ONE CAN HEAR A DISCRETE RECTANGULAR TORUS

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Communicated by A.V. PYATKIN

Abstract: In the present paper, we prove that two discrete rectangular tori are isospectral if and only if they are isomorphic.

Keywords: Laplacian matrix, isospectral manifolds, isospectral graphs, discrete torus.

1 Introduction

In 1964, in his one-page paper "Eigenvalues of the Laplace Operator on Some Manifolds", J. Milnor [21] described an example of two 16-dimensional flat tori that are isospectral but not isometric. Milnor's article inspired Mark Kac to write his famous paper [14] entitled "Can one hear the shape of a drum?". Since these classical works, the question of what geometric properties of a manifold are determined by its Laplace operator has been the source of much research. Wolpert [28] showed that a generic Riemann surface is determined by its Laplace spectrum. Nevertheless, pairs of isospectral non-isometric Riemann surfaces in every genus ≥ 4 are known. See papers by Buser [5], Brooks and Tse [2], and others. There are also examples of isospectral non-isometric surfaces of genus two and three with variable curvature Barden and Hyunsuk Kang [1]. At the same time, R. Brooks [3] proved that

isospectral genus one Riemann surfaces (flat tori) are isometric. A similar result for the Klein bottle was obtained by R. Isangulov [13].

As we have seen, one-dimensional isospectral flat tori are always isometric, so a natural question is, What is the lowest dimension in which there are isospectral non-isometric flat tori? Following Milnor, the search for pairs of isospectral but non-isometric flat tori became a race towards the lowest possible dimension. Kneser found a 12-dimensional example in 1967 [16]. Ten years later, Kitaoka [15] reduced this to eight. In 1986, Conway and Sloane [6] found five- and six-dimensional examples. In 1990, Schiemann [25] constructed a four-dimensional example. Independently, and using a different approach, Shiota [26] found another example one year later in 1991. The same year, Earnest and Nipp [8] contributed one more pair.

Similar results are also known for graphs. The problem of how graphs are defined by their spectrum is the subject of many studies (see survey by E.R.van Dam and W.H.Haemers [7]). Peter Buser [4] posed an interesting problem: are two isospectral Riemann surfaces of genus two isometric? Up to our knowledge the problem is still open but, quite probably, can be solved positively. Because of the intrinsic link between Riemann surfaces and graphs, we hope that our result will be helpful to make a progress in solution of the Buser problem. More precisely, in the papers [20] and [18] the discrete counterpart for genus two Riemann surface was considered. This role is played by the so-called theta graphs. In these papers, In these papers, it was shown that two isospectral theta graphs are isomorphic.

Recently, the following result was obtained. Consider a *rectangular lattice* Γ in $\mathbb{R}^n$ with diagonal basis matrix. Define a *rectangular flat torus* as the associated flat torus $\mathbb{R}^n/\Gamma$. Then, by ([23], Theorem 2.35), two rectangular flat tori are isospectral if and only if they are isometric.

The aim of the present paper is to establish the similar result for discrete rectangular tori.

2   Preliminary results and definitions

2.1. Laplacian for graphs and isospectrality.   The graphs in this paper are undirected. Denote by $V(G)$ and $E(G)$, respectively, the set of vertices and edges of a graph G. For each $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between vertices u and v. The matrix $A = A(G) = \{a_{uv}\}_{u,v \in V(G)}$ is called the *adjacency matrix* of the graph G. Let $d(v)$ denote the valency of $v \in V(G)$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.

The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. The Laplacian matrix $L(G)$ is a $n \times n$ square matrix, where $n = |V(G)|$ is the number of vertices in G. Throughout the paper we shall denote by $\chi(G, x)$ the characteristic polynomial of $L(G)$. That is $\chi(G, \lambda) = \det(L(G) - \lambda I_n)$, where I_n is the identity $n \times n$ matrix. For brevity, we will call $\chi(G, x)$ the *Laplacian polynomial* of G. The roots of $\chi(G, x)$ will be called the *Laplacian*

eigenvalues (or just eigenvalues) of G. They are all non-negative numbers and can be arranged as follows

$$0 = \lambda_1(G) \le \lambda_2(G) \le \ldots \le \lambda_n(G),$$

repeated according to their multiplicity. Recall that for connected graph G we always have $\lambda_1(G) = 0$ and $\lambda_2(G) > 0$.

Two graphs G and H are called *isospectral* if their Laplacian polynomials coincide $\chi(G, \lambda) = \chi(H, \lambda)$.

2.2. Cartesian product and uniqueness of decomposition. The Cartesian product of graphs G and H is a graph, denoted $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and h is adjacent to h' in H, or $h = h'$ and g is adjacent to g' in G. A non-trivial graph G is called *prime* if $G = G_1 \times G_2$ implies that either G_1 or G_2 is a one vertex graph K_1. It is easy to see that every non-trivial finite graph has a prime factorization with respect to the Cartesian product. The number of prime factors is at most $\log_2 |V(G)|$.

The uniqueness of the prime factor decomposition of connected graphs with respect to the Cartesian product was first shown by Sabidussi [24], and independently by Vizing [27]. More precisely, the following theorem holds.

Theorem 1 (Sabidussi-Vizing). *A connected graph has a unique representation as a product of prime graphs, up to isomorphism and the order of the factors.*

See book ([12], Charter 6) for the proof of this theorem.

2.3. Theta function of graphs. Let G be a finite graph. Denote by S_G the Laplacian spectrum of G. Define $\Theta_G(t)$ function of graph G in the following way

$$\Theta_G(t) = \sum_{\lambda \in S_G} e^{-\lambda t}.$$

According to R. Brooks [3], we note that function $\Theta_G(t)$ completely defines spectrum of G. Indeed, suppose that

$$\Theta_G(t) = c_1 e^{-\mu_1 t} + c_2 e^{-\mu_2 t} + \ldots + c_\ell e^{-\mu_\ell t}, \ 0 \le \mu_1 < \mu_2 < \ldots < \mu_\ell.$$

Here μ_j are all elements of S_G taken without repetitions and c_j theirs respective multiplicities.

The multiplicities c_j of eigenvalues μ_j are completely defined by $\Theta_G(t)$, because all functions $e^{-\mu_j t}$ are linearly independent. To find exact values of c_j and μ_j we consider the expression

$$e^{\mu t} \Theta_G(t) = c_1 e^{(\mu - \mu_1)t} + c_2 e^{(\mu - \mu_2)t} + \ldots + c_\ell e^{(\mu - \mu_\ell)t}.$$

We can conclude that $\mu_1 = \max_\mu \{ \lim_{t \to \infty} (e^{\mu t} \Theta_G(t)) \}$ is finite. Putting $\mu = \mu_1$ we have $\lim_{t \to \infty} e^{\mu t} \Theta_G(t) = c_1$.

Assume that we already found the initial values $(c_1, \mu_1), (c_2, \mu_2), \ldots, (c_k, \mu_k)$ for $k < \ell$. Then μ_{k+1} is equal to $\max_{\mu}\{\lim_{t\to\infty}(e^{\mu t}(\Theta_G(t) - \sum_{j=1}^{k} c_j e^{-\mu_j t}))\}$ is finite.

Its corresponding multiplicity c_{k+1} is equal to $\lim_{t\to\infty} e^{\mu_{k+1} t}(\Theta_G(t) - \sum_{j=1}^{k} c_j e^{-\mu_j t})$.

As a result, we get the following lemma.

Lemma 1. *Two finite graphs G_1 and G_2 are isospectral if and only if $\Theta_{G_1}(t) = \Theta_{G_2}(t)$.*

Moreover, we can derive one more important property of Θ-function for graphs.

Lemma 2. *Let G_1 and G_2 are two finite graphs. Denote by $G_1 \times G_2$ the Cartesian product of G_1 and G_2. Then $\Theta_{G_1 \times G_2}(t) = \Theta_{G_1}(t)\,\Theta_{G_2}(t)$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the spectrum of G_1 and $\mu_1, \mu_2, \ldots, \mu_n$ be the spectrum of G_2 By [22] the spectrum of Cartesian product $G_1 \times G_2$ is given by the set $\{\lambda_i + \mu_j, 1 \le i \le m, 1 \le j \le n\}$.

Let $G = G_1 \times G_2$. We have

$$\Theta_G(t) = \sum_{\lambda \in S_G} e^{-\lambda t} = \sum_{i=1}^{m} \sum_{j=1}^{n} e^{-(\lambda_i + \mu_j)t} = \sum_{i=1}^{m} e^{-\lambda_i t} \sum_{j=1}^{n} e^{-\mu_j t} = \Theta_{G_1}(t)\,\Theta_{G_2}(t).$$

$\square$

2.4. Discrete tori. We define a *discrete rectangular torus* as the Cartesian product $C_{m_1} \times C_{m_2} \times \ldots \times C_{m_p}$, where C_m is a cyclic graph on m vertices. Excluding trivial factors we assume that $m_i \ge 2$, $i = 1, 2, \ldots, p$. In what follows, instead of the long term "discrete rectangular torus", we will freely use the terms "discrete torus" or simply "torus".

The Cartesian product of graphs is commutative and associative in the following sense. For any two graphs G_1, G_2 the products $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic. Given graphs G_1, G_2 and G_3, the map $((x_1, x_2), x_3) \to (x_1, (x_2, x_3))$ is an isomorphism $(G_1 \times G_2) \times G_3 \to G_1 \times (G_2 \times G_3)$.

Associativity and commutativity gives us permission to omit parentheses when dealing with products with more than two factors. This allows to present a discrete rectangular torus in the following form

$$T = T_{m_1, \ldots, m_p} = C_{m_1} \times C_{m_2} \times \ldots \times C_{m_p}, \text{ where } 1 < m_1 \le m_2 \le \ldots \le m_p.$$

By making use of Sabidussi-Vizing theorem, one can deduce that the latter representation of T is unique. Indeed, by Unique Square Lemma ([12], Lemma 6.3), every nontrivial Cartesian product $G_1 \times G_2$ contains a square subgraph $C_4 = K_2 \times K_2$. Thus, a cyclic graph C_m is prime for any $m \neq 4$. Then, by replacing C_4 with two copies of K_2 in the above representation, we obtained an irreducible Cartesian product of prime graphs. According to the Sabidussi-Vizing theorem, it is unique. Replacing back the double

product $K_2 \times K_2$ with one copy of C_4 we have the uniqueness of the required expression.

For convenience, we will call the parameter p the *dimension* of the torus T. As a result, we have the following important property.

Lemma 3. *Any discrete rectangular torus T of dimension p can be uniquely represented in the form*

$$T = C_{m_1} \times C_{m_2} \times \ldots \times C_{m_p}, \ \ where \ 1 < m_1 \leq m_2 \leq \ldots \leq m_p.$$

2.5. Spectrum of discrete torus. We will use the following observation from [22]. The Laplacian eigenvalues of the Cartesian product $G_1 \times G_2$ of graphs G_1 and G_2 are equal to all the possible sums of eigenvalues of the two factors: $\lambda_i(G_1) + \lambda_j(G_2)$, $i = 1, \ldots, |V(G_1)|$, $j = 1, \ldots, |V(G_2)|$.

Let G be a connected graph on n vertices and

$$0 = \lambda_1(G) < \lambda_2(G) = a(G) \leq \lambda_3(G) \leq \ldots \leq \lambda_n(G)$$

be its Laplacian spectrum. Let us call the second smallest eigenvalue $a(G)$ of the spectrum an *algebraic connectivity* of the graph G. Following to [9], from the above observation we note that $a(G_1 \times G_2) = \min(a(G_1), a(G_2))$.

We set $G_i = C_{m_i}$, for $i = 1, \ldots, p$. Then for any i spectrum of G_i is given by the list

$$\{4\sin^2(\frac{\pi j}{m_i}), \ j = 0, 1, \ldots, m_i - 1\}.$$

Hence, $a(G_i) = 4\sin^2(\frac{\pi}{m_i})$. Now, consider the discrete torus

$$T = T_{m_1, \ldots, m_p} = C_{m_1} \times C_{m_2} \times \ldots \times C_{m_p}, \ where \ 1 < m_1 \leq m_2 \leq \ldots \leq m_p.$$

Then, because of associativity of Cartesian product, spectrum of T is given by

$$\{\sum_{i=1}^{p} 4\sin^2(\frac{\pi j_i}{m_i}), \ i = 1, \ldots, p, \ j_i = 0, 1, \ldots, m_i - 1\}.$$

Also,

$$a(T) = \min(a(G_1), a(G_2), \ldots, a(G_p)) = 4\sin^2(\frac{\pi}{m_p}).$$

2.6. One can hear the dimension of a discrete torus. In this point, we prove that one can hear the dimension of a discrete torus. In other words, if two tori are isospectral, then they have the same dimension. More precisely, we get the following statement.

Proposition 1. *Consider two isospectral disctere rectangular tori $T_{m_1, \ldots, m_p}$, $1 < m_1 \leq \ldots \leq m_p$ and $T_{\tilde{m}_1, \ldots, \tilde{m}_{\tilde{p}}}$, $1 < \tilde{m}_1 \leq \ldots \leq \tilde{m}_{\tilde{p}}$. Then $p = \tilde{p}$.*

Proof. If $p = \tilde{p}$, we have what needs to be done. Without loss of generality, we can assume that $p < \tilde{p}$. Since the tori $T = T_{m_1, \ldots, m_p}$ and $\tilde{T} = T_{\tilde{m}_1, \ldots, \tilde{m}_{\tilde{p}}}$ are isospectral, they share the algebraic connectivity $a(T) = a(\tilde{T})$, which is certainly a spectral invariant. Therefore, $4\sin^2(\frac{\pi}{m_p}) = 4\sin^2(\frac{\pi}{\tilde{m}_{\tilde{p}}})$, that gives us $m_p = \tilde{m}_{\tilde{p}}$.

Recall the number of vertices $m_1 \cdot \ldots \cdot m_p$ of the torus $T_{m_1,\ldots,m_p}$ is also a spectral invariant.

We represent tori T and $\tilde{T}$ as the Cartesian product $T = T_{m_1,\ldots,m_{p-1}} \times C_{m_p}$ and $\tilde{T} = T_{\tilde{m}_1,\ldots,\tilde{m}_{\tilde{p}-1}} \times C_{m_p}$. Since T and $\tilde{T}$ are isospectral, by Lemma 1 we have $\Theta_T(t) = \Theta_{\tilde{T}}(t)$. By Lemma 2, $\Theta_T(t) = \Theta_A(t)\Theta_{C_{m_p}}(t)$ and $\Theta_{\tilde{T}}(t) = \Theta_B(t)\Theta_{C_{m_p}}(t)$, where $A = T_{m_1,\ldots,m_{p-1}}$ and $B = T_{\tilde{m}_1,\ldots,\tilde{m}_{\tilde{p}-1}}$. Since $\Theta_{C_{m_p}}(t)$ is strictly positive, we have $\Theta_A(t) = \Theta_B(t)$. That is, graph A and B are isospectral and $a(A) = a(B)$. Hence, $m_{p-1} = \tilde{m}_{\tilde{p}-1}$ and, by the same arguments, $T_{m_1,\ldots,m_{p-2}}$ and $T_{\tilde{m}_1,\ldots,\tilde{m}_{\tilde{p}-2}}$ are isospectral. Repeating the process, we obtain that $m_1 = \tilde{m}_{\tilde{p}-p+1}$ and tori T_{m_1} and $T_{\tilde{m}_1,\ldots,\tilde{m}_{\tilde{p}-p+1}}$ are isospectral. The latter is impossible, since the number of vertices m_1 of the first torus is strictly less than $\tilde{m}_1 \cdot \ldots \cdot \tilde{m}_{\tilde{p}-p+1}$ of the second one. $\qquad\square$

3   Main results

The main result of the paper is the following theorem.

Theorem 2. *Two discrete rectangular tori are isospectral if and only if they are isomorphic.*

Proof. If two tori are isomorphic, then up to the enumeration of vertices they have the same Laplace matrix and, therefore, have the same Laplace spectrum. That is, they are isospectral.

Conversely, suppose that two discrete tori $T_{m_1,\ldots,m_p}$, $1 < m_1 \leq \ldots \leq m_p$ and $T_{\tilde{m}_1,\ldots,\tilde{m}_{\tilde{p}}}$, $1 < \tilde{m}_1 \leq \ldots \leq \tilde{m}_{\tilde{p}}$ are isospectral. From Proposition 1 we already know that $p = \tilde{p}$.

Now, to proof the isomorphism between $T = T_{m_1,\ldots,m_p}$ and $\tilde{T} = T_{\tilde{m}_1,\ldots,\tilde{m}_p}$ we can use induction on the dimension p. For $p = 1$, isospectral tori $T_{m_1} = C_{m_1}$ and $T_{\tilde{m}_1} = C_{\tilde{m}_1}$ are cyclic graphs with the same number of vertices $m_1 = \tilde{m}_1$ and, therefore, are isomorphic.

By induction, suppose that the theorem is proved for all $k < p$, where $p \geq 2$. Let us prove it for $p = k$. We represent tori T and $\tilde{T}$ as the Cartesian product $T = A \times C_{m_p}$ and $\tilde{T} = B \times C_{\tilde{m}_p}$, where $A = T_{m_1,\ldots,m_{p-1}}$ and $B = T_{\tilde{m}_1,\ldots,\tilde{m}_{p-1}}$ are respective $(p-1)$-dimensional tori. Repeating the arguments in the proof of Proposition 1, we note that the minimal non-zero eigenvalue is common to graphs T and $\tilde{T}$ and is equal to $4\sin^2(\frac{\pi}{m_p}) = 4\sin^2(\frac{\pi}{\tilde{m}_p})$. Hence, $m_p = \tilde{m}_p$. Since T and $\tilde{T}$ are isospectral, by Lemma 1 we have $\Theta_T(t) = \Theta_{\tilde{T}}(t)$. By Lemma 2, $\Theta_T(t) = \Theta_A(t)\Theta_{C_{m_p}}(t)$ and $\Theta_{\tilde{T}}(t) = \Theta_B(t)\Theta_{C_{m_p}}(t)$. Since $\Theta_{C_{m_p}}(t) > 0$, we have $\Theta_A(t) = \Theta_B(t)$. That is, graph A and B are isospectral. By inductive hypothesis, they are isomorphic. Then $T = A \times C_{m_p}$ and $\tilde{T} = B \times C_{m_p}$ are also isomorphic. $\qquad\square$

4   Conclusion

The simplicity of the proof of Theorem 2 is deceptive. Such statements do not hold true even for graphs with a simple cyclic structure. For example, we

can consider circulant graph $G = C_n(s_1, s_2, \cdots, s_k)$, where $0 < s_1 < s_2 < \ldots < s_k < n/2$. Its Laplacian spectrum is well known. It is given by the list $S_G = \{\sum_{i=1}^{k} 4\sin^2(\frac{\pi s_i j}{n}), j = 0, 1, \ldots, n-1\}$. It was shown by Godsil, Holton and McKay [11], that graphs $C_1 = C_{20}(2, 3, 4, 7)$ and $C_2 = C_{20}(3, 6, 7, 8)$ are isospectral but not isomorphic. Moreover, there are no cospectral circulant graphs on less than twenty vertices.

The discrete rectangular torus considered in this paper is the simplest discrete analogue of a continuous multidimensional torus. There is a more general aspect of the discrete torus that emulates the general continuous torus coming from a multidimensional parallelepiped. See, for instance papers [19] and [17]. The structure of Laplacian operator as well as its spectrum is known in this case. Up to our knowledge, the problem of isospectrality of such tori has not been solved.

The same problem naturally arises for connection Laplacian in vector bundles on discrete tori. The properties of the connection Laplacian are the subject of investigation of many recent papers [10], [17] and others.

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