Referee's report on the manuscript "On algebras of binary formulas for weakly circularly minimal theories of finite convexity rank" by B. SH. Kulpeshov and S. V. Sudoplatov

This manuscript continues a series of papers that analyze the algebras of distributions of binary isolating formulas in various tame model theoretic settings. Given a non-algebraic 1-type $p(x)$ over $\emptyset$ realized in a structure M, one considers the formulas $\phi(x, y)$ such that for some (any) realization a of $p(x)$, $\phi(a, y)$ isolates a complete extension of $p(y)$ over $\{a\}$. Then one considers an equivalence relation on such formulas (logical equivalence of $\phi(a, y)$ and $\psi(a, y)$ for a realizing $p(x)$). The classes of this equivalence are then embedded in an algebra of *labels* with a composition-like operation. Specifically, composing two (equivalence classes of) binary isolating formulas $\phi(x, y)$ and $\psi(y, z)$ results in the set of all (classes of) isolating formulas $\theta(x, z)$ such that $\theta(x, z)$ implies $\exists y(\phi(x, y) \wedge \psi(y, z))$.

The authors consider the class of $\aleph_0$-categorical circularly ordered structures with the condition of weak circular minimality, a tameness condition similar to the classical notion of (weak) o-minimality. Further restrictions placed on the class are the finiteness of convexity rank, triviality of the definable closure, definability of a monotonic-to-right function, and the absence of a nontrivial equivalence relation with finitely many convex classes. The main result of the paper is the characterization of the algebras of binary isolating formulas under the conditions above. In particular, a sharp bound is obtained for the maximal size of a product of two labels.

I recommend this paper for publication in SEMR with minor revisions primarily aimed at improving the presentation and readability. My main suggestion is to provide more motivation for studying the algebras of binary formulas, the importance of the restrictions placed on the structures, and to provide more examples of the settings analyzed in this paper to clarify the very technical definitions and results. Are there any conjectures/open questions regarding commutativity/determinacy of the binary algebra? It would also be helpful if the authors explained how does describing the algebra help in classifying the structures satisfying the tameness conditions?

Below I list the line-by-line comments, corrections and suggestions.

- Abstract, line 5: not having
- 145^4: continued in [3-22]
- 145_{16}: circularly ordered structures that are not linearly ordered
- 145_{13}: What is K here? How is it defined in terms of L?
- 145_2: a better wording could be "was introduced in [15]"
- 146_{19}: Further, we need
- 146_7: (essentially,
- 147^{15}: Can the authors explain/motivate the notion of equivalence - generating convex-to-right formula? An example would also help the reader.
- 147^{24}: not necessarily in M
- 147^{26}: An additional explanation/motivation of the meaning of E^* would be very helpful.
- 148^{17}: What exactly is the characterization up to binarity, in case when the two structures have potentially different languages? Does it mean that both structures have the $\emptyset$-definable binary relations denoted as in the statement of the theorem, and the reducts to these relations are isomorphic?
- 148^{22}: Is property $dcl(\{a\}) = \{a\}$ referred to as *triviality* of dcl in this paper? Typically, the triviality of a closure operator cl refers to the property $cl(A) = A$ for any set, not just singletons. Is the term used in such a way because we are working in the binary context? Clarification would be helpful.

- 148_{11}: What is the order induced on the E_i-classes? Is it the one defined on page 147, line 5 from above? If so, a reference to the definition would help.

- 148_5: The last paragraph of Section 1 is hard to read (the last sentence is especially long). It would be helpful to outline the numerous conditions placed on the structures in form of a numbered/bulleted list, emphasizing the progression of complexity from the earlier work to the current paper. In addition, it would be really helpful to give a detailed definition of the algebra, the operation on labels and some basic examples.

- 149^{10}: and not having

- 149_{16}: Does the assertion made here follow directly from the description in Theorem 1, or is there a reference? It is not clear where these formulas come from. An explanation would help.

- 150^1: It would really help if the calculations were carried out in detail for selected products to illustrate the concept and the process.

- 150_{16}: Again, it is not clear how the list of isolating formulas is obtained. As in Example 2, in the proof of Theorem 2, it would improve the presentation if some special cases were treated in greater detail, while some similar ones were omitted. In its current form, the proof is hard to follow. Maybe splitting it into lemmas would help the presentation.

- 151^5: We will now prove that

- 151^7: First,