

TO THE IWASAWA AND HUPPERT THEOREMS ON
SUPERSOLVABILITY OF FINITE GROUPSV.S. MONAKHOV AND I.L. SOKHOR *Communicated by* P.P. PETROV

Abstract: Let $G \neq 1$ be a finite group and let $\mathbb{P}$ be the set of all primes. A chain $1 = M_0 < M_1 < \dots < M_{n-1} < M_n = G$ such that M_i is a maximal subgroup of M_{i+1} for every i is called a maximal chain of G . Every chain is associated with a sequence of non-negative integers $j_1, j_2, \dots, j_n$, where $j_i = |M_i : M_{i-1}|$. A maximal chain is a $\mathbb{P}$ -chain if $j_i \in \mathbb{P}$ for every i . We say that a $\mathbb{P}$ -chain is a $\mathbb{P}^<$ -chain ($\mathbb{P}^>$ -chain) if $j_1 \leq j_2 \leq \dots \leq j_n$ ($j_1 \geq j_2 \geq \dots \geq j_n$, respectively). We investigate finite groups in which some maximal chains are $\mathbb{P}$ -chains. In particular, we obtain the following criteria for finite groups to be supersolvable: a group G is supersolvable if and only if there are a $\mathbb{P}^>$ -chain and $\mathbb{P}^<$ -chain in G ; a group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^<$ -chain in G . The obtained results are used for characterization of generally supersolvable groups.

Keywords: finite group, maximal subgroup, chain of subgroups, subgroup index, supersolvable group.

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1 Introduction

All groups in this paper are finite. We use the standard notation and terminology of [1].

Let $G \neq 1$ be a group. A subgroup chain

$$1 = M_0 < M_1 < \dots < M_i < M_{i+1} < \dots < M_{n-1} < M_n = G \quad (1)$$

such that M_i is a maximal subgroup of M_{i+1} for every i is called a maximal chain of G , and n is the length of this chain. Every chain is associated with a sequence of non-negative integers $j_1, j_2, \dots, j_n$, where $j_i = |M_i : M_{i-1}|$ is the index of M_{i-1} in M_i , $i = 1, \dots, n$. We use $\mathbb{P}$ to denote the set of all primes.

In 1941, Iwasawa proved the following result.

Theorem (Iwasawa). *A group is supersolvable if and only if its maximal chains have the same length, [2].*

Huppert proved the following fundamental theorem 13 years later.

Theorem (Huppert). *A group is supersolvable if and only if all its maximal subgroups are of prime indices, [3, Theorem 9].*

It follows from the Huppert Theorem that all indices of every maximal chain in a supersolvable group are primes. Therefore, the following concepts are quite natural.

Definition 1. *Let G be a group. If $j_i \in \mathbb{P}$ for every i , then a maximal chain (1) is called a $\mathbb{P}$ -chain of G . We say that a $\mathbb{P}$ -chain is a $\mathbb{P}^<$ -chain ($\mathbb{P}^>$ -chain) if $j_1 \leq j_2 \leq \dots \leq j_n$ ($j_1 \geq j_2 \geq \dots \geq j_n$, respectively). A maximal chain of G is said to be a monotone $\mathbb{P}$ -chain if it is a $\mathbb{P}^<$ -chain or a $\mathbb{P}^>$ -chain.*

In this paper, we investigate groups with $\mathbb{P}$ -chains. We enumerate groups in which all maximal chains of every proper subgroup are monotone $\mathbb{P}$ -chains, Theorem 1. In Theorem 2, we indicate the properties of the class of all groups with $\mathbb{P}^<$ -chain. The following supersolvability criteria follow from this theorem: a group G is supersolvable if and only if there are a $\mathbb{P}^>$ -chain and $\mathbb{P}^<$ -chain in G ; a group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^<$ -chain in G . The obtained results are used for characterization of generally supersolvable groups, Theorems 3, 4.

2 Used notation and concepts.

If X is a subgroup (proper subgroup, maximal subgroup) of a group Y , then we write $X \leq Y$ ($X < Y$, $X \triangleleft Y$, respectively). We use $A \leq_t B$ if $A \leq B$ and $|B : A| = t$. We write $r = \max \pi(G)$ ($r = \min \pi(G)$) to indicate that r is the greatest prime divisor (the lowest prime divisor, respectively) of the order of a group G . Here and later, $\pi(G)$ is the set of all prime divisors of $|G|$. We use $\mathbb{P}$ and $\mathbb{N}$ to denote the sets of all primes and all non-negative integers, respectively; A_n and S_n denotes alternating and symmetric groups

of degree n , respectively; C_n is a cyclic group of order n , and C_n^t denotes a direct product of t copies of C_n . We use $A \rtimes B$ to denote the semidirect product of a normal subgroup A and a subgroup B .

Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, $p_i \in \mathbb{P}$, $\alpha_i \in \mathbb{N}$, $i = 1, \dots, n$. We say that G has a *Sylow tower* if G has a normal series

$$1 = G_0 < G_1 < \dots < G_{n-1} < G_n = G \quad (2)$$

such that $|G_i| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$ for every i . In that case, G_{i+1}/G_i is isomorphic to a Sylow p_i -subgroup of G for every i . If $p_1 > p_2 > \dots > p_n$, then we say that G has a *Sylow tower of supersolvable type*, and if $p_1 < p_2 < \dots < p_n$, then G has a *Sylow tower of anti-supersolvable type*. The classes of all groups with Sylow towers of supersolvable type and anti-supersolvable type are denoted by $\mathfrak{D}$ and $\overline{\mathfrak{D}}$, respectively. It is easy to check $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ are subgroup-closed hereditary Fitting formations.

Recall a Schmidt group is a non-nilpotent group with all proper subgroups nilpotent. A group G is a minimal non-supersolvable group if G is not supersolvable but every proper subgroup of G is supersolvable. The properties of Schmidt groups and minimal non-supersolvable groups are well known, see, for example, [4, p. 26] or [5, 6]. A group is primary if it is of prime power order.

3 Groups with monotone $\mathbb{P}$ -chains

Since composition factors of a solvable group $G \neq 1$ have prime orders, every composition series of a solvable group is a $\mathbb{P}$ -chain.

Example 1. ex1 (1) In A_4 , there is no subgroup of index 2, hence A_4 has no $\mathbb{P}^>$ -chains, but in A_4 , there is a $\mathbb{P}^<$ -chain: $1 <_2 C_2 <_2 C_2^2 <_3 A_4$.

(2) In $G = C_3^2 \rtimes C_4$ [7, SmallGroup(36,9)], there is no subgroup of index 3, hence G has no $\mathbb{P}^<$ -chains, but G has a $\mathbb{P}^>$ -chain: $1 <_3 C_3 <_3 C_3^2 <_2 C_3^2 \rtimes C_2 <_2 G$.

(3) In $G = C_3^2 \rtimes SL(2, 3)$ [7, SmallGroup(216,153)], there is no subgroup of index 2, hence G has no $\mathbb{P}^>$ -chain. Among the maximal chains of G , only three are $\mathbb{P}$ -chains:

$$1 <_2 C_2 <_3 S_3 <_3 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3, 2) <_3 G,$$

$$1 <_3 C_3 <_2 S_3 <_3 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3, 2) <_3 G,$$

$$1 <_3 C_3 <_3 C_3^2 <_2 C_3 \rtimes S_3 <_2 C_3^2 \rtimes C_4 <_2 PSU(3, 2) <_3 G,$$

but each of them is not monotone. In particular, in $G = C_3^2 \rtimes SL(2, 3)$, there is a $\mathbb{P}$ -chain, but there are no $\mathbb{P}^>$ -chain and no $\mathbb{P}^<$ -chain.

(4) In A_6 , there is no maximal subgroup of prime index, hence A_6 has no $\mathbb{P}$ -chain.

Lemma 1. *lpd Let G be a group. The following statements hold.*

(1) *G contains a $\mathbb{P}^>$ -chain if and only if G has a Sylow tower of supersolvable type.*

(2) If G has a Sylow tower of anti-supersolvable type, then there is a $\mathbb{P}^<$ -chain in G .

(3) In any supersolvable group, there is a $\mathbb{P}^<$ -chain.

Proof. (1) Assume that in G , there is a $\mathbb{P}^>$ -chain (1). If $n = 1$, then G is a group of prime order and the statement is true. Therefore we can assume that $n > 1$. Use induction on n . By induction, M_{n-1} has a Sylow tower of supersolvable type. Hence a Sylow r -subgroup R of M_{n-1} is normal in M_{n-1} for $r = j_1 = \max \pi(M_{n-1})$. If $r = j_n$, then G is an r -group, and the statement is true. Therefore, $r > j_n$ and R is normal in G in view of the Sylow Theorem. By induction, $G/R \in \mathfrak{D}$. Consequently, $G \in \mathfrak{D}$.

Conversely, let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, $p_1 > p_2 > \dots > p_n$, with a Sylow tower of supersolvable type. In that case, G has a normal series (2) such that $|G_i| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i}$ for every i . In particular, G_1 is a normal Sylow p_1 -subgroup of G for $p_1 = \max \pi(G)$. We have that G_1 contains a chain

$$1 < G_1^1 < \dots < G_1^{\alpha_1} = G_1, |G_1^j : G_1^{j-1}| = p_1, j = 1, 2, \dots, \alpha_1.$$

Since G/G_1 has a Sylow tower of supersolvable type, in G/G_1 there is a maximal $\mathbb{P}^>$ -chain

$$1 = G_1/G_1 < M_1/G_1 < \dots < M_m/G_1 = G/G_1$$

by induction. Now, $1 < G_1^1 < \dots < G_1^{\alpha_1} < M_1 < M_2 < \dots < M_m = G$ is a $\mathbb{P}^>$ -chain of G .

(2) Use induction on $|G|$. Let $G \in \overline{\mathfrak{D}}$ and let $r = \max \pi(G)$. In that case, G has a normal Hall r' -subgroup H . By induction, there is a $\mathbb{P}^<$ -chain in H . Continuing this chain to G , we obtain a $\mathbb{P}^<$ -chain.

(3) Use induction on $|G|$. Let G be supersolvable and let $r = \max \pi(G)$. Then G contains a subgroup H of index r . By induction, there is a $\mathbb{P}^<$ -chain in H . Hence in G , there is a $\mathbb{P}^<$ -chain. $\square$

Example 2. In S_4 , A_5 , $PSL(2, 7)$, there are $\mathbb{P}^<$ -chains, but these groups have no Sylow towers. Therefore, Statement (2) of Lemma 1 is not converse.

Theorem 1. *tmmp* Assume that in all proper subgroups of a group G , every maximal chain is a monotone $\mathbb{P}$ -chain. Then $1 \leq |\pi(G)| \leq 3$ and G is a group of one of the following type.

- (1) G is a supersolvable group of order p^n , pq , p^2q or pqr ;
- (2) G is a p -closed Schmidt group with Frattini subgroup of prime order p ;
- (3) G is a non-supersolvable Schmidt group with identity Frattini subgroup.

Conversely, in all proper subgroups of the enumerated groups, every maximal chain is a monotone $\mathbb{P}$ -chain.

Proof. Assume that every maximal chain of G is a monotone $\mathbb{P}$ -chain. In that case, all maximal subgroups of G have prime indices, and G is supersolvable by the Huppert Theorem. If G is not primary, then in G , there is a normal subgroup C_p of prime order p and a subgroup C_q of prime order q , $p \neq q$. Suppose that $H = C_p C_q$ is a proper subgroup of G . If $p > q$, the chain

$1 \triangleleft_q C_q \triangleleft_p H$ is a $\mathbb{P}^<$ -chain and the chain $1 \triangleleft_p C_p \triangleleft_q H$ is a $\mathbb{P}^>$ -chain. Fix a monotone $\mathbb{P}$ -chain from H to G : $H \triangleleft H_1 \triangleleft \dots \triangleleft H_{n-1} \triangleleft H_n = G$. In that case, one of the chains

$$\begin{aligned} 1 \triangleleft_q C_q \triangleleft_p H \triangleleft H_1 \triangleleft \dots \triangleleft H_{n-1} \triangleleft H_n = G, \\ 1 \triangleleft_p C_p \triangleleft_q H \triangleleft H_1 \triangleleft \dots \triangleleft H_{n-1} \triangleleft H_n = G \end{aligned}$$

are not monotone, a contradiction. Therefore, $G = C_p C_q$. Thus, if every maximal chain of G is monotone $\mathbb{P}$ -chain, then either G is primary or G has order pq .

In the sequel, we can assume that G contains a maximal chain that is not a monotone $\mathbb{P}$ -chain. By the above, every proper subgroup of G is either primary or of order pq . Let G be a supersolvable. Then $1 \leq |\pi(G)| \leq 3$, and $|G| = pqr$ if $|\pi(G)| = 3$. If $|\pi(G)| = 2$, then $|G| = p^2q$. Let G be non-supersolvable. In that case, G is a minimal non-supersolvable group, $G = P \rtimes Q$, where Q is a supplement to the normal Sylow p -subgroup P of G and $|P/\Phi(P)| = p^n > p$ by [8, Lemma 2.1]. If $Q_1 \triangleleft Q$, $|Q_1| \neq 1$, then $|PQ_1| = p^n|Q_1| \neq pq$, a contradiction. Consequently, $|Q| = q \in \mathbb{P}$. If $\Phi(P) = 1$, then $\Phi(G) = 1$ and G is a Schmidt group with identity Frattini subgroup, i.e. G is a group from Statement (3) of the theorem. If $\Phi(P) \neq 1$, then $|\Phi(P)Q| = pq$ and $M = \Phi(P)Q \triangleleft G$. Suppose that $M = \Phi(P) \rtimes Q$ is not nilpotent. It follows q divides $p - 1$. In view of $M_G = \Phi(P)$, we have $G/M_G = P/M_G \rtimes M/M_G$ is a Schmidt group with identity Frattini subgroup. Hence $|G : M| = |P/M_G| = p^n$, where n is the multiplicative order of p modulo q , and $n = 1$, a contradiction. Hence M is nilpotent, and G is a p -closed Schmidt group with Frattini subgroup of order p , i.e. G is a group from Statement (2) of the theorem.

It remains to show that in proper subgroups of the enumerated groups, every maximal chain is a monotone $\mathbb{P}$ -chain. Obviously, the statement is true for primary groups, groups of order pq , p^2q , or pqr . If G is a p -closed Schmidt group with Frattini subgroup of prime order p , then $G = P \rtimes Q$, $|P| = p^{1+m}$, $|Q| = q$, where m is the multiplicative order of p modulo q , and maximal subgroups of G have order p^{1+m} or pq . It is clear that in all proper subgroups of G , every maximal chain is a monotone $\mathbb{P}$ -chain. In a non-supersolvable Schmidt group with identity Frattini subgroup, all proper subgroups are primary. Therefore such a group also satisfies the conditions of the theorem. $\square$

4 On the class of groups with $\mathbb{P}^<$ -chains.

Let $\mathcal{P}^<$ ($\mathcal{P}^>$) be the class of all groups in which there is a $\mathbb{P}^<$ -chain (respectively, $\mathbb{P}^>$ -chain). By $\mathfrak{U}$ and $\mathfrak{S}$ we denote the formations of all supersolvable and all solvable groups, respectively. In view of Lemma 1, we have $\mathcal{P}^> = \mathfrak{D}$, $\overline{\mathfrak{D}} \subseteq \mathcal{P}^<$ and $\mathfrak{U} \subseteq \mathcal{P}^<$. In particular, $\mathcal{P}^>$ is a subgroup-closed hereditary Fitting formation. The class $\mathcal{P}^<$ is more complicated.

Theorem 2. *tx The following statements hold.*

- (1) $\mathcal{P}^<$ is closed under taking normal subgroups, quotients, and direct products.
- (2) $\mathcal{P}^< \cap \mathfrak{S}$ is a subgroup-closed formation.
- (3) $\mathcal{P}^< \cap \mathfrak{D} = \mathcal{P}^< \cap \mathcal{P}^> = \mathfrak{U}$.
- (4) If a simple non-abelian group $G \in \mathcal{P}^<$, then $G \in \{PSL(2, 7), PSL(2, 11), PSL(2, 2^n)\}$, where $2^n + 1$ is a Fermat prime.

Proof. (1) Let $G \in \mathcal{P}^<$ and let N be a normal subgroup of G . Then in G , there is a $\mathbb{P}^<$ -chain (1) with prime indices $j_1 \leq j_2 \leq \dots \leq j_n$. Put $K_i = M_i \cap N$ and consider a chain

$$1 = K_0 \leq K_1 \leq \dots \leq K_{n-1} \leq K_n = N. \quad (3)$$

Since N is normal in G , we get K_{i+1} is normal in M_{i+1} and $M_i \leq K_{i+1}M_i \leq M_{i+1}$. In view of $|M_{i+1} : M_i| = j_{i+1} \in \mathbb{P}$, either $M_i = K_{i+1}M_i$ or $K_{i+1}M_i = M_{i+1}$. If $M_i = K_{i+1}M_i$, then $K_i = K_{i+1}$. If $K_{i+1}M_i = M_{i+1}$, then $j_{i+1} = |M_{i+1} : M_i| = |K_{i+1}M_i : M_i| = |K_{i+1} : K_i|$. In chain (3), we leave only one of the coincide subgroups in chain (3) and we get a $\mathcal{P}^<$ -chain of N , i. e. $N \in \mathcal{P}^<$. Thus, $\mathcal{P}^<$ is closed under taking normal subgroups.

It is clear that there is a chain

$$N/N = M_0N/N \leq \dots \leq M_{n-1}N/N \leq M_nN/N = G/N. \quad (4)$$

Note that $|M_{i+1}N/N : M_iN/N| = |M_{i+1}N : M_iN| = |M_{i+1} : M_i| / |(M_{i+1} \cap N) : (M_i \cap N)|$. In view of $|M_{i+1} : M_i| = j_{i+1} \in \mathbb{P}$, we have either $|(M_{i+1} \cap N) : (M_i \cap N)| = |M_{i+1} : M_i|$ and $|M_{i+1}N/N : M_iN/N| = 1$ or $M_{i+1} \cap N = M_i \cap N$ and $|M_{i+1}N/N : M_iN/N| = |M_{i+1} : M_i| = j_{i+1}$. In chain (4), we leave only one of the coincide subgroups in chain (4) and we get a $\mathcal{P}^<$ -chain of G/N , i. e. $G/N \in \mathcal{P}^<$. Thus, $\mathcal{P}^<$ is closed under taking quotients.

Now, we check that $\mathcal{P}^<$ is closed under taking direct products. If $A, B \in \mathcal{P}^<$, then there are $\mathbb{P}^<$ -chains

$$1 = A_0 < A_1 < \dots < A_{n-1} < A_n = A, \quad (5)$$

$$a_i = |A_i : A_{i-1}| \in \mathbb{P}, \quad a_1 \leq a_2 \leq \dots \leq a_n,$$

$$1 = B_0 < B_1 < \dots < B_{m-1} < B_m = B, \quad (6)$$

$$b_i = |B_i : B_{i-1}| \in \mathbb{P}, \quad b_1 \leq b_2 \leq \dots \leq b_m.$$

Use induction on $|A \times B|$. Let $r = \max\{a_n, b_m\}$. In that case, in chain (5), A_α is a Hall r' -subgroup of A for some α . Similarly, in chain (6), B_β is a Hall r' -subgroup of B for some β . By induction, $A_\alpha \times B_\beta$ contains a $\mathbb{P}^<$ -chain. It is clear that $A_\alpha \times B_\beta$ is a Hall r' -subgroup of $A \times B$, and we can construct a chain from $A_\alpha \times B_\beta$ to $A \times B$ with indices equal to r . So, $A \times B \in \mathcal{P}^<$.

- (2) Let $G \in \mathcal{P}^<$ and let G be solvable. Then in G , there is a $\mathbb{P}^<$ -chain

$$1 \triangleleft M_1 \triangleleft \dots \triangleleft M_{n-1} = M \triangleleft M_n = G.$$

Hence $|G : M| = r = \max \pi(G)$, and by induction, there is a $\mathbb{P}^<$ -chain in every subgroup of M . Let $H \triangleleft G$. If H and M are conjugate, then H contains a $\mathbb{P}^<$ -chain. Assume that H and M are not conjugate. Since G is solvable, we have $G = MH$ and $|G : H| = |H : H \cap M| = r = \max \pi(G)$. In view of

$H \cap M \leq M$, we deduce that $H \cap M$ has a $\mathbb{P}^<$ -chain, and so, H also has a $\mathbb{P}^<$ -chain. Thus, there is a $\mathbb{P}^<$ -chain in every maximal subgroup of G . By induction, in every subgroup of G , there is a $\mathbb{P}^<$ -chain, therefore the class $\mathcal{P}^< \cap \mathfrak{S}$ is closed under taking subgroups, i.e. it is subgroup-closed. From Statement (1), we deduce that $\mathcal{P}^< \cap \mathfrak{S}$ is a subgroup-closed formation.

(3) By Lemma 1 (2-3), $\mathcal{P}^< \cap \overline{\mathfrak{D}} = \overline{\mathfrak{D}}$ and $\mathfrak{U} \subseteq \mathcal{P}^<$. Since $\mathfrak{U} \subseteq \mathfrak{D}$, we get $\mathfrak{U} \subseteq \mathcal{P}^< \cap \mathfrak{D}$. Suppose that $\mathcal{P}^< \cap \mathfrak{D} \not\subseteq \mathfrak{U}$, and let G be a group of least order such that $G \in \mathcal{P}^< \cap \mathfrak{D} \setminus \mathfrak{U}$. In that case, G is a minimal non-supersolvable group and G has a Sylow tower of supersolvable type. Therefore, $|G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})| = p^n > p$ for $p = \max \pi(G)$ by [8, Lemma 2.1] and G has no subgroup of index p . Hence G has no $\mathbb{P}^<$ -chain, i.e. $G \notin \mathcal{P}^<$, a contradiction. Consequently, $\mathcal{P}^< \cap \mathfrak{D} \subseteq \mathfrak{U}$ and $\mathcal{P}^< \cap \mathfrak{D} = \mathfrak{U}$.

(4) Simple non-abelian groups with $\mathbb{P}$ -chains were enumerated in [9, Theorem 3.2]. From these groups only $PSL(2, 7)$, $PSL(2, 11)$ and $PSL(2, 2^n)$, where $2^n + 1$ is a Fermat prime, have $\mathbb{P}^<$ -chains. $\square$

Corollary 1. *ctx A group G is supersolvable if and only if there are a $\mathbb{P}^<$ -chain and a $\mathbb{P}^>$ -chain in G .*

Proof. If G is supersolvable, then G has a Sylow tower of supersolvable type [1, VI.9.1(c)], and by Lemma 1 (1) (3), there are a $\mathbb{P}^>$ -chain and a $\mathbb{P}^<$ -chain in G . Conversely, assume that in G , there are a $\mathbb{P}^<$ -chain and a $\mathbb{P}^>$ -chain. In that case, $G \in \mathcal{P}^<$ by the definition of the class $\mathcal{P}^<$, and $G \in \mathfrak{D}$ by Lemma 1 (1). Thus, $G \in \mathfrak{D} \cap \mathcal{P}^<$ and G is supersolvable by Theorem 2 (3). $\square$

Corollary 2. *ctx2 A group G is supersolvable if and only if G has a Sylow tower of supersolvable type and there is a $\mathbb{P}^<$ -chain in G .*

Proof. If G is supersolvable, then G has a Sylow tower of supersolvable type [1, VI.9.1(c)], and there is a $\mathbb{P}^<$ -chain in G by Lemma 1 (3). Conversely, if G has a Sylow tower of supersolvable type [1, VI.9.1(c)] and there is a $\mathbb{P}^<$ -chain in G , then $G \in \mathfrak{D} \cap \mathcal{P}^<$. According to Theorem 2 (3), G is supersolvable. $\square$

Example 3. The group $G = C_5^2 \rtimes D_8$ [7, SmallGroup(200,43)] contains subgroups A and B such that $A \cong B \cong D_{10}^2$ and $|G : A| = |G : B| = 2$. These subgroups are supersolvable, therefore they have $\mathbb{P}^<$ -chain. In G , there is no maximal subgroups of index 5, hence G contains no $\mathbb{P}^<$ -chain. Thus, the normal subgroup-closed class $\mathcal{P}^<$ is not a Fitting class.

5 Characterisations of w- and v-supersolvable groups

In [9], the following concept was proposed.

Definition 2. *A subgroup H of a group G is $\mathbb{P}$ -subnormal in G if either $G = H$ or there is a chain*

$$H = H_0 < H_1 < \dots < H_{n-1} < H_n = G \quad (7)$$

such that $|H_{i+1} : H_i| \in \mathbb{P}$ for every i .

It follows from the Huppert Theorem that we can define the formation $\mathfrak{U}$ of all supersolvable groups as the class of all groups in which every subgroup is $\mathbb{P}$ -subnormal. The classes of all groups with $\mathbb{P}$ -subnormal Sylow subgroups and with $\mathbb{P}$ -subnormal cyclic primary subgroups are denoted by $w\mathfrak{U}$ and $v\mathfrak{U}$, respectively. These classes were quit well investigated [8, 9, 10, 11]. In particular, $w\mathfrak{U}$ and $v\mathfrak{U}$ are subgroup-closed hereditary formations, $\mathfrak{U} \subset w\mathfrak{U} \subset v\mathfrak{U} \subset \mathfrak{D}$, and all inclusions are proper. Groups from $w\mathfrak{U}$ is called *w-supersolvable*, groups from $v\mathfrak{U}$ is called *v-supersolvable*.

In the context of this paper, the following concepts are quite natural.

Definition 3. A subgroup H of a group G is $\mathbb{P}^<$ -subnormal ($\mathbb{P}^>$ -subnormal) in G , if either $G = H$ or there is a chain (7) such that $|H_{i+1} : H_i| \in \mathbb{P}$ and $|H_i : H_{i-1}| \leq |H_{i+1} : H_i|$ (respectively, $|H_i : H_{i-1}| \geq |H_{i+1} : H_i|$) for every i .

It is clear that every $\mathbb{P}^<$ -subnormal and every $\mathbb{P}^>$ -subnormal subgroup is $\mathbb{P}$ -subnormal. The converse is not true in general, see Example 1 (3).

In this section, supersolvable, w- and v-supersolvable groups are characterized by the existence or unexistence of $\mathbb{P}^<$ - or $\mathbb{P}^>$ -subnormal subgroups in them.

Lemma 2. *Let H be a $\mathbb{P}^<$ -subnormal subgroup of a group G , let K be a subgroup of G , and let N be a normal subgroup of G . The following statements hold.*

- (1) $(H \cap N)$ is $\mathbb{P}^<$ -subnormal in N .
- (2) HN is $\mathbb{P}^<$ -subnormal in G .
- (3) HN/N is $\mathbb{P}^<$ -subnormal in G/N .

Proof. For $\mathbb{P}$ -subnormal subgroups, all statements are true, see [10, Lemma 1]. By hypothesis, there is a chain (7) with indices $p_1 \leq p_2 \leq \dots \leq p_n$.

- (1) It is clear that there is a chain

$$H \cap N = H_0 \cap N \leq H_1 \cap N \leq \dots \leq H_{n-1} \cap N \leq H_n \cap N = N.$$

Since N is normal in G , we deduce that $|H_i \cap N : H_{i-1} \cap N|$ divides $p_i = |H_i : H_{i-1}|$ for every i . We leave only one of the coincide subgroups in the chain and we get that $H \cap N$ is $\mathbb{P}^<$ -subnormal in N .

- (2) Since N is normal in G , there is a chain

$$HN = H_0N \leq H_1N \leq \dots \leq H_{n-1}N \leq H_nN = GN.$$

In view of

$$|H_{i+1}N : H_iN| = \frac{|H_{i+1} : H_i|}{|H_{i+1} \cap N : H_i \cap N|} \in \{1 \cup \mathbb{P}\},$$

we leave only one of the coincide subgroups in the chain and we get that HN is $\mathbb{P}^<$ -subnormal in G .

- (3) Since N is normal in G , there is a chain

$$HN/N = H_0N/N \leq H_1N/N \leq \dots \leq H_{n-1}N/N \leq H_nN/N = GN/N.$$

In view of

$$|H_{i+1}N/N : H_iN/N| = \frac{|H_{i+1} : H_i|}{|H_{i+1} \cap N : H_i \cap N|} \in \{1 \cup \mathbb{P}\},$$

we leave only one of the coincide subgroups in the chain and we get that HN/N is $\mathbb{P}^<$ -subnormal in G/N . $\square$

We can repeat the proof of Lemma 2 with replacing « $\mathbb{P}^<$ -subnormality» by « $\mathbb{P}^>$ -subnormality» to get the similar properties of $\mathbb{P}^>$ -subnormal subgroups.

Lemma 3. *td> The following statements hold.*

- (1) *If $G \in \mathfrak{D}$, then every $\mathbb{P}$ -subnormal subgroup of G is $\mathbb{P}^>$ -subnormal.*
- (2) *In supersolvable group, every subgroup is $\mathbb{P}^>$ -subnormal.*
- (3) *If $G \in \mathfrak{D}$, then every $\mathbb{P}$ -subnormal subgroup of G is $\mathbb{P}^<$ -subnormal.*

Proof. (1) Let H be a $\mathbb{P}$ -subnormal subgroup of a group $G \in \mathfrak{D}$. Use induction on $|G : H|$. Assume that R is a Sylow r -subgroup of G for $r = \max \pi(G)$. In view of $G \in \mathfrak{D}$, R is normal in G by the definition of the class $\mathfrak{D}$. If $G = HR$, then H is $\mathbb{P}^>$ -subnormal in G . Assume that HR is a proper subgroup of G . In view of [10, Lemma 4], HR is $\mathbb{P}$ -subnormal in G and H is $\mathbb{P}$ -subnormal in HR . By induction, HR is $\mathbb{P}^>$ -subnormal in G , hence there is a $\mathbb{P}^>$ -chain $HR <_{q_1} U_1 <_{q_2} U_2 <_{q_3} \dots <_{q_{n-1}} U_{n-1} <_{q_n} U_n = G$, $q_1 \geq q_2 \geq \dots \geq q_{n-1} \geq q_n$.

In a $\mathbb{P}$ -chain from H to HR , all indices are equal to $r > q_1$. Therefore, H is $\mathbb{P}^>$ -subnormal in G .

(2) It follows from Statement (1) because in a supersolvable group, every subgroup is $\mathbb{P}$ -subnormal.

(3) We repeat the proof of Statement (1) with replacing « $r = \max \pi(G)$ » by « $r = \min \pi(G)$ » and we get that the statement is true. $\square$

Theorem 3. *tu Let G be a group. The following statements hold.*

- (1) *If G is supersolvable, then every Hall subgroup of G is $\mathbb{P}^>$ -subnormal and $\mathbb{P}^<$ -subnormal in G .*
- (2) *G is supersolvable if and only if every Sylow subgroup of G is $\mathbb{P}^<$ -subnormal in G .*
- (3) *G is w-supersolvable if and only if every Sylow subgroup of G is $\mathbb{P}^>$ -subnormal in G .*

Proof. (1) Let G be supersolvable and let H be a Hall subgroup of G . In view of Lemma 3(2), H is $\mathbb{P}^>$ -subnormal in G . Put

$$\pi = \pi(G) \setminus \pi(H) = \{q_1, q_2, \dots, q_n\}.$$

Then there is Sylow q_i -subgroups Q_i such that $HQ_1Q_2 \dots Q_i$ is a subgroup of G for every i . Since G is supersolvable, it follows from the Huppert Theorem that a chain

$$H < HQ_1 < HQ_1Q_2 < \dots < HQ_1Q_2 \dots Q_{n-1} < G$$

can be compacted so that all indices of new chain are prime. It is true for any ordering π . For $q_1 < q_2 < \dots < q_n$, a new chain shows that H is $\mathbb{P}^<$ -subnormal in G .

(2) If G is supersolvable, then every Sylow subgroup of G is $\mathbb{P}^<$ -subnormal in G by Statement (1). Conversely, assume that every Sylow subgroup of G is

$\mathbb{P}^<$ -subnormal in G . In that case, $G \in \mathbf{w}\mathfrak{U} \subset \mathfrak{D}$. In view of Lemma 2 (3) and by induction, G/N is supersolvable for every non-trivial normal subgroup N of G . Therefore G is a primitive group, and $G = R \rtimes M$, where $R = F(G) = C_G(R)$ is a Sylow r -subgroup of G for $r = \max \pi(G)$ and a unique minimal normal subgroup, and $M \triangleleft G$. Let Q be a Sylow q -subgroup of M . Then $q \neq r$ and Q is a Sylow q -subgroup of G . By hypothesis, Q is $\mathbb{P}^<$ -subnormal in G . Hence, there is a subgroup H of G such that $Q \leq H$ and $|G : H| = r$. Consequently, $M \leq H^g \leq G$ for some $g \in G$. Since $M \triangleleft G$, we have $M = H^g$ and $|R| = |G : M| = r$. Therefore, $M \cong G/R = N_G(R)/C_G(R)$ is a cyclic group of order dividing $r - 1$, and G is supersolvable.

(3) Since every w-supersolvable group G has a Sylow tower of supersolvable type, then every Sylow subgroup of G is $\mathbb{P}^>$ -subnormal in G by Lemma 3 (1). Conversely, if every Sylow subgroup of G is $\mathbb{P}^>$ -subnormal in G , then every Sylow subgroup of G is $\mathbb{P}$ -subnormal in G , and G is w-supersolvable by definition. $\square$

Theorem 4. *Let G be a group. The following statements hold.*

(1) *If every cyclic primary subgroup of G is $\mathbb{P}^<$ -subnormal in G , then G is supersolvable.*

(2) *G is v-supersolvable if and only if every cyclic primary subgroup of G is $\mathbb{P}^>$ -subnormal in G .*

Proof. (1) Let every cyclic primary subgroup of G be $\mathbb{P}^<$ -subnormal in G . In that case, $G \in \mathbf{v}\mathfrak{U} \subset \mathfrak{D}$, and by Lemma 1 (1), G contains a $\mathbb{P}^>$ -chain. Let $q = \min \pi(G)$. By hypothesis, a subgroup of prime order q is $\mathbb{P}^<$ -subnormal in G . Therefore G contains a $\mathbb{P}^<$ -chain. By Corollary 1, G is supersolvable.

(2) If $G \in \mathbf{v}\mathfrak{U}$, then $G \in \mathfrak{D}$, and by Lemma 3 (1), every cyclic primary subgroup is $\mathbb{P}^>$ -subnormal in G . Conversely, if every cyclic primary subgroup A of G is $\mathbb{P}^>$ -subnormal in G , then A is $\mathbb{P}$ -subnormal in G , and $G \in \mathbf{v}\mathfrak{U}$. $\square$

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