Referees report on the article
A.D. Mednykh, I.A. Mednykh, and G.K. Sokolova,
'One can hear a discrete rectangular torus'

The reviewed article 'One can hear a discrete rectangular torus' belongs to the topic of Spectral Graph Theory, where the key concept is the spectrum $Sp(G)$ of the graph G, namely the multiset of eigenvalues of its Laplacian matrix. The spectrum of a graph is not a complete invariant. The first example of cospectral or isospectral nonisomorphic undirected ordinary graphs was given by Collatz and Sinogowitz in 1957: $Sp\,(K_{1,4}) = Sp\,(C_4 \cup K_1)$ and $K_{1,4} \ncong C_4 \cup K_1$ [1]. On this topic, one can refer, for example, to the review [2].

The spectra of such classical objects as a complete graph, a simple path and cycle graph, a complete bipartite graph, etc. are well known. For example, in [3] the values of the spectrum of a simple n-vertex cycle graph C_n were given. Thus, the spectrum $Sp(C_n)$ consists of the values $4\sin^2\frac{\pi k}{n}$, $k = 1, 2, \ldots, n$. In other words, the spectrum $Sp(C_n)$ consists of the following values:

(i) for odd $n \geq 3$ from 0 and roots $\lambda_k = 4\sin^2\frac{\pi k}{n}$, $k = 1, 2, \ldots, \frac{n-1}{2}$ of multiplicity 2. In this case $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{\frac{n-1}{2}} < 4$.

(ii) for even $n \geq 4$ from $0, 4$ and roots $\lambda_k = 4\sin^2\frac{\pi k}{n}$, $k = 1, 2, \ldots, \frac{n}{2} - 1$ of multiplicity 2. In this case $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{\frac{n}{2}-1} < 4$.

(iii) in the reviewed article it is not specified, although it is implied, that by 2-vertex cycle graph C_2 the authors mean a 2-vertex graph with exactly 2 edges. The formula for the values of its spectrum is similar to the case $n \geq 3$: $Sp(C_2)$ also consists of the values $\lambda_k = 4\sin^2\frac{\pi k}{n}$, $k = 0, 1$ and $\lambda_0 = 0 < \lambda_1 = 4$.
Further, for the minimal positive value $4\sin^2\frac{\pi}{n}$ of the spectrum of n-vertex cycle graph C_n we will use the notation μ_n.

The Cartesian product of graphs is also well studied in both classical and spectral Graph Theory. In particular, the spectrum of a graph $G = G_1 \times G_2 \times \cdots \times G_m$ consists of the values: $Sp(G) = \{\lambda_1 + \lambda_2 + \cdots + \lambda_m \mid \lambda_i \in Sp(G_i), i = 1, 2, \ldots, m\}$ (see, for example, [2, 4]). For brevity, here and below for multisets we use the standard notations for sets and set-theoretic operations, implying the possibility of multiplicity of element values.

The article under review is devoted to establishing the equivalence of cospectrality and isomorphism for discrete tori. The justification of this fact, in a more straightforward way, is given in the next few lines.

Let us consider a discrete torus $T = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_m}$, where $n_i \geq 2$ for $i = 1, \ldots, m$. Let $n_{k_1} \geq n_{k_2} \geq \ldots \geq n_{k_m}$ and n be the number of elements of multiset $Sp(T)$. In steps $s = 1, 2, \ldots$ we define at step s an multiset S_s and express the value $\mu_{n_{k_s}}$ through the spectrum $Sp\,(T)$ and the multisets S_i, $i < s$. As a result, at step s we find the value n_{k_s}. Let $\lambda_0 = 0$.

Step $s \geq 1$. Let $\lambda = \min Sp(T) \setminus \{\lambda_0 + \lambda_1 + \cdots + \lambda_{s-1} | \lambda_i \in S_i \text{ if } 1 \leq i < s\}$. It is easy to see that $\lambda = \mu_{n_{k_s}}$. Now, knowing the root $\mu_{n_{k_s}} = 4\sin^2(\frac{\pi}{n_{k_s}})$, we calculate n_{k_s}. Next, we set $S_s = Sp(C_{n_{k_s}})$.

We calculate integers $n_{k_1}, n_{k_2}, \ldots$ step by step until their product becomes equal to n.
Thus, the spectrum $Sp(T)$ completely determines the values $n_{k_1}, n_{k_2}, \ldots, n_{k_m}$. Hence, cospectral discrete tori are isomorphic. Besides, the converse is valid in general.

The above justification for the statement of the reviewed article does not go beyond an university course on Spectral Graph Theory and is one of the advanced level exercises. This statement can be published in a tutorial for the university course on Spectral Graph Theory, for example, as an exercise. As for the publication in the scientific journal SEMR, the reviewer leaves the editorial board the possibility of this decision taking into account the conclusions presented.

References

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[2] B Mohar, The Laplacian spectrum of graphs, Graph theory, combinatorics, and applications, Vol. 2, Ed. Y. Alavi, G. Chartrand, O.R. Oellermann, A.J. Schwenk, 871-898, 1991.

[3] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, Linear and Multilinear Algebra, 18(2), 141–145, 1985.

[4] M. Fiedler, Algebraic connectivity of graphs, Czech. Math. J. 23(98), 298-305, 1973.