{J(M_n(R; s))} \cong \frac{A}{J(A)} \times \frac{R}{J(R)}.$$

Finally, the induction hypothesis and Proposition 4.10 yield the claim after all. □

A *Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is called *trivial*, if the context products are trivial, i.e., $MN = 0$ and $NM = 0$. We now have

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N),$$

where $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is a trivial *Morita context* consulting with [12].

Corollary 4.14. *The trivial Morita context* $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ *is a 2-ΔU ring if, and only if, both A and B are 2-ΔU.*

Proof. It is apparent to see that the isomorphisms

$$\begin{pmatrix} A & M \\ N & B \end{pmatrix} \cong T(A \times B, M \oplus N) \cong \begin{pmatrix} A \times B & M \oplus N \\ 0 & A \times B \end{pmatrix}$$

are fulfilled. Then, the rest of the proof follows combining Proposition 4.6(i) and 3.4. □

As usual, for an arbitrary ring R and an arbitrary group G, the symbol RG stands for the *group ring* of G over R. Standardly, $\varepsilon(RG)$ designates the kernel of the classical *augmentation map* $\varepsilon : RG \to R$, defined by $\varepsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$, and this ideal is called the *augmentation ideal* of RG. A group G is called a *p-group* if every element of G is a power of the prime number p. Moreover, a group G is said to be *locally finite* if every finitely generated subgroup is finite.

Lemma 4.15. [26, Lemma 2]. *Let p be a prime with $p \in J(R)$. If G is a locally finite p-group, then $\varepsilon(RG) \subseteq J(RG)$.*

Proposition 4.16. *(i) If RG is a 2-ΔU ring, then R is also a 2-ΔU ring.*
(ii) If R is a 2-ΔU ring and G is a locally finite p-group, where p is a prime number such that $p \in J(R)$, then RG is a 2-ΔU ring.

Proof. (i) Assume $u \in U(R)$, then $u \in U(RG)$. Thus, $u^2 = 1 + r$ where $r \in \Delta(RG)$. Since $r = 1 - u^2 \in R$, it suffices to show that $r \in \Delta(R)$, which is obvious because for any $v \in U(R) \subseteq U(RG)$, we have $v - r \in U(RG) \cap R \subseteq U(R)$. Therefore, $r \in \Delta(R)$.

(ii) By Lemma 4.15, we have $\varepsilon(RG) \subseteq J(RG)$. On the other hand, since $RG/\varepsilon(RG) \cong R$, by Theorem 3.10, we conclude that RG is a 2-ΔU ring. $\square$

Proposition 4.17. *If the group ring RG is a 2-ΔU ring with $2 \in \Delta(RG)$, then G is a 2-group.*

Proof. We first consider two claims:

Claim 1: Every element $g \in G$ has a finite order.

Assume there exists $g \in G$ with infinite order. Since RG is a 2-ΔU ring, we have $1 - g^2 \in \Delta(RG)$. Given $2 \in \Delta(RG)$, we then have $1 + g^2 \in \Delta(RG)$, implying $1 + g + g^2 \in U(RG)$. Therefore, there exist integers $n < m$ and a_i with $a_n \neq 0 \neq a_m$ such that

$$(1 + g + g^2) \sum_{n}^{m} a_i g^i = 1.$$

This leads to a contradiction, thus every element $g \in G$ must have finite order.

Claim 2: For any $g \in G$ and $k \in \mathbb{N}$, we have $\sum_{i=0}^{n=2k} g^i \in U(RG)$.

We will prove this for $k = 1, 2$, and the rest follows similarly.

For $k = 1$, for any $g \in G$, we have $1 - g^2 \in \Delta(RG)$. Since $2 \in \Delta(RG)$, we then have $1 + g^2 \in \Delta(RG)$ and hence $1 + g + g^2 \in U(RG)$.

For $k = 2$, for any $g \in G$, since $g, g^2 \in U(RG)$, we have:

$1 - g^2 \in \Delta(RG)$ and hence $1 + g^2 \in \Delta(RG)$. Thus, $g + g^3 \in \Delta(RG)$. On the other hand $1 - g^4 \in \Delta(RG)$ and hence $1 + g^4 \in \Delta(RG)$.

Since $\Delta(RG)$ is closed under addition, it follows that $2 + g + g^2 + g^3 + g^4 \in \Delta(RG)$, implying $g + g^2 + g^3 + g^4 \in \Delta(RG)$ and thus $1 + g + g^2 + g^3 + g^4 \in U(RG)$.

Continuing this process, we can show that $\sum_{i=0}^{n=2k} g^i \in U(RG)$.

If $g \in G$ has an order p that does not divide 2, then p is odd, hence $p - 1 = 2k$. Consequently, $\sum_{i=0}^{2k} g^i \in U(RG)$, and since $(1 - g)(\sum_{i=0}^{2k} g^i) = 0$, we have $1 - g = 0$, which is a contradiction. Thus, G must be a 2-group. $\square$

5. Open Questions

We finish our work with the following questions which allude us.

A ring R is called UQ if $U(R) = 1 + QN(R)$ (see [7]).

Problem 5.1. Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + q$ where $q \in QN(R)$ (i.e., 2-UQ rings).

Problem 5.2. Characterize regular (or semi-regular) 2-UQ rings.

A ring R is called UNJ if $U(R) = 1 + Nil(R) + J(R)$ (see [14]).

Problem 5.3. Examine those rings R whose for each $u \in U(R)$, $u^2 = 1 + n + j$, where $n \in Nil(R)$ and $j \in J(R)$ (i.e., 2-UNJ rings).

Problem 5.4. Characterize regular (or semi-regular) 2-UNJ rings.

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