DEFINABLE DINI'S THEOREM AND NETS IN SOME STRUCTURES

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Abstract

In this paper, we are going to prove a new formula of Dini's Theorem in a definably complete expansion of an ordered group, afterwards we will give some properties for cluster points of a given definable net.

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1. Introduction

Local o-minimal structures localizes the definition of o-minimal structures. They were first studied by Toffalori and Vozoris [5]. We recall the well known formula of Dini's theorem : If X is a compact topological space, and $(f_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence (meaning $f_n(x) \leq f_{n+1}(x)$ for each $n \in \mathbb{N}$ and $x \in X$) of continuous real-valued functions on X which converges pointwise to a continuous function $f\colon X \to \mathbb{R}$, then the convergence is uniform. In the third section, we will give and prove a new Dini's formula in case of definably complete expansion of an ordered group (see Corollary 3.2 bellow).

The nets are generalizations of sequences: rather than being defined on a countable linearly ordered set $(\mathbb{N}, \leq)$. Recall that a preorder is a transitive and reflexive binary relation. A downward directed set $(\Omega, \leq)$ is a nonempty set with a preorder $\leq$ such that, for any finite subset Ω' of Ω there exists $v \in \Omega$ satisfying the relation $v \leq u$ for all $u \in \Omega'$. The dual notion of upward directed set is defined analogously. We mainly consider a downward directed set in this paper.

We simply call it a directed set.
Let $\mathcal{M} = (M, <, ...)$ be an expansion of a dense linear order without endpoints. A definable preordered set is a definable set $\Omega \subseteq M^n$ together with a definable preorder $\leq$ on Ω. A definable direct set is a definable preordered set which is directed. A net is defined on an arbitrary directed set.

In the last section, we are going to characterize some definable nets and their cluster points as elements of their sets of left convergences (see Proposition 3.2 bellow), also we will study some properties of such nets and the sets of their cluster points in some definable topological spaces by adopting the notations and the definitions given in [2].

2. Preliminaries

Definition 2.1. ([5]). A densely linearly ordered structure without endpoints $\mathcal{M} = (M, <, ..)$ is locally o-minimal if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals.

Definition 2.2. ([4]). An expansion of a densely linearly ordered set without endpoints $\mathcal{M} = (M, <, ...)$ is definably complete if any definable subset X of M has the supremum and infimum in $M \cup \{\pm\infty\}$.

Definition 2.3. ([1]). Consider an expansion of an ordered field $\mathcal{M} = (M, <, +, \cdot, ...)$. Let $C \subseteq M$ be a definable subset and s be a positive element in M. A family of functions $\{f_t : C \to M\}_{0<t<s}$ with the parameter variable t is said to be uniformly convergent if for all positive $\epsilon > 0$, there exists $s' > 0$ such that $|f_t(x) - f_{t'}(x)| < \epsilon$ for all $x \in C$ and $t, t' \in]0, s'[$.

Definition 2.4. For a set X, a family F of subsets of X is called a filtered collection if, for any $B_1, B_2 \in F$ there exists $B_3 \in F$ with $B_3 \subseteq B_1 \cap B_2$.
Consider an expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, ...)$. Let T be a definable set of M^m. A parameterized family $\{S_t\}_{t\in T}$ of subsets of M^n is called definable if the union $\cup_{t\in T}\{t\} \times S_t \subseteq M^{m+n}$ is definable.
A parameterized family $\{S_t\}_{t\in T}$ of subsets of M^n is a definable filtered collection if it is simultaneously definable and a filtered collection.

Definition 2.5. Let $\mathcal{M} = (M, <)$ be an expansion of a dense linear order without endpoints. Consider a definable topological space (X, τ).

A definable net in X is a definable map from a definable directed set $(\Omega, \leq)$ to X. A definable net $\gamma : (\Omega, \leq) \to (X, \tau)$ converges to $x \in X$ if, for every definable neighborhood A of x, there exists $u_A \in \Omega$ such that $\gamma(u) \in A$ whenever $u \leq u_A$.
A definable subnet $\gamma^{'}$ of a definable net γ is a definable net of the from $\gamma^{'} = \gamma \circ f$ where $f : (\Omega^{'}, V^{'}) \to (\Omega, \leq)$ is a definable downward cofinal map (the witnessing map).

3. Definable Dini's Theorem

The target of this section is to exhibit a formula of Dini's Theorem for an arbitrary definably complete expansion of an ordered group.

A definable set X is definably compact if every definable filtered collection of closed nonempty subsets of X has a nonempty intersection. This definition is found in [3, Section 8.4].

Theorem 3.1. *(Definable Dini's Theorem). Consider a definably complete expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definably compact definable set and I be an open interval. Let $f : C \times I \to M$ be a definable function satisfying the following conditions:*

1. *For any $t \in I$, the definable function $f_t : C \to M$ defined by $f_t(x) = f(x, t)$ is continuous.*
2. *For any $x \in C$, the inequality $f(x, t_1) \leq f(x, t_2)$ holds whenever $t_1 \leq t_2$.*
3. *For any $x \in C$, we have $\sup\{f(x, t) \mid t \in I\} < \infty$ and the definable function $g : C \to M$ defined by $g(x) = \sup\{f(x, t) | t \in I\}$ is continuous.*

Then, for any positive $\epsilon > 0$, there exists $c \in I$ such that $g(x) - f(x, t) < \epsilon$ whenever $x \in C$ and $t > c$.

Proof. Consider the definable function $h_t : C \to M$ defined by $h_t(x) = g(x) - f(x, t)$ for each $t \in I$.
Note that $h_t(x)$ is nonnegative by condition 3 and it is continuous by conditions 1 and 3. Fix $\epsilon > 0$. For each $t \in C$, we define $D_t := \{x \in C \mid h_t(x) \geq \epsilon\}$. We want to show that D_c is an empty set for some $c \in I$. Assume for contradiction that D_t is not empty for each $t \in I$.

We prove that the definable parametrized family $\{D_t \mid t \in I\}$ is a definable filtered collection of nonempty definable closed sets. Since h_t

is continuous, D_t is closed. When $t_1 \leq t_2$, we have $h_{t_1}(x) - h_{t_2}(x) = f_{t_2}(x) - f_{t_1}(x) \geq 0$ for each $x \in C$. It means that $h_{t_1}(x) - h_{t_2}(x)$ whenever $t_1 \leq t_2$ and $x \in C$. It implies that $D_{t_2} \subseteq D_{t_1}$ whenever $t_1 \leq t_2$. We have shown that $\{D_t | t \in I\}$ is a definable filtered collection of nonempty definable closed sets.

Since C is definably compact, there exists $x^* \in \cap_{t \in I} D_t$. It implies that $h_t(x^*) = g(x^*) - f(x^*, t) \geq \epsilon$ for every $t \in I$, which contradicts the definition of g given in 3.

□

Corollary 3.2. *If in a definably complete expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$ conditions 1 and 2 of Theorem 3.1 are satisfied, suppose there exists $s > 0$ such that the set $\{t \in (0, s); f(x, t)\}$ is bounded and that $g(x) := \inf\{t \in (0, s); f(x, t)\}$ is continuous on a definably compact definable set C, then the function $f(x, t)$ converges uniformly to $g(x)$ as $t \to 0^+$.*

Proof. Set $I = (0, s)$ and $J = \{-t \mid t \in I\}$, and by taking the function F definable over $C \times J$ by

$F := -f(x, -t)$, we clearly gain the assumptions (1) and (2) of the previous theorem.

For condition 3; we have that $\sup\{t \in J; -f(x, -t)\} = -\inf\{t \in J; f(x, -t)\} < +\infty$.

So, by theorem 3.1 for any positive $\epsilon > 0$, there exists $c \in J$ such that $g(x) + f(x, -t) < \epsilon$ whenever $x \in C$ and $t > c > 0$.

Then for any positive $\epsilon > 0$, there exists $c \in J$ such that $g(x) + f(x, -t) < \epsilon$ whenever $x \in C$ and $-c > -t > 0$.

Consequently, for any positive $\epsilon > 0$, there exists $c \in I =]0, s[$ such that $g(x) + f(x, t) < \epsilon$, whenever $x \in C$ and $c > t > 0$.

And we get the result according to the definition of uniform convergence of $-f$ and so does f.

□

4. The cluster points of a definable net

A definable curve is a definable map defined on an interval or its image. Note that we do not require that a definable curve is continuous in this paper. We denote by $\mathcal{V}(x)$ the set of all definable neighborhood of x; and (X, τ) a definable topological space.

We define the set of left convergences of a definable curve $\gamma : (a, b) \to (X, \tau)$ with $a \in M \cup \{-\infty\}$ and $b \in M \cup \{+\infty\}$ as follows:

$\mathrm{Conv}_{left}(\gamma) = \{x \in X \mid \forall t \in (a,b)$, for all definable neighborhood of x, $\gamma((a,t)) \cap A \neq \emptyset\}$.

A point $x \in X$ is called a cluster point of a definable net γ if, there exists a definable subnet γ' of γ which converges to x.

Lemma 4.1. *Let $\gamma : (\Omega, \leq) \to (X, \tau)$ be a definable net in an expansion of a dense linear order without endpoints that converges to x, then so does every definable subnet.*

Proof. Every definable subnet is of the form $\gamma^{'} = \gamma o f$; where, f is a downward cofinal definable map $f : (\Omega^{'}, \leq') \to (\Omega, \leq)$.
As γ converges to x, we have that

$$\forall V \in \mathcal{V}(x), \exists a_1 \in \Omega \text{ such that for all } a \leq a_1, \text{ we have } \gamma(a) \in V \quad (*).$$

As $a_1 \in \Omega$, f is downward cofinal, $\exists a' \in \Omega^{'}$ such that if $y \leq' a'$, we have $f(y) \leq a_1$. So $\gamma o f(y) \in V$ by $(*)$. This proves the lemma.

□

Proposition 4.2. *Let $\mathcal{M} = (M, <, ...)$ be an expansion of a dense linear order without endpoints; (X, τ) a definably compact topological space and $\gamma : (\Omega, \leq) \to (X, \tau)$ a definable net that has unique cluster point, then γ converges to x.*

Proof. Assume otherwise, $\exists V \in \mathcal{V}(x)$ such that for all $y \in \Omega$, $\exists x_y \leq y$ such that $\gamma(x_y) \notin V$; by definable choice in [2, Lemma 2.3], the map $\alpha : (\Omega, \leq) \to (\Omega, \leq), y \mapsto x_y$ is definable such that

$$\gamma(\alpha(x)) \notin V; \quad (1).$$

Let $b \in \Omega$, then if $y \leq b$, then $\alpha(y) = x_y \leq y \leq b$. So α is a a downward cofinal map and therefore $\gamma^{'} = \gamma \circ \alpha$ is a definable subnet of γ. By [2, theorem 4.6], $\gamma^{'}$ admits a subnet $\gamma^{'} \circ \beta = \gamma \circ \alpha \circ \beta$ (where $\beta : (\Omega_1, \leq^1) \to (\Omega, \leq)$) which converges also to x thanks to Lemma 4.1.

So $\forall V \in \mathcal{V}(x), \exists a \in \Omega_1$, $\forall y \leq^1 a$, we have that

$$\gamma \circ \alpha \circ \beta(y) \in V; \quad (2).$$

So $\gamma \circ \alpha(\beta(a)) \in V$. By taking in (1) $x = \beta(a) \in \Omega$.
Which is a contradiction by (1) and (2).

□

The following Proposition generalizes [2, Lemma 4.20] to an arbitrary definable topological space (X, τ).

Proposition 4.3. *Let $\mathcal{M} = (M, <, +, 0, ...)$ be a definably complete locally o-minimal expansion of an ordered group. Let $\gamma : ((a, b), \leq) \to (X, \tau)$ be a definable net then, x is a cluster point of the net γ if and only if $x \in Conv_{left}(\gamma)$.*

Proof. $\Leftarrow)$: Assume that $x \in \text{Conv}_{left}(\gamma)$, and let's find a definable subnet γ' that converges to x.
Set $D_x = \{U \in \tau : x \in U\}$ be the collection of all definable basis of neighborhoods of X at the point x.
Clearly $(D_x, \supseteq)$ is a directed set. Define the set:
$\mathcal{E} := \{(d, U) \in (a, b) \times D_x : \gamma(d) \in U\}$

We endow the set $\mathcal{E}$ with the ordering $(d, U) \leq_0 (e, V)$ if and only if $d \leq e$ and $U \subseteq V$. So $(\mathcal{E}, \leq_0)$ is clearly reflexive and transitive.

For directedness, suppose we have (d_1, U_1) and (d_2, U_2) in $(\mathcal{E}, \leq_0)$. Let $V = U_1 \cap U_2$, and note that $x \in V$. By [2, Lemma 3.2], we get the directedness of $((a, b), \leq)$, so there exists an $e \in (a, b)$ such that $e \leq d_1$ and $e \leq d_2$. Since $\text{Conv}_{left}(\gamma) \neq \emptyset$, there exists an $a < e' < e$ such that $\gamma(e^{'}) \in V$. Therefore $(e', V) \in \mathcal{E}$, and $(e', V) \leq_0 (d_1, U_1)$ and $(e', V) \leq_0 (d_2, U_2)$. So $(\mathcal{E}, \leq_0)$ is a directed set.

Define $h : (\mathcal{E}, \leq_0) \to ((a, b), \leq)$ by $h(d, U) = d$. h is cofinal. In fact, let $d_1 \in (a, b)$; X is a definable neighborhood of x, so $(d_1, X) \in \mathcal{E}$. So for all $(d, U) \leq_0 (d_1, X)$, we get $h(d, U) = d \leq d_1$. Let's define the subnet $\gamma' : (\mathcal{E}, \leq_0) \to (X, \tau)$ by $\gamma'(d, U) := \gamma(h(d, U)) = \gamma(d) \in X$. So $\gamma' = \gamma \circ h$.
The set $\mathcal{E} \times (a, b)$ is clearly definable, and so is the function h as h is surjective.

Claim : γ' converges to x. In fact, set U a definable neighborhood x, we have $x \in \text{Conv}_{left}(\gamma)$, so there is $d \in (a, b)$ such that $\gamma(d) \in U$. But then $(d, U) \in \mathcal{E}$ and if $(e, V) \in \mathcal{E}$ such that $(e, V) \leq_0 (d, U)$, then in particular $V \subseteq U$, so $\gamma'(e, V) = \gamma(e) \in V \subseteq U$.

$\Rightarrow)$: By the proof of [2, Theorem 4.6] (condition (2) implies condition (1)).

□

Corollary 4.4. *Let $\mathcal{M} = (M, <, +, 0, ...)$ be a definably complete locally o-minimal expansion of an ordered group and $\gamma : ((a, b), \leq) \to (X, \tau)$ be a definable net and V be a definable neighborhood of x, if x*

is a cluster point of the net γ, then the set $A := \{y \in (a,b); \gamma(y) \in V\}$ is infinite.

Proof. $\Rightarrow$): By proposition 4.3, $x \in \text{Conv}_{left}(\gamma)$. So for all $a < t < b$, $\gamma((a,t)) \cap V \neq \phi$, so $\exists x_{1,t} \in (a,t)$ such that $\gamma(x_{1,t}) \in V$, we repeat the same with $(a, x_{1,t})$ we get $\exists x_{2,t} \in (a, x_{1,t})$ such that $\gamma(x_{2,t}) \in V$, and so on... We finally get an infinite set of elements in A.

□

Remark 4.1. The converse of Corollary 4.4 is not true. In fact,
Let's take $X = \mathbb{R}$, τ the usual topology, and γ the usual inclusion.
Set $x = 0, V = \mathbb{R}, (a,b) = (1,2)$, and $<$ the usual order.
We get that A is infinite, but x is not a cluster point.

Proposition 4.5. *Consider $\mathcal{M} = (M, <, ...)$ be an expansion of a dense linear order without endpoints. If every definable net in a definable topological space (X,τ) converges to at most one point, then (X,τ) is Hausdorff.*

Proof. We prove this by contrapositive. Suppose (X,τ) is not Hausdorff, and we will show that there is a definable net converging to more than one point. There are two points $x, y \in X$ such that any pair of definable neighborhoods U and V containing x and y, respectively, intersect one another. Again using the idea of the directed set $D_x = \{U \in \tau : x \in U\}$ of all definable basis of neighboorhood of x with the superset relation, define the directed set $D = D_x \times D_y$, with $(U_1, V_1) \leq (U_2, V_2)$ if and only if $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$.

For each $U \in D_x$ and $V \in D_y$, applying [2, Lemma 2.3] and let's fix a point $x_{U,V} \in U \cap V$. Then the function $f : D \to X$ given by $f(U,V) = x_{U,V}$ is a definable net that converges to both x and y. In fact, Take a definable neighborhood U_0 of x. Then $U_0 \in D_x$, and $(U_0, X) \in D$. Note that X is itself an open set containing y. Take any $(U,V) \in D$ such that $(U,V) \leq (U_0, X)$. Certainly $U \subseteq U_0$. Then $x_{U,V} \in U \cap V \subseteq U_0$. Therefore f converges to x.
We can similarly show that f converges to y.

□

We end this paper with this Proposition that generalizes Corollary 4.4 to an arbitrary directed definable set $(\Omega, \leq)$.

Proposition 4.6. *Let $\mathcal{M} = (M, <, ...)$ be an expansion of a dense linear order without endpoints. Let (X,τ) be a definable topological space and $\gamma : (\Omega, \leq) \to (X,\tau)$ such that x is a cluster point, then for all definable neighborhood U of x, the set $\{x \in \Omega\,;\, \gamma(x) \in U\}$ is infinite.*

Proof. Fix an arbitrary definable open set U containing x, and $d \in \Omega$. We want to show that there is an element $e \leq d$ such that $\gamma(e) \in U$. Let h be the function witnessing that γ' is a subnet of γ; that is, $\gamma' = \gamma \circ h$.
Since h is cofinal, there is an $d_1 \in \Omega$ such that for all $x \leq d_1$, we have $h(x) \leq d$.
γ' converges to x, so $\exists c \in \Omega$ sch that $\gamma'(u) \in U$ for all $u \leq c$.
d_1 and c are elements of Ω, so $\exists \delta \in \Omega$ such that $\delta \leq d_1$ and $\delta \leq c$.
So $e = h(\delta)$ and $\gamma(e) = \gamma(h(\delta)) = \gamma'(\delta) \in U$ as $\delta \leq c$.
Therefore, $e = h(\delta) \leq d$ as $\delta \leq d_1$.

□

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